Canad. Math. Bull. Vol. **52** (3), 2009 pp. 403–406

Shaken Rogers's Theorem for Homothetic Sections

Dedicated to Ted Bisztriczky, on his sixtieth birthday.

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Abstract. We shall prove the following shaken Rogers's theorem for homothetic sections: Let K and L be strictly convex bodies and suppose that for every plane H through the origin we can choose continuously sections of K and L, parallel to H, which are directly homothetic. Then K and L are directly homothetic.

1 Introduction

Let K be a convex body in affine 3-space and let p_0 be a point. Suppose that every section of K through p_0 is centrally symmetric, then Rogers proved [6] that K is centrally symmetric, although p_0 may not be the centre of K. If this is the case, Aitchison, Petty, and Rogers [1] and Larman [2] proved that K must be an ellipsoid. Suppose now that for every direction we can choose continuously a section of K that is centrally symmetric. If, in addition, K is strictly convex, then Montejano [4] proved that K must be centrally symmetric.

Rogers's obtained his result as a corollary of the following more interesting theorem [6].

Theorem *Let K and L be convex bodies in affine 3-space and let $p \in \operatorname{int} K$ and $q \in \operatorname{int} L$. If for every* plane H through the origin, *the section $K \cap (H + p)$ is directly homothetic to the section $L \cap (H + q)$, then K is directly homothetic to L.*

Montejano's result [3] is a corollary of the following shaken version of Rogers's Theorem for strictly convex bodies and translations instead of homothety.

Theorem *Let K and L be strictly convex bodies in affine 3-space. If for every plane H through the origin we can choose continuously sections of K and L, parallel to H, which are translated copies one of each other, then K is a translate of L.*

The purpose of this paper is to prove the shaken version of Rogers's theorem for homothetic sections. That is, we prove the following result.

Theorem *Let K and L be strictly convex bodies in affine 3-space and suppose that for every* plane H through the origin *we can choose continuously sections of K and L, parallel to H, which are directly homothetic. Then K and L are directly homothetic.*

Received by the editors January 7, 2008.
The authors' research was supported by CONACYT MOD-ORD-21-07, PCI-155-05-07
AMS subject classification: 52A15.
Keywords: convex bodies, homothetic bodies, sections and projections, Rogers's Theorem.

We shall also generalize Rogers's and Montejano's theorems for centrally symmetric sections as follows.

Theorem *Let K be a strictly convex body in affine 3-space. Suppose that for every* plane H through the origin *we can choose continuously two sections of K, parallel to H, which are inversely homothetic. Then K is centrally symmetric.*

2 The Main Theorem

Let $\delta\colon \mathbb{S}^2 \to \mathbb{R}$ be a continuous function such that $\delta(-x) = -\delta(x)$. Let us denote by $[\delta]$ the following set of hyperplanes in $\mathbb{E}^3$:

$$[\delta] = \{H_y^\delta = \{x \in \mathbb{E}^3 \mid \langle x, y\rangle = \delta(y)\}\}_{y\in\mathbb{S}^2}.$$

If this is the case, we shall say that $[\delta]$ is a 2-cycle of planes in $\mathbb{E}^3$. This 2-cycle should be considered as a subset of the Grassmannian manifold $G(3,4) = \mathbb{P}^3$ (identifying $\mathbb{E}^3$ with a hyperplane of $\mathbb{E}^4$ that does not contain the origin, and every plane of $\mathbb{E}^3$ with the hyperplane of $\mathbb{E}^4$ that passes through the origin and contains the plane). The cohomology ring $H^*(G(3,4),\mathbb{Z}_2) = \{\mathbb{Z}_2[\varkappa]\ ;\ \varkappa^4 = 0\}$, where the generator $\varkappa \in H^1(G(3,4),\mathbb{Z}_2) = \mathbb{Z}_2$, by duality can be realized through every 2-cycle of planes. The interested reader may consult [7] for more information about the cohomology ring.

