LATTICE AUTOMORPHISMS OF SEMI-SIMPLE LIE ALGEBRAS

Dedicated to the memory of Hanna Neumann

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1. Introduction

Let L be a Lie algebra over the field F. A lattice automorphism of L is an automorphism $\phi: \mathscr{L}(L) \to \mathscr{L}(L)$ of the lattice $\mathscr{L}(L)$ of all subalgebras of L. We seek to describe the lattice automorphisms in terms of maps $\sigma: L \to L$ of the underlying algebra. A semi-automorphism σ of L is an automorphism of the algebraic system consisting of the pair (F, L), that is, a pair of maps $\sigma: F \to F$, $\sigma: L \to L$ preserving the operations. Thus $\sigma: F \to F$ is an automorphism of F and $(x + y)^\sigma = x^\sigma + y^\sigma$, $(xy)^\sigma = x^\sigma y^\sigma$, $(\lambda x)^\sigma = \lambda^\sigma x^\sigma$ for all $x, y \in L$ and $\lambda \in F$. Clearly, any semi-automorphism of L induces a lattice automorphism. To study a given lattice automorphism ϕ, we select a semi-automorphism σ such that $\phi\sigma^{-1}$ fixes certain subalgebras, and so we reduce the problem to the investigation of lattice automorphisms leaving these subalgebras fixed.

Except where otherwise stated, we suppose that F is an algebraically closed field of characteristic 0 and that L is semi-simple of rank $r \geqq 3$. For $x \in L$, we denote by x_s, x_n the semi-simple and nil parts[1] respectively of x. We shall prove

THEOREM. *Let L be a semi-simple Lie algebra of rank $\geqq 3$ over an algebraically closed field F of characteristic* 0. *For $t \in F$, $t \neq 0$, let $\tau(t): L \to L$ be the map defined by $x^{\tau(t)} = x_s + tx_n$ for $x \in L$. Let ϕ be a lattice automorphism of L. Then there exists a unique semi-automorphism σ of L and a unique $t \in F$, $t \neq 0$ such that $\sigma\tau(t)$ induces ϕ. Conversely, for all semi-automorphisms σ and all $t \in F$, $t \neq 0$, the map $\sigma\tau(t)$ induces a lattice automorphism of L.*

If A is a subset of L, we denote by $\langle A \rangle$ the subspace spanned by A. We denote by $\langle\langle A \rangle\rangle$ the subalgebra generated by A. We say that the lattice automorphism ϕ fixes the subalgebra A if $A^\phi = A$. Let A be any subspace of L. We say that ϕ is the identity on A if $U^\varphi = U$ for all subalgebras U of L contained in A, that is, if $\langle a \rangle^\phi = \langle a \rangle$ for all $a \in A$. We say that ϕ is linear (semi-linear) on A if

1. See Jacobson [3] p. 98.

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there exists a linear (respectively semi-linear) transformation $\sigma: A \to A$ such that $\langle a\rangle^\phi = \langle a^\sigma\rangle$ for all $a \in A$.

We note that, if A is a subalgebra fixed by ϕ, dim $A \geqq 3$, and if every subspace of A is a subalgebra, then by the Fundamental Theorem of Projective Geometry, ϕ is semi-linear on A. If ϕ is semi-linear on A and there exists a 2-dimensional subspace $B \subseteq A$ on which ϕ is linear, then ϕ is linear on A.

2. Reduction

Let H be a Cartan subalgebra of L. We denote by h_α the scaled star vector defined by

$$(h, h_\alpha) = \alpha(h)/(\alpha, \alpha)$$

for all $h \in H$. This departure from the usual notation is convenient for our purposes. For any choice of root vector e_α, we have $h_\alpha e_\alpha = e_\alpha$. We now choose root vectors e_α such that $e_{-\alpha}e_\alpha = h_\alpha$ and such that all the multiplication constants $v_{\alpha,\beta}$ (defined when $\alpha, \beta, \alpha+\beta$ are roots by $e_\alpha e_\beta = v_{\alpha,\beta}\, e_{\alpha+\beta}$) are rational. We choose an order relation $<$ on the roots in the usual way. We put $K_\alpha = \ker \alpha$.

DEFINITION 2.1. We say that the lattice automorphism ϕ is reduced with respect to H, $\{e_\alpha \mid \alpha \text{ root}\}$, $<$ if ϕ is the identity on H and fixes $\langle h_\alpha + e_\alpha\rangle$ for every fundamental root α.

In §4, we shall show that the concept of a reduced lattice automorphism is in fact independent of the choices of H, $\{e_\alpha \mid \alpha \text{ root}\}$ and of $<$. If $\langle e_\gamma\rangle$ is a root space for H, then $\langle e_\gamma\rangle^\phi$ is a root space for H^ϕ by Barnes [1], Lemma 4. If ϕ is reduced, we have that ϕ fixes $\langle h_\alpha\rangle$, $\langle h_\alpha + e_\alpha\rangle$ and so also $\langle e_\alpha\rangle$ for each fundamental root α. Since ϕ also preserves sums of roots ([1], proof of Theorem 2), it follows that ϕ fixes $\langle e_\gamma\rangle$ for all roots γ.

