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A Fritz John theorem in complex space

B.D. Craven and B. Mond

Necessary conditions of the Fritz John type are given for a class of nonlinear programming problems over polyhedral cones in finite dimensional complex space.

Consider the problem to

$$\text{MINIMIZE } f(z, \bar{z}) \text{ SUBJECT TO } g(z, \bar{z}) \in S$$

where S is a polyhedral cone in C^m, and $f : C^{2n} \to C$, $g : C^{2n} \to C^m$ are differentiable functions. A necessary condition for a feasible point z_0 to be optimal is that there exist $\tau \geq 0$, $v \in S^*$, $(\tau, v) \neq 0$, such that

$$\overline{\tau\nabla_z f(z, \bar{z})} + \tau\nabla_{\bar{z}} f(z_0, \bar{z}_0) = \overline{v^T\nabla_z g(z_0, \bar{z}_0)} + v^H\nabla_{\bar{z}} g(z_0, \bar{z}_0)$$

and

$$\operatorname{Re} v^H g(z_0, \bar{z}_0) = 0 .$$

Introduction

In [1], Abrams and Ben-Israel gave a complex version of the well-known [6] Kuhn and Tucker necessary conditions for the existence of an optimal solution to the problem of minimizing a function subject to inequality constraints. Here we give a complex version of the Fritz John necessary conditions [5].

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Notation and preliminaries

Denote by C^n (R^n) n-dimensional complex (real) space, with hermitian (euclidean) norm $|\ |$. If A is a matrix or vector, then A^T, $\bar{A}$, and A^H denote its transpose, complex conjugate, and conjugate transpose. R_+ denotes the half line $[0, \infty)$. $S \subset C^m$ is a polyhedral cone if it is the finite intersection of closed half-spaces in C^m, each containing 0 in its boundary. The polar S^* of S is defined by

$$S^* = \{y \in C^m : x \in S \Rightarrow \mathrm{Re}(y^H x) \geq 0\} .$$

We shall make use of the following [1], [2]: If S and T are polyhedral cones in C^m, then

$$(S \times T)^* = S^* \times T^* ,$$

$$(S \cap T)^* = \mathrm{cl}(S^* + T^*) ,$$ where cl denotes closure.

If $S = R_+$, then $S^* = R_+$. If $S = C^m$, then $S^* = \{0\}$.

Define the set

$$Q = \left\{ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \in C^{2n} : w^2 = \bar{w}^1 \right\} .$$

Then Q is the polyhedral cone in C^{2n} generated by the vectors

$$\begin{pmatrix} e_j \\ e_j \end{pmatrix}, \begin{pmatrix} -e_j \\ -e_j \end{pmatrix}, \begin{pmatrix} ie_j \\ -ie_j \end{pmatrix}, \begin{pmatrix} -ie_j \\ ie_j \end{pmatrix} \quad (j = 1, 2, \ldots, n)$$

where e_j is the j-th unit vector in R^n. The polar of Q is easily seen [1] to be the cone

$$Q^* = \left\{ \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \in C^{2n} : w^2 = -\bar{w}^1 \right\} .$$

The functions $f : Q \to C$ and $g : Q \to C^m$ are *differentiable* at $(z_0, \bar{z}_0) \in Q$ if

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$$f(z, \bar{z}) - f(z_0, \bar{z}_0) = \nabla_z f(z_0, \bar{z}_0)(z-z_0) + \nabla_{\bar{z}} f(z_0, \bar{z}_0)(\bar{z}-\bar{z}_0) + o(|z-z_0|)$$

and

$$g(z, \bar{z}) - g(z_0, \bar{z}_0) = \nabla_z g(z_0, \bar{z}_0)(z-z_0) + \nabla_{\bar{z}} g(z_0, \bar{z}_0)(\bar{z}-\bar{z}_0) + o(|z-z_0|)$$

where $\nabla_z f(z_0, \bar{z}_0)$ and $\nabla_{\bar{z}} f(z_0, \bar{z}_0)$ denote, respectively, the row vectors of partial derivatives

$$\frac{\partial f(z_0, \bar{z}_0)}{\partial w_i^1} \quad \text{and} \quad \frac{\partial f(z_0, \bar{z}_0)}{\partial w_i^2} .$$

$\nabla_z g(z_0, \bar{z}_0)$ and $\nabla_{\bar{z}} g(z_0, \bar{z}_0)$ denote, respectively, the $m \times n$ matrices whose i, j-th elements are

$$\frac{\partial g_i(z_0, \bar{z}_0)}{\partial w_j^1} \quad \text{and} \quad \frac{\partial g_i(z_0, \bar{z}_0)}{\partial w_j^2} ,$$

and $o(|z-z_0|)/|z-z_0| \to 0$ as $z \to z_0$.