Let $K \subset \mathbb{E}^3$ be a convex body and for every $y \in \mathbb{S}^2$ let $K_y^\delta = K \cap H_y^\delta$. We say that $\{K_y^\delta\}_{y\in\mathbb{S}^2}$ is a 2-cycle of sections of K if $K_x^\delta \cap K_y^\delta \cap \operatorname{int} K \neq \phi$ for every $x, y \in \mathbb{S}^2$.

Lemma 1 *Let $[\delta_1]$ and $[\delta_2]$ be two 2-cycles of planes in $\mathbb{E}^3$ and let $p \in \mathbb{E}^3$. Then there is $x_0 \in \mathbb{S}^2$ such that $p \in H_{x_0}^{\delta_1} = H_{x_0}^{\delta_2}$.*

Proof Let $[\delta_3]$ be the 2-cycle of planes that pass through p. It is enough to prove that $[\delta_1]\cap[\delta_2]\cap[\delta_3] \neq \phi$. That is, there exist $x_0 \in \mathbb{S}^2$ such that $H_{x_0}^{\delta_1} = H_{x_0}^{\delta_2} = H_{x_0}^{\delta_3}$, but this is true because any 2-cycle of planes realizes the generator $\varkappa \in H^1(G(3,4),\mathbb{Z}_2) = \mathbb{Z}_2$ and we know that $\varkappa^3$, the generator of $H^3(G(3,4),\mathbb{Z}_2) = \mathbb{Z}_2$, is not zero. ■

Lemma 2 *Let $[\delta]$ be a 2-cycle of planes in $\mathbb{E}^3$ and let L be a line in $\mathbb{E}^3$. Then there is $x_0 \in \mathbb{S}^2$ such that $L \subset H_{x_0}^\delta$.*

Proof Let p_1 and p_2 be two different points of L and let $[\delta_i]$ be the 2-cycle of planes that pass through p_i, $i = 1, 2$. As in the above lemma, we have that $[\delta_1]\cap[\delta_2]\cap[\delta] \neq \phi$, that is, there exists $x_0 \in \mathbb{S}^2$ such that $L \subset H_{x_0}^\delta$. ■

Theorem 3 *Let K and L be strictly convex bodies in affine 3-space and let $[\delta_i]$ be such that $\{K_y^{\delta_1}\}_{y\in\mathbb{S}^2}$ is a 2-cycle of sections of K and $\{L_y^{\delta_2}\}_{y\in\mathbb{S}^2}$ is a 2-cycle of sections of L. Suppose that for every $x \in \mathbb{S}^2$, $K_x^{\delta_1}$ is directly homothetic to $L_x^{\delta_2}$. Then K is directly homothetic to L.*

Proof Suppose, by contradiction, that K is not directly homothetic to L. For every $x \in \mathbb{S}^2$, let $\Omega_x\colon \mathbb{E}^3 \to \mathbb{E}^3$ be the homothety such that $\Omega_x(K_x^{\delta_1}) = L_x^{\delta_2}$. We shall prove that $\partial(\Omega_x K) \cap \partial L = \partial L_x^{\delta_2}$.

https://doi.org/10.4153/CMB-2009-043-8 Published online by Cambridge University Press

For that purpose, let us fix the point $x \in \mathbb{S}^2$. We may assume, without loss of generality, that $\Omega_x K = K$. If this is the case, then $K_x^{\delta_1} = L_x^{\delta_2}$ and $\Omega_x = \mathrm{Id}_{\mathbb{E}^3}$ is the identity. So, we need to prove that $\partial K \cap \partial L = \partial L_x^{\delta_2} = \partial K_x^{\delta_1}$.

At this point of the proof, it is convenient to give a name for those points which are at the same time in the boundary of K and in the boundary of L. So we shall say that a point $p \in \partial K \cup \partial L$ is a *double* point if $p \in \partial K \cap \partial L$. Hence every point in $\partial L_x^{\delta_2} = \partial K_x^{\delta_1}$ is double. Suppose that there is another double point $p \in \partial K \cap \partial L$, but $p \notin H_x^{\delta_1} = H_x^{\delta_2}$. By Lemma 1, there is $x_0 \in \mathbb{S}^2$ such that $p \in H_{x_0}^{\delta_1} = H_{x_0}^{\delta_2}$. The convex figures $K_{x_0}^{\delta_1}$ and $L_{x_0}^{\delta_2}$ lie in the same plane $H_{x_0}^{\delta_i}$, $i = 1, 2$, and are directly homothetic. Notice now that the directly homothetic convex figures $K_{x_0}^{\delta_1}$ and $L_{x_0}^{\delta_2}$ share at least three points of their boundaries; the point p and, since $K_x^{\delta_1} \cap K_{x_0}^{\delta_1} \cap \operatorname{int} K \neq \phi$, the two double points of $H_x^{\delta_1} \cap H_{x_0}^{\delta_1} \cap \partial K = H_x^{\delta_1} \cap H_{x_0}^{\delta_1} \cap \partial L$. So, keeping in mind that $K_{x_0}^{\delta_1}$ and $L_{x_0}^{\delta_2}$ are strictly convex, we have that $K_{x_0}^{\delta_1} = L_{x_0}^{\delta_2}$.

Since $K \neq L$, we can consider $q \in \partial L - \partial K$. Again, by Lemma 1, there is $x_1 \in \mathbb{S}^2$ such that $q \in H_{x_1}^{\delta_1} = H_{x_1}^{\delta_2}$. The section $K_{x_1}^{\delta_1}$ shares at least two double points of its boundary with the section $K_x^{\delta_1} = L_x^{\delta_2}$. This is so because $K_x^{\delta_1} \cap K_{x_1}^{\delta_1} \cap \operatorname{int} K \neq \phi$. Similarly, the section $K_{x_1}^{\delta_1}$ shares at least two double points of its boundary with the section $K_{x_0}^{\delta_1} = L_{x_0}^{\delta_2}$. So either $K_{x_1}^{\delta_1}$ shares at least three double points of its boundary with the section $L_{x_1}^{\delta_2}$, or the plane $H_{x_1}^{\delta_1} = H_{x_1}^{\delta_2}$ contains the line $H_x^{\delta_i} \cap H_{x_0}^{\delta_i}$, $i = 1, 2$. In the first case, $K_{x_1}^{\delta_1} = L_{x_1}^{\delta_2}$, because two distinct, planar, strictly convex figures that are directly homothetic share at most two points of its boundary. In the second case also $K_{x_1}^{\delta_1} = L_{x_1}^{\delta_2}$, but this time the reason is that two distinct, planar, strictly convex figures that are homothetic cannot have two common boundary points with common supporting line. Effectively, we shall next show that $K_{x_1}^{\delta_1}$ and $L_{x_1}^{\delta_2}$ share two supporting lines. Let $\{a, b\} = H_x^{\delta_i} \cap H_{x_0}^{\delta_i} \cap \partial K$, and let Γ_a and Γ_b be supporting planes of K at a and b, respectively. Notice that Γ_a and Γ_b are also supporting planes of L at a and b, respectively, because $K_x^{\delta_1} = L_x^{\delta_2}$ and $K_{x_0}^{\delta_1} = L_{x_0}^{\delta_2}$. Now $\{a, b\} \subset \partial K_{x_1}^{\delta_1} \cap \partial L_{x_1}^{\delta_2}$ and $\Gamma_a \cap H_{x_1}^{\delta_i}$ is a supporting line of both $K_{x_1}^{\delta_1}$ and $L_{x_1}^{\delta_2}$ at a and similarly $\Gamma_b \cap H_{x_1}^{\delta_i}$ is a supporting line of both $K_{x_1}^{\delta_1}$ and $L_{x_1}^{\delta_2}$ at b. In any case, $K_{x_1}^{\delta_1} = L_{x_1}^{\delta_2}$, which is a contradiction, since $q \in \partial L - \partial K$. Therefore $\partial K \cap \partial L = \partial L_x^{\delta_2} = \partial K_x^{\delta_1}$.

So, we have proved that for every $x \in \mathbb{S}^2$, $\partial(\Omega_x K) \cap \partial L = \partial L_x^{\delta_2}$, where $\Omega_x \colon \mathbb{E}^3 \to \mathbb{E}^3$ is the homothety with the property that $\Omega_x(K_x^{\delta_1}) = L_x^{\delta_2}$.

Let us define the following continuous function $\Psi \colon \mathbb{S}^2 \to \mathbb{R}$ as follows: for every $x \in \mathbb{S}^2$ let Δ^x be the closed half space bounded by the plane $H_x^{\delta_2}$ which has the vector x as an outer normal vector. Define, for every $x \in \mathbb{S}^2$,

$$\Psi(x) = \mathrm{Vol}[(L - \Omega_x K) \cap \Delta^x] - \mathrm{Vol}[(\Omega_x K - L) \cap \Delta^x].$$

Notice that by the above either $(L - \Omega_x K) \cap \Delta^x \neq \phi$ or $(\Omega_x K - L) \cap \Delta^x \neq \phi$. Note also that $\Psi(x) \neq 0$, otherwise $\partial(\Omega_x K) \cap \partial L \neq \partial L_x^{\delta_2}$. Consequently, either $\Psi(x) > 0$ for every $x \in \mathbb{S}^2$ or $\Psi(x) < 0$, for every $x \in \mathbb{S}^2$.

Suppose first that $\Psi(y)$ and $\Psi(-y)$ are greater than zero, for some $y \in \mathbb{S}^2$, then $\Omega_y K \subset L$. Fix the point $y \in \mathbb{S}^2$ and assume, without loss of generality, that $\Omega_y K = K$ and δ_1 is such that $K_y^{\delta_1} = L_y^{\delta_2}$ and $\Omega_y = \mathrm{Id}_{\mathbb{E}^3}$.

Let $p \in \partial L - \partial K$. Again, by Lemma 1, there is $x_2 \in \mathbb{S}^2$ such that $p \in H_{x_2}^{\delta_1} = H_{x_2}^{\delta_2}$. The section $K_{x_2}^{\delta_1}$ shares at least two double points of its boundary with the section

https://doi.org/10.4153/CMB-2009-043-8 Published online by Cambridge University Press

$K_y^{\delta_1} = L_y^{\delta_2}$ and $K_{x_2}^{\delta_1} \subset L_{x_2}^{\delta_2}$, because $K \subset L$. Since $K_{x_2}^{\delta_1}$ and $L_{x_2}^{\delta_2}$ are planar, homothetic, strictly convex figures, we have that $K_{x_2}^{\delta_1} = L_{x_2}^{\delta_2}$, contradicting the fact that $p \in \partial L - \partial K$. Similarly, there is a contradiction when $\Psi(x) < 0$, for every $x \in \mathbb{S}^2$. Therefore, K must be directly homothetic to L. This completes the proof of the theorem. ■

Theorem 4 *Let K be a strictly convex body in affine 3-space and let $[\delta_i]$ be such that $\{K_y^{\delta_1}\}_{y\in\mathbb{S}^2}$ and $\{K_y^{\delta_2}\}_{y\in\mathbb{S}^2}$ are two 2-cycles of sections of K. Suppose that for every $x \in \mathbb{S}^2$, $K_x^{\delta_1}$ is inversely homothetic to $K_x^{\delta_2}$. Then K is centrally symmetric.*

Proof Let $L = -K$ and let $\delta_3 = -\delta_2$. Therefore, $\{K_y^{\delta_1}\}_{y\in\mathbb{S}^2}$ is a 2-cycle of sections of K and $\{L_y^{\delta_3}\}_{y\in\mathbb{S}^2}$ is a 2-cycle of sections of L with the property that for every $x \in \mathbb{S}^2$, $K_x^{\delta_1}$ is directly homothetic to $L_x^{\delta_3}$. Then by Theorem 3 K is directly homothetic to $L = -K$, and therefore K is centrally symmetric. ■

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https://doi.org/10.4153/CMB-2009-043-8 Published online by Cambridge University Press