LEMMA 2.2. *Let ϕ be a lattice automorphism of L. Then there exists a unique semi-automorphism σ of L such that $\phi^{\sigma-1}$ is reduced with respect to given H, $\{e_\alpha\}$, $<$.*

PROOF. H^ϕ is a Cartan subalgebra, $\langle e_\alpha\rangle^\phi$ is the root space of a root α^ϕ for H^ϕ, and the correspondence $\alpha \to \alpha^\phi$ preserves sums. Therefore there exists an automorphism σ_1 of L such that $H^{\sigma'} = H^\phi$ and $\langle e_\alpha^{\sigma'}\rangle = \langle e_\alpha\rangle^\phi$ for all roots α. Thus $\phi\sigma_1^{-1}$ fixes H and all the root spaces $\langle e_\alpha\rangle$. This implies that $\phi\sigma_1^{-1}$ fixes $\langle h_\alpha\rangle$ for all α. Since H is abelian and dim $H \geqq 3$, $\phi\sigma_1^{-1}$ is semi-linear on H. Let $\alpha_1, \cdots, \alpha_r$ be the fundamental roots and put $h_i = h_{\alpha_i}$. The semi-linear transformation fixes all the $\langle h_i\rangle$, and so has the form

$$\sum_{i=1}^{r} x_i h_i \to \sum_{i=1}^{r} \lambda_i x_i^\tau h_i$$

where τ is an automorphism of F and $\lambda_1, \cdots, \lambda_r \in F$. We now prove $\lambda_i = \lambda_j$ for all i, j.

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Suppose $\alpha_i + \alpha_j = \beta$ is a root. Then $h_\beta = xh_{\alpha_i} + yh_{\alpha_j}$, where $x = (\alpha_i, \alpha_i)/(\beta, \beta)$ and $y = (\alpha_j, \alpha_j)/(\beta, \beta)$. Since x, y are rational, $x^\tau = x$, $y^\tau = y$ and the semi-linear transformation sends h_β to $\lambda_i x h_{\alpha_i} + \lambda_j y h_{\alpha_j}$. But $\langle h_\beta \rangle$ is fixed by $\phi\sigma_1^{-1}$ and therefore $\lambda_i = \lambda_j$.

Suppose $\alpha_i + \alpha_j$ is not a root. Then $e_{\alpha_i} e_{\alpha_j} = 0$, $\langle e_{\alpha_i}, e_{\alpha_j} \rangle$ is a subalgebra fixed by $\phi\sigma_1^{-1}$ and it follows that $\phi\sigma_1^{-1}$ permutes the subalgebras $\langle e_{\alpha_i} + xe_{\alpha_j} \rangle$ for $x \neq 0$. The subalgebra $\langle h_i + yh_j \rangle$ generates with $\langle e_{\alpha_i} + xe_{\alpha_j} \rangle$ a 2-dimensional subalgebra ff and only if $y = 1$. Since this property is preserved by $\phi\sigma_1^{-1}$, $\langle h_i + h_j \rangle$ is fixed by $\phi\sigma_1^{-1}$ and we have $\lambda_i = \lambda_j$.

We have now shown that $\phi\sigma_1^{-1}$ is given on H by the semi-linear map

$$\sum_{i=1}^{r} x_i h_i \to \sum_{i=1}^{r} x_i^\tau h_i.$$

We define $\sigma_2 : L \to L$ by

$$\left(\sum_{i=1}^{r} x_i h_i + \sum_{\gamma \text{root}} y_\gamma e_\gamma \right)^{\sigma_2} = \sum_{i=1}^{r} x_i^\tau h_i + \sum_{\gamma \text{root}} y_\gamma^\tau e_\gamma.$$

Since the multiplication constants are rational, they are fixed under τ and σ_2 is a semi-automorphism. We now have that $\phi\sigma_1^{-1}\sigma_2^{-1}$ is the identity on H and fixes all the $\langle e_\gamma \rangle$.

Let $\lambda_1, \cdots, \lambda_r$ be non-zero elements of F. For each root

$$\gamma = \sum_{i=1}^{r} m_i \alpha_i, \ (m_i \in \mathbb{Z},) \text{ put } \lambda_\gamma = \prod_{i=1}^{r} \lambda_i^{m_i}.$$

Then the map $\sigma_3 : L \to L$ defined by

$$(*) \qquad \left(h + \sum_\gamma x_\gamma e_\gamma \right)^{\sigma_3} = h + \sum_\gamma x_\gamma \lambda_\gamma e_\gamma$$

for $h \in H$, $x_\gamma \in F$, is an automorphism of L and every automorphism of L which is the identity on H has this form. We have $\langle h_i + e_{\alpha_i} \rangle^{\varphi\sigma_1^{-1}\sigma_2^{-1}} = \langle h_i + \mu_i e_{\alpha_i} \rangle$ for some $\mu_i \neq 0$. Put $\lambda_i = 1/\mu_i$ $(i = 1, 2, \cdots, r)$ and then $\sigma = \sigma_3 \sigma_2 \sigma_1$ is a semi-automorphism with the required properties.

Suppose σ' is another such semi-automorphism. Then $\sigma'\sigma^{-1}$ is a semi-automorphism of L. For all $h \in H$,

$$\langle h \rangle^\phi = \langle h \rangle^\sigma = \langle h \rangle^{\sigma'}$$

and $\sigma'\sigma^{-1}$ is a scalar multiple of the identity on H. As $\sigma'\sigma^{-1}$ is linear on H, it is also linear on L and so is an automorphism of L. Since we also have $\langle h_i + e_{\alpha_i} \rangle^{\sigma'\sigma^{-1}} = \langle h_i + e_{\alpha_i} \rangle$, it follows that the permutation of the roots given by $\sigma'\sigma^{-1}$ is the identity and hence, that $h^{\sigma'\sigma^{-1}}$ is the scaled star vector of the root α_i, that is $h_i^{\sigma'\sigma^{-1}} = h_i$ and $\sigma'\sigma^{-1}$ is the identity on H. Thus $\sigma'\sigma^{-1}$ is an automorphism of the form (*) above, and, since $\langle h_i + e_{\alpha_i} \rangle^{\sigma'\sigma^{-1}} = \langle h_i + e_{\alpha_i} \rangle$, it follows that $\sigma'\sigma^{-1} = 1$.

3. The algebra $\mathfrak{A}_1$

Suppose the lattice automorphism ϕ of L is reduced with respect to H. If α is a root, then $\langle h_\alpha, e_\alpha, e_{-\alpha}\rangle$ is a subalgebra of type $\mathfrak{A}_1$ and ϕ gives a lattice automorphism of $\langle h_\alpha, e_\alpha, e_{-\alpha}\rangle$. Consequently it is of interest to determine the lattice automorphisms of $\mathfrak{A}_1$.

Let S be the Lie algebra $\langle h, e, f\rangle$ with defining relations $he = e$, $hf = -f$, $ef = h$. We consider the projective plane whose points and lines are the 1 and 2-dimensional subspaces of S. The conic defined by the vanishing of the Killing form is the set of 1-dimensional subalgebras which are not Cartan subalgebras. The 2-dimensional subalgebras are the tangents to the conic, each point of the conic being a root space for the Cartan subalgebras on the tangent at the point.

Let ϕ be a lattice automorphism of S. Then ϕ permutes the points of the conic. Since each point not on the conic is the intersection of two tangents, this permutation determines ϕ completely. Since the tangents are the only 2-dimensional subalgebras, every permutation of the points of the conic gives a lattice automorphism of S.

LEMMA 3.1. *Let ϕ be a lattice automorphism of S which fixes $\langle h\rangle$ and is linear on $\langle h, e\rangle$. Suppose $\langle h + e\rangle^\phi = \langle h + \lambda e\rangle$. Then ϕ is given by*

$$\langle xh + ye + zf\rangle^\phi = \langle xh + \lambda ye + \frac{1}{\lambda} zf\rangle$$

for x, y, $z \in F$.

PROOF. For $u = xh + ye + zf$, $(u, u) = 2(x^2 + 2yz)$. Thus the conic can be expressed parametrically

$$x = \theta,\ y = -\tfrac{1}{2}\theta^2,\ z = 1.$$

The tangent to the conic at $(\theta, -\frac{1}{2}\theta^2, 1)$ has equation $\theta x + y - \frac{1}{2}\theta^2 z = 0$ and meets $\langle h, e\rangle$ in $\langle h + \theta e\rangle$. But $\langle h + \theta e\rangle^\phi = \langle h + \lambda\theta e\rangle$. Thus $(\theta, -\frac{1}{2}\theta^2, 1)^\phi = (\lambda\theta, -\frac{1}{2}\lambda^2\theta^2, 1)$. The tangents from (x_0, y_0, z_0) meet the conic at its points of intersection with the polar line $x_0 x + z_0 y + y_0 z = 0$, that is at the points $(\theta, -\frac{1}{2}\theta^2, 1)$ for θ satisfying $x_0\theta - \frac{1}{2}z_0\theta^2 + y_0 = 0$. The tangents from $(x_1, y_1, z_1) = (x_0, y_0, z_0)^\phi$ meet the conic in the points $(\theta', -\frac{1}{2}\theta'^2, 1)$ for θ' satisfying $x_0\theta'/\lambda + z_0$

$$x_0\frac{\theta'}{\lambda} - \tfrac{1}{2}z_0\left(\frac{\theta'}{\lambda}\right)^2 + y_0 = 0.$$

Therefore

$$x_1 : y_1 : z_1 = x_0/\lambda : y_0 : z_0/\lambda^2,$$

and

$$\langle x_0 h + y_0 e + z_0 f\rangle^\phi = \langle x_0 h + \lambda y_0 e + \frac{1}{\lambda} z_0 f\rangle.$$

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4. Reduced lattice automorphisms

Throughout this section, we suppose that ϕ is reduced with respect to $H, \{e_\alpha \mid \alpha \text{ root}\}, <$.

LEMMA 4.1. *Let α be a root. Then there exists a root β such that $(\alpha, \beta) = 0$ and $\alpha + \beta$ is not a root.*

PROOF. We need only consider the case where α is a fundamental root at the end of a component of the Dynkin diagram since, for any α, there exists an element of the Weyl group which transforms α into such a root. For such a root α, there is a point β of the Dynkin diagram not joined to α and the result follows.

LEMMA 4.2. *For each root α, there is $\lambda_\alpha \in F$ such that $\langle h_\alpha + xe_\alpha \rangle^\varphi = \langle h_\alpha + \lambda_\alpha x e_\alpha \rangle$ for all $x \in F$, and $\lambda_\alpha \lambda_{-\alpha} = 1$.*

PROOF. We choose a root β such that $(\alpha, \beta) = 0$ and $\alpha + \beta$ is not a root. There exist $k \in K_\alpha \cap K_\beta$ and $n \in K_\alpha$ such that k and n are linearly independent. Then $\langle k, n, e_\alpha \rangle$ is an abelian 3-dimensional subalgebra fixed under ϕ. Therefore ϕ is semi-linear on $\langle k, n, e_\alpha \rangle$. But ϕ is linear on $\langle k, n \rangle$ and therefore also on the fixed subalgebra $\langle k, e_\alpha \rangle$. Since $\langle k, e_\alpha, e_\beta \rangle$ is also a 3-dimensional abelian fixed subalgebra, it follows that ϕ is linear on $\langle e_\alpha, e_\beta \rangle$. Put $h = h_\alpha + h_\beta$. Every subspace of $\langle h, e_\alpha, e_\beta \rangle$ is a subalgebra and it follows that ϕ is also linear on $\langle h, e_\alpha, e_\beta \rangle$. Since ϕ fixes $\langle h \rangle$, $\langle e_\alpha \rangle$ and $\langle e_\beta \rangle$, ϕ is given on $\langle h, e_\alpha, e_\beta \rangle$ by

$$\langle xh + ye_\alpha + ze_\beta \rangle^\phi = \langle xh + \lambda_\alpha y e_\alpha + \lambda_\beta z e_\beta \rangle$$

for some λ_α, $\lambda_\beta \in F$.

In the 3-dimensional projective geometry of subspaces of $\langle h_\alpha, h_\beta, e_\alpha, e_\beta \rangle$, $\langle h_\alpha, e_\alpha \rangle$ and $\langle h_\beta, e_\beta \rangle$ are skew lines. There is a unique transversal to them through $\langle h + xe_\alpha + ye_\beta \rangle$. This transversal $\langle h_\alpha + xe_\alpha, h_\beta + ye_\beta \rangle$ is a subalgebra since $(\alpha, \beta) = 0$ and $\alpha + \beta$ is not a root. Since $\langle h_\alpha + xe_\alpha, h_\beta + ye_\beta \rangle^\phi$ is the transversal through $\langle h + xe_\alpha + ye_\beta \rangle^\phi = \langle h + \lambda_\alpha x e_\alpha + \lambda_\beta y e_\beta \rangle$,

$$\langle h_\alpha + xe_\alpha,\, h_\beta + ye_\beta \rangle^\phi = \langle h_\alpha + \lambda_\alpha x e_\alpha,\, h_\beta + \lambda_\beta y e_\beta \rangle \text{ and}$$

$$\begin{aligned}\langle h_\alpha + xe_\alpha \rangle^\phi &= \langle h_\alpha, e_\alpha \rangle \cap \langle h_\alpha + xe_\alpha, h_\beta + ye_\beta \rangle^\phi \\ &= \langle h_\alpha + \lambda_\alpha x e_\alpha \rangle.\end{aligned}$$

By Lemma 3.1, we have $\lambda_\alpha \lambda_{-\alpha} = 1$.

COROLLARY 4.3. *The concept of a reduced lattice automorphism is independent of the choice of the root vectors e_α.*

PROOF. If ϕ is reduced and α is a fundamental root, then $\lambda_\alpha = 1$ and ϕ is the identity on $\langle h_\alpha, e_\alpha \rangle$. For any choice e'_α of the root vector, we have $\langle h_\alpha + e'_\alpha \rangle^\phi = \langle h_\alpha + e'_\alpha \rangle$ as required.

LEMMA 4.4. *For all* $h \in H - K_\alpha$, $\langle h + e_\alpha \rangle^\phi = \langle h + \lambda_\alpha e_\alpha \rangle$.

PROOF. If $h \in H - K_\alpha$, then $h = uh_\alpha + k$ for some $k \in K_\alpha$ and $u \in F$, $u \neq 0$. By Lemma 4.2, the result holds for $k = 0$. Suppose $k \neq 0$. Then

$$\langle h + e_\alpha \rangle = \langle uh_\alpha + e_\alpha, k \rangle \cap \langle uh_\alpha + k, e_\alpha \rangle$$

and

$$\langle h + e_\alpha \rangle^\phi = \langle uh_\alpha + \lambda_\alpha e_\alpha, k \rangle \cap \langle uh_\alpha + k, e_\alpha \rangle = \langle h + \lambda_\alpha e_\alpha \rangle.$$

LEMMA 4.5. *Let* α, β *be independent roots and let* $N_{\alpha,\beta} = \langle e_\gamma \mid \gamma = r\alpha + s\beta$, $r \geqq 0$, $s \geqq 0 \rangle$. Then

$$\langle xe_\alpha + ye_\beta, N'_{\alpha,\beta} \rangle^\phi = \langle \lambda_\alpha xe_\alpha + \lambda_\beta ye_\beta, N'_{\alpha,\beta} \rangle$$

for all x, $y \in F$.

PROOF. Put

$$p = (\alpha,\alpha) \frac{(\beta,\beta) - (\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta) - (\alpha,\beta)^2}, \quad q = (\beta,\beta) \frac{(\alpha,\alpha) - (\alpha,\beta)}{(\alpha,\alpha)(\beta,\beta) - (\alpha,\beta)^2}$$

and $h = ph_\alpha + qh_\beta$. Then $he_\alpha = e_\alpha$ and $he_\beta = e_\beta$. Put $N = N_{\alpha,\beta}$ and $U = \langle h, N \rangle$. Then N is a fixed nilpotent subalgebra. Since N' is normalised by H, N' is a sum of root spaces and is therefore fixed. But N' is an ideal of U and every subspace of U/N' is a subalgebra. Hence ϕ is semi-linear on U/N'. By Lemma 4.4, we have

$$\langle zh + xe_\alpha \rangle^\phi = \langle zh + \lambda_\alpha xe_\alpha \rangle, \ \langle zh + ye_\beta \rangle^\phi = \langle zh + \lambda_\beta ye_\beta \rangle$$

and it follows that

$$\langle zh + xe_\alpha + ye_\beta, N' \rangle^\phi = \langle zh + \lambda_\alpha xe_\alpha + \lambda_\beta ye_\beta, N' \rangle$$

for all $x, y, z \in F$.

LEMMA 4.6. *There exists* $t = t(H) \in F$, $t \neq 0$, *independent of* α, *such that* $\langle k + e_\alpha \rangle^\phi = \langle k + t\lambda_\alpha e_\alpha \rangle$ *for all* $k \in K_\alpha$.

PROOF. Since $\langle K_\alpha, e_\alpha \rangle$ is a fixed abelian subalgebra of dimension $\geqq 3$, ϕ is semi-linear on $\langle K_\alpha, e_\alpha \rangle$. But ϕ is the identity on K_α and fixes $\langle e_\alpha \rangle$. Hence ϕ is given on $\langle K_\alpha, e_\alpha \rangle$ by $\langle k + e_\alpha \rangle^\phi = \langle k + \mu_\alpha e_\alpha \rangle$ for all $k \in K_\alpha$ and some $\mu_\alpha \in F$, $\mu_\alpha \neq 0$ independent of k. We have to prove that $\mu_\alpha / \lambda_\alpha$ is independent of α.

Let α, β be independent roots. Take $k \in K_\alpha \cap K_\beta$, $k \neq 0$, and put $V = \langle k, N \rangle$ where $N = N_{\alpha,\beta}$. Then N' is an ideal of V, V/N' is abelian, and it follows that ϕ is given on V/N' by

$$\langle zk + xe_\alpha + ye_\beta, N' \rangle^\phi = \langle zk + \mu_\alpha xe_\alpha + \mu_\beta ye_\beta, N' \rangle.$$

Comparing this with the description of ϕ on $\langle e_\alpha, e_\beta, N' \rangle$ given by Lemma 4.5, we get $\mu_\alpha / \lambda_\alpha = \mu_\beta / \lambda_\beta$.

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LEMMA 4.7. *Let $\sigma = \exp(\theta \operatorname{ad} e_\beta)$ where $\theta \in F$. Suppose $\lambda_\beta = 1$. Then ϕ is reduced with respect to H^σ and $<^\sigma$, $\lambda_{\alpha^\sigma} = \lambda_\alpha$ for all roots α of H, and $t(H^\sigma) = t(H)$.*

PROOF. $H = \langle h_\beta, K_\beta \rangle$ and $H^\sigma = \langle h_\beta + \theta e_\beta, K_\beta \rangle$. We have that ϕ is the identity on K_β and, by Lemma 4.4, $\langle h_\beta + k + \theta e_\beta \rangle^\phi = \langle h_\beta + k + \lambda_\beta \theta e_\beta \rangle$ for all $k \in K_\beta$. Since $\lambda_\beta = 1$, ϕ is the identity on H^σ. It follows that, for every root γ of H^σ, either $\langle e_\gamma \rangle^\phi = \langle e_\gamma \rangle$ or $\langle e_\gamma \rangle^\phi = \langle e_{-\gamma} \rangle$. But $\gamma = \alpha^\sigma$ for some root α of H and $e_\gamma = e_\alpha^\sigma \in N_{\alpha,\beta}$. Since $e_\alpha^\sigma \notin N_{-\alpha,\beta}$, $\langle e_\gamma \rangle^\phi \neq \langle e_{-\gamma} \rangle$ and we have $\langle e_\gamma \rangle^\phi = \langle e_\gamma \rangle$ for all roots γ of H^σ.

By Lemma 4.4, for some $\lambda_\gamma \in F$, we have $\langle h + e_\gamma \rangle^\phi = \langle h + \lambda_\gamma e_\gamma \rangle$ for all $h \in H^\sigma - K_\gamma$. (The proof of Lemma 4.4 makes no use of the assumption that $\lambda_\alpha = 1$ for all fundamental roots α.) Trivially, $\lambda_{\beta^\sigma} = \lambda_\beta$.

For $\alpha \neq \pm \beta$, take $h \in K_\beta$, $h \notin K_\alpha$. Then $h \in H^\sigma - K_{\alpha^\sigma}$ and

$$\begin{aligned} \langle h + \lambda_{\alpha^\sigma} e_\alpha^\sigma, N'_{\alpha,\beta} \rangle &= \langle h + e_\alpha^\sigma, N'_{\alpha,\beta} \rangle^\phi = \langle h + e_\alpha, N'_{\alpha,\beta} \rangle^\phi \\ &= \langle h + \lambda_\alpha e_\alpha, N'_{\alpha,\beta} \rangle. \end{aligned}$$

Therefore $\lambda_{\alpha^\sigma} = \lambda_\alpha$. In particular, if α^σ is fundamental with respect to $<^\sigma$, then $\lambda_{\alpha^\sigma} = 1$. Thus ϕ is reduced with respect to H^σ and $<^\sigma$.