Results

We make use of the complex version of Gordan's Transposition Theorem [4] given by Ben-Israel [3].

LEMMA. *Let* $B \in C^{p \times q}$, $v \in C^p$, $w \in C^q$, *and* $S \subset C^q$ *a convex polyhedral cone with non-empty interior. Then exactly one of the following two systems has a solution:*

(i) $-Bw \in \operatorname{int} S$,

(ii) $B^H v = 0$, $v \in S^*$, $v \neq 0$.

THEOREM (Complex Fritz John). *Let* $f : C^{2n} \to C$ *and* $g : C^{2n} \to C^m$ *be differentiable mappings, and let* $S \subset C^m$ *be a polyhedral convex cone with nonempty interior. Let* (P) *denote the problem*

(P): MINIMIZE $\operatorname{Re} f(z, \bar{z})$ SUBJECT TO $g(z, \bar{z}) \in S$.

A necessary condition for z_0 *to be a local minimum of* (P) *is that there exist* $\tau \in R$ *and* $v \in S^*$, *not both zero, such that*

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(2) $\tau\overline{\nabla_z f(z_0, \bar{z}_0)} + \tau\nabla_{\bar{z}} f(z_0, \bar{z}_0) - v^T\overline{\nabla_z g(z_0, \bar{z}_0)} - v^H\nabla_{\bar{z}} g(z_0, \bar{z}_0) = 0$,

(3) $$\mathrm{Re}\, v^H g(z_0, \bar{z}_0) = 0 .$$

Proof. Equation (3) can be written as

$$\tfrac{1}{2}v^T\overline{g(z_0, \bar{z}_0)} + \tfrac{1}{2}v^H g(z_0, \bar{z}_0) = 0 .$$

If there is no non-zero (τ, v) , $\tau \in R_+$, $v \in S^*$ satisfying (2) and (3), it follows that there is no solution to the system

$$\begin{pmatrix} \nabla_z f(z_0, \bar{z}_0) + \overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)} & 0 \\ -\nabla_z g(z_0, \bar{z}_0) & g(z_0, \bar{z}_0) \\ -\overline{\nabla_{\bar{z}} g(z_0, \bar{z}_0)} & \overline{g(z_0, \bar{z}_0)} \end{pmatrix}^H \begin{pmatrix} \tau \\ v_1 \\ v_2 \end{pmatrix} = 0 ,$$

$$0 \neq \begin{pmatrix} \tau \\ v_1 \\ v_2 \end{pmatrix} \in R_+ \times [(S^* \times \bar{S}^*) \cap Q]$$

where $\bar{S}^* = \{\bar{w} : w \in S^*\}$, and Q is defined by (1).

By the lemma, there exist $p \in C^n$, $q \in C$ such that

$$-\begin{pmatrix} \nabla_z f(z_0, \bar{z}_0) + \overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)} & 0 \\ -\nabla_z g(z_0, \bar{z}_0) & g(z_0, \bar{z}_0) \\ -\overline{\nabla_{\bar{z}} g(z_0, \bar{z}_0)} & \overline{g(z_0, \bar{z}_0)} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \in \mathrm{int}\{R_+ \times \mathrm{cl}\{[(S \times \bar{S}) + Q^*]\}\}$$

$$= \mathrm{int}R_+ \times \mathrm{int}[(S \times \bar{S}) + Q^*] .$$

Thus

$$\nabla_z f(z_0, \bar{z}_0)p + \overline{\nabla_{\bar{z}} f(z_0, \bar{z}_0)p} < 0$$

or

(4) $$\mathrm{Re}\left[\nabla_z f(z_0, \bar{z}_0)p + \nabla_{\bar{z}} f(z_0, \bar{z}_0)\bar{p}\right] < 0 .$$

Also, any vector in $\mathrm{int}[(S \times \bar{S}) + Q^*]$ is of the form $\begin{pmatrix} s+\lambda \\ \bar{r}-\bar{\lambda} \end{pmatrix}$, where

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$s, r \in \operatorname{int}S$ and $\lambda \in C^m$. Hence

(5) $$\nabla_z g(z_0, \bar{z}_0)p - g(z_0, \bar{z}_0)q = s + \lambda ,$$

(6) $$\overline{\nabla_{\bar{z}} g(z_0, \bar{z}_0)p} - \overline{g(z_0, \bar{z}_0)q} = \bar{r} - \bar{\lambda} .$$

Conjugating (6) and adding to (5) yields

$$\nabla_z g(z_0, \bar{z}_0)p + \nabla_{\bar{z}} g(z_0, \bar{z}_0)\bar{p} - g(z_0, \bar{z}_0)(q+\bar{q}) = s + