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ON THE BLOCK OF DEFECT ZERO

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1. Let G be a finite group and let p be a fixed prime number. If D is any p-subgroup of G, then the problem whether there exists a p-block with D as its defect group is reduced to whether $N_G(D)/D$ possesses a p-block of defect 0. Some necessary or sufficient conditions for a finite group to possess a p-block of defect 0 have been known (Brauer-Fowler [1], Green [3], Ito [4] [5]). In this paper we shall show that the existences of such blocks depend on the multiplicative structures of the p-elements of G. Namely, let $\mathfrak{p}$ be a prime divisor of p in an algebraic number field which is a splitting one for G, $\mathfrak{o}$ the ring of $\mathfrak{p}$-integers and $k = \mathfrak{o}/\mathfrak{p}$, the residue class field. Then,

THEOREM 1. *Let c denote the sum of the p-elements of G including the identity in the group ring kG. Then c^2 is equal to the sum of the block idempotents of p-defect* 0.

2. Proof of the Theorem and Corollaries

First we note,

LEMMA 1. *Let A be a quasi Frobenius algebra with minimum conditions and let B be a block of A, i.e. a two sided direct summand of A which is indecomposable. If there exists a primitive idempotent e in B such that Ae is a minimal left ideal, then B is a simple algebra.*

Proof. Let f be any primitive idempotent in B. It suffices to show that $Ae \cong Af$ by assuming Ae and Af have a composition factor in common, since e and f are linked ([2] § 55). Then there exists an injection $0 \to Ae \to Af/M$ for a suitable A-submodule M of Af. But since Ae is injective, this map splits. Therefore there exists an epimorphism $Af \to Ae \to 0$, which

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implies $Af \cong Ae$, since Ae is projective and Af is indecomposable. This completes the proof.

Proof of the THEOREM 1.

Let $\mathfrak{n}$ and $r(\mathfrak{n})$ denote the Jacobson radical of kG and its annihilator ideal respectively. In [6] we have shown that $r(\mathfrak{n})$ contains c. Now let B be any block of kG with δ the block idempotent. First we assume B is of positive defect and let e be a primitive idempotent in B. If $\mathfrak{n}e = 0$, then $(kG)e$ is minimal and so B is simple by Lemma 1. Then B has defect 0, which contraries to the assumption (see [2] (86.5)). Therefore $\mathfrak{n}e \neq 0$. Then it contains the unique minimal submodule $r(\mathfrak{n})e$. Since $r(\mathfrak{n})$ contains c, it follows ce is contained in $\mathfrak{n}e$ and in particular it follows $c^2e = 0$. Thus we have $c^2\delta = 0$, since e is arbitary. Next suppose B is of defect 0 and let ϕ be the unique linear character such that $\phi(\delta) = 1$ and takes zero on any other block idempotent different from δ. Since B is central simple and ϕ vanishes on any p-irregular class, we have $c\delta = \phi(c)\delta = \delta$. Therefore, $c^2\delta = \delta$, completing the proof of Theorem 1.

By definition, if G possesses a p-block of defect 0, then it contains an element of p-defect 0, *i.e.* one which commutes with no non trivial p-element. More precisely we have,

COROLLARY 1. *If G possesses a p-block of defect* 0, *then there exist p-elements η_1, η_2 of G such that $\eta_1\eta_2$ is of p-defect* 0.

Proof. Since any block idempotent of defect 0 is a sum of p-regular elements of defect 0, our assertion is clear from Theorem 1.

Of course the converse of the above Corollary is not true in general. We shall show the converse in the following special case.

THEOREM 2. *Suppose a 2-Sylow subgroup of G is elementary abelian. Then G possesses a 2-block of defect 0 if and only if there exist an involution η and an element σ of 2-defect 0 such that $\eta^{-1}\sigma\eta = \sigma^{-1}$.*

Proof. "only if" part is clear from Corollary 1 by taking η_1 as η and $\eta_1\eta_2$ as σ. Note that every non trivial 2-element is an involution. Let I be the set of the pairs of involutions (η_1, η_2) such that $\eta_1\eta_2 = \sigma$. If $|I|$, the number of the elements in I, is odd, then $c^2 \neq 0$. Hence G possesses a 2-block of defect 0 by Theorem 1. A direct computation shows that $|I|$ is equal to the number of the 2-elements in $C_G^*(\sigma) - C_G(\sigma)$, where $C_G^*(\sigma) = \{\tau \in G \mid \tau^{-1}\sigma\tau = \sigma^{-1} \text{ or } \sigma\}$ (see *e.g.* [1]). On the other hand, since $C_G^*(\sigma)$ contains η, its order is even, so that the number of the 2-elements in $C_G^*(\sigma)$ is even by the general theory of finite groups and that of $C_G(\sigma)$ is 1 by assumption. Thus $|I|$ is odd. This completes the proof.

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