Take $k \in K_{\beta^\sigma} = K_\beta$. Since $e_{\beta^\sigma} = e_\beta$, we have

$$\langle k + e_{\beta^\sigma} \rangle^\phi = \langle k + e_\beta \rangle^\phi = \langle k + t\lambda_\beta e_\beta \rangle = \langle k + t\lambda_{\beta^\sigma} e_{\beta^\sigma} \rangle$$

and $t(H^\sigma) = t(H)$.

LEMMA 4.8. *For all α, $\lambda_\alpha = 1$. The concept of a reduced lattice automorphism is independent of the ordering of the roots.*

PROOF. It is sufficient to show that $\lambda_\alpha = \lambda_\beta = 1$ implies $\lambda_{\alpha+\beta} = 1$. Since we may suppose $\alpha + \beta$ is a root (otherwise we have nothing to prove), we have $e_\alpha e_\beta = v e_{\alpha+\beta}$ and $v \neq 0$. Put $\sigma = \exp(1/v \operatorname{ad} e_\beta)$ and $N = N_{\alpha,\beta}$. Since the term $\Gamma^3 N$ of the descending central series of N is normalised by H, it is a sum of root spaces and is therefore fixed. Since $\lambda_\alpha = 1$ and $N'_{\alpha,\alpha+\beta} \subseteq \Gamma^3 N$, we have by Lemma 4.5 that $\langle e_\alpha + e_{\alpha+\beta}, \Gamma^3 N \rangle^\phi = \langle e_\alpha + \lambda_{\alpha+\beta} e_{\alpha+\beta}, \Gamma^3 N \rangle$. But

$$\begin{aligned} \langle e_\alpha + e_{\alpha+\beta}, \Gamma^3 N \rangle^\phi &= \langle e_\alpha^\sigma, \Gamma^3 N \rangle^\phi = \langle e_\alpha^\sigma, \Gamma^3 N \rangle \\ &= \langle e_\alpha + e_{\alpha+\beta}, \Gamma^3 N \rangle. \end{aligned}$$

Since $e_{\alpha+\beta} \notin \Gamma^3 N$, it follows that $\lambda_{\alpha+\beta} = 1$.

LEMMA 4.9. *Let H, H^* be Cartan subalgebras of L, and suppose ϕ is reduced with respect to H. Then ϕ is reduced with respect to H^* and $t(H) = t(H^*)$. The concept of a reduced lattice automorphism is independent of the choice of Cartan subalgebra.*

PROOF. By Lemmas 4.7, 4.8, it is sufficient to prove the existence of a chain of Cartan subalgebras $H_0 = H, H_1, \cdots, H_n = H^*$ and root vectors e_α^i of H_i $(i = 0, 1, \cdots,$

$n-1$) such that $H_{i+1} = H_i \exp(\lambda_i \operatorname{ad} e_\alpha^i)$. By Jacobson [3] p. 288 Exercise 18, there exist roots $\alpha_1, \cdots, \alpha_n$ of H, not necessarily distinct, and $\lambda_i \in F$ such that

$$H^* = \exp(\lambda_1 \operatorname{ad} e_{\alpha_1}) \exp(\lambda_2 \operatorname{ad} e_{\alpha_2}) \cdots \exp(\lambda_n \operatorname{ad} e_{\alpha_n}).$$

Put

$$\sigma_i = \exp(\lambda_{n-i} \operatorname{ad} e_{\alpha_{n-i}}) \exp(\lambda_{n-i+1} \operatorname{ad} e_{\alpha_{n-i+1}}) \cdots \exp(\lambda_n \operatorname{ad} e_{\alpha_n}),$$

$H_{i+1} = H^{\sigma_i}$ and $e_\alpha^{i+1} = e_{\alpha_{n-i-1}}^{\sigma_i}$. Then $\operatorname{ad} e_\alpha^{i+1} = \sigma_i^{-1}(\operatorname{ad} e_{\alpha_{n-i-1}})\sigma_i$ and $\sigma_i \exp(\lambda_{i+1} \operatorname{ad} e_\alpha^{i+1}) = \sigma_{i+1}$. Thus we have e_α^i a root vector for H_i, $H_{i+1} = H_i \exp(\lambda_i \operatorname{ad} e_\alpha^i)$ and $H_n = H^*$.

LEMMA 4.10. *Suppose ϕ is a reduced lattice automorphism of L. Let $t = t(H)$ for some Cartan subalgebra H of L. Then $\langle x \rangle^\phi = \langle x_s + tx_n \rangle$ for all $x \in L$.*

PROOF. (a) Suppose x is semi-simple. Then there exists a Cartan subalgebra H of L such that $x \in H$. By Lemma 4.9, ϕ is reduced with respect to H and therefore $\langle x \rangle^\phi = \langle x \rangle$.

(b) Suppose x is nil. Then there exist $h, y \in L$ such that $hx = x$, $hy = -y$ and $xy = h$. All the elements $h + \lambda x$ are semi-simple. Therefore all the subalgebras $\langle h + \lambda x \rangle$ are fixed, and it follows that $\langle x \rangle$ is fixed.

(c) Suppose x_s and x_n are non-zero. Take a Cartan subalgebra H such that $x_s \in H$. Then $x_n = \sum_\alpha \mu_\alpha e_\alpha$ where $\mu_\alpha \in F$ and e_α is a root vector for the root α of H. For some ordering of the roots, the roots α for which $\mu_\alpha \neq 0$ are all positive. Since $x_s x_n = 0$, if $\mu_\alpha \neq 0$, then $\alpha(x_s) = 0$. Let γ be the largest root for which $\gamma(x_s) = 0$. Then $\langle x_s + \lambda e_\gamma \rangle^\phi = \langle x_s + t\lambda e_\gamma \rangle$ by Lemma 4.6. Thus the result holds if $x_n = \lambda e_\gamma$ for some $\lambda \in F$. If x_n and e_γ are linearly independent, then $\langle x_s, x_n, e_\gamma \rangle$ is a fixed 3-dimensional abelian subalgebra. Every element of $\langle x_n, e_\gamma \rangle$ is nil. Thus ϕ is the identity on $\langle x_n, e_\gamma \rangle$. From ϕ linear on $\langle x_s, x_n, e_\gamma \rangle$ and $\langle x_s + e_\gamma \rangle^\phi = \langle x_s + te_\gamma \rangle$, it now follows that $\langle x_s + x_n \rangle^\phi = \langle x_s + tx_n \rangle$.

This completes the proof of the direct part of the theorem.

5. The converse

In this section, we suppose that L is a semi-simple Lie algebra over a (not necessarily algebraically closed) field of characteristic 0. We take some $t \in F$, $t \neq 0$ and put $\tau = \tau(t)$ and we prove that τ induces a reduced lattice automorphism of L.

LEMMA 5.1. *Let $A = \langle\langle a, b \rangle\rangle$ be the subalgebra of L generated by a, b. Then*

(a) $a_s b_s$, $a_s b_n$, $a_n b_s$, $a_n b_n \in A'$.

(b) *There exists $u \in A'$ such that $a_s(b + u) = 0$.*

(c) *If A is soluble, there exist $c, d \in A$ such that $c - a$, $d - b \in A'$ and $c_s d = cd_s = 0$.*

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PROOF. (a) For all polynomials $f(x)$ without constant term, $bf(\mathrm{ad}\, a) \in A'$. In particular, $b \,\mathrm{ad}\, a_s \in A'$ and $b \,\mathrm{ad}\, a_n \in A'$. Thus $a_s b$, $a_n b \in A'$. Therefore $a_s g(\mathrm{ad}\, b) \in A'$ $a_n\ g(\mathrm{ad}\, b) \in A'$ for all polynomials $g(x)$ without constant term. Therefore $a_s b_s$, $a_s b_n$, $a_n b_s$, $a_n b_n \in A'$.

(b) Let $m(x) = x f(x)$ be the minimum polynomial of ad a_s. Then x does not divide $f(x)$, and there exist polynomials $p(x)$, $q(x)$ such that $xp(x) + f(x)q(x) = 1$. Put $u = -b \,\mathrm{ad}\, a_s p(\mathrm{ad}\, a_s)$. Then $u \in A'$, $b + u = b\, f(\mathrm{ad}\, a_s)\, q(\mathrm{ad}\, a_s)$ and $a_s(b + u) = 0$.

(c) By (b), there exist $u \in A'$ such that $(a + u)b_s = 0$. Put $c = a + u$. Since ad b_s commutes with ad c, it commutes with all polynomials in ad c, in particular with ad c_s. By the argument of (b), there exists a polynomial $f(x)$ such that, for $v = b_n\ f(\mathrm{ad}\, c_s)$, we have $c_s(b_n + v) = 0$. Clearly $v \in A'$. Let K be the algebraic closure of F and let L_K be the algebra obtained from L by extension of the field. Since A is soluble, A is contained in a Borel subalgebra B of L_K. B' is precisely the set of all nil elements in B. It follows that every element of A' is nil and that the sum of two nil elements of A is again nil. Thus $b_n + v$ is nil. Since ad b_s commutes with ad c_s, $v\ b_s = b_n\ b_s\ f(\mathrm{ad}\, c_s) = 0$. Thus $b_s(b_n + v) = 0$ and b_s and $b_n + v$ are the semi-simple and nil parts respectively of $d = b + v$.

LEMMA 5.2. *Let U be a subset of L, $P = \langle\langle U \rangle\rangle$ and $Q = \langle\langle U^\tau \rangle\rangle$. Then $P' = Q'$.*

PROOF. For all $u, v \in U$, we have

$$u^\tau v^\tau = (u_s + tu_n)\ (v_s + tv_n) = u_s v_s + t(u_n v_s + u_s v_n) + t^2 u_n v_n$$

and by Lemma 5.1(a), $u^\tau v^\tau \in P'$. For all $u \in U$ and $a \in P'$, we have $u^\tau a = u_s a + tu_n a \in P'$ by Lemma 5.1 (a). It follows that $Q' \subseteq P'$. Applying this to $\tau^{-1} = \tau(1/t)$, we get $P' \subseteq Q'$.

LEMMA 5.3. *Let A be a soluble subalgebra of L. Then A^τ is a subalgebra of L and* $\dim A^\tau = \dim A$.

PROOF. Since all elements of A' are nil, $A'^\tau = A'$. A^τ is closed with respect to multiplication since, by Lemma 5.2, $\langle\langle A^\tau \rangle\rangle' = A'$. We have to prove that A^τ is a subspace of the same dimension as A.

(1) We use induction on dim A to prove that $(a + b)^\tau - a^\tau - b^\tau \in A'$ for all $a, b \in A$. The assertion is trivial if $\dim A = 1$. We can work in $\langle\langle a, b \rangle\rangle$, so we may suppose $A = \langle\langle a, b \rangle\rangle$.

CASE 1. Suppose $b \in A'$. If $b = 0$, the result is trivial, so we suppose $b \neq 0$. Since $A' \neq 0$, we have $A' > A''$. As $A = \langle a, A' \rangle$ and $b \in A' = aA' + A''$, we have $b = au + v$ for some $u \in A'$, $v \in A''$. Put

$$c = a \exp(\mathrm{ad}\, u) = a + au + \tfrac{1}{2}(au)u + \cdots.$$

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Then $c = a + b + w$ where $w \in A''$. We have

$$c_s = a_s \exp(\operatorname{ad} u) \equiv a_s \bmod A'$$
$$c_n = a_n \exp(\operatorname{ad} u) \equiv a_n \bmod A',$$

and

$$c^\tau - a^\tau = (c_s - a_s) + t(c_n - a_n) \equiv 0 \bmod A'.$$

Since $w \in A'' < A'$, $\langle\langle c, w\rangle\rangle < A$ and by induction, $(c-w)^\tau - c^\tau + w^\tau \in \langle\langle c, w\rangle\rangle'$. But b^τ, $w^\tau \in A'$ and therefore

$$(a+b)^\tau - a^\tau - b^\tau = (c-w)^\tau - a^\tau - b^\tau \equiv c^\tau - a^\tau \equiv 0 \bmod A'.$$

CASE 2. We now consider general a, b. By Lemma 5.1(c), there exist $c, d \in A$ such that $c - a$, $d - b \in A'$ and $c_s d = c d_s = 0$. Put $u = c - a$, $v = d - b$. Since $c_s d_s = 0$, $c_s + d_s$ is semi-simple. Since c_n, d_n are nil and A is soluble, $c_n + d_n$ is nil. But $(c_s + d_s)(c_n + d_n) = 0$. Therefore $(c_s + d_s)$ and $(c_n + d_n)$ are the semi-simple and nil parts of $c + d$, and we have $(c+d)^\tau - c^\tau - d^\tau = 0$. Trivially, $(u+v)^\tau = u^\tau + v^\tau$ since $u, v, u+v$ are nil. By Case 1, we have

$$\begin{aligned} ((c+d) - (u+v))^\tau &\equiv (c+d)^\tau - (u+v)^\tau \\ a^\tau &= (c-u)^\tau \equiv c^\tau - u^\tau \\ b^\tau &= (d-v)^\tau \equiv d^\tau - v^\tau \bmod A'. \end{aligned}$$

Thus

$$\begin{aligned} (a+b)^\tau - a^\tau - b^\tau &= ((c+d) - (u+v))^\tau - a^\tau - b^\tau \\ &\equiv (c+d)^\tau - (u+v)^\tau - c^\tau + u^\tau - d^\tau + v^\tau \\ &\equiv 0 \bmod A'. \end{aligned}$$

(2) For any $a \in A$, by (1) we have $(a + A')^\tau \subseteq a^\tau + A'$. Applying this to τ^{-1} and the soluble subalgebra $\langle a^\tau, A'\rangle$, we get $(a + A')^\tau = a^\tau + A'$. Put $B = \langle A^\tau\rangle$. Then τ induces a map $\tau_1 : A/A' \to B/A'$. By (1), τ_1 is linear and im τ_1 is a subspace of B/A'. But $\langle a^\tau + A' \mid a \in A\rangle = B/A'$. Thus im $\tau_1 = B/A'$ and it follows that $A^\tau = B$. Since τ_1 is invertible, $\dim B/A' = \dim A/A'$ and the result follows.

LEMMA 5.4. *Let A be any subalgebra of L. Then A^τ is a subalgebra of L and* $\dim A^\tau = \dim A$.

PROOF. Let R be the radical of A and let S be a Levi factor. Then $A = R \oplus S$ as vector space. By Lemma 5.3, R^τ is a subalgebra of L and R^τ is soluble since $R^{\tau\prime} = R'$. Since S is semi-simple, we have $S = S' = \langle\langle S^\tau\rangle\rangle'$ and so $S^\tau \subseteq S$. Applying this to $\tau^{-\tau}$, we get $S^{\tau^{-1}} \subseteq S$ and so $S^\tau = S$.

For any $s \in S$, $\langle s, R\rangle$ is a soluble subalgebra and $\langle s, R\rangle' \subseteq R$. By Lemma 5.3 $\langle s, R\rangle^\tau$ is a subalgebra and

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$$\langle s, R\rangle^{\tau\prime} = \langle s, R\rangle' = \langle s, R\rangle'^{\tau} \subseteq R^{\tau}.$$

Thus $s^{\tau}R^{\tau} \subseteq R^{\tau}$ and $R^{\tau} + S$ is a subalgebra. Since R^{τ} is a soluble ideal of $R^{\tau} + S$ $R^{\tau} \cap S = 0$ and $\dim (R^{\tau} + S) = \dim (R + S)$.

Suppose $a \in A$. Then $a = r + s$ for some $r \in R$, $s \in S$, and $a \in \langle s, R\rangle$. Hence $a^{\tau} \in \langle s, R\rangle^{\tau} \subseteq R^{\tau} + S$ since $\langle s, R\rangle^{\tau}$ is a subalgebra. Thus $(R + S)^{\tau} \subseteq R^{\tau} + S$. Applying this to τ^{-1} gives the reverse inequality. Therefore $A^{\tau} = R^{\tau} + S$ and so is a subalgebra.

This completes the proof that τ induces a reduced lattice automorphism of L, and so completes the proof of the theorem.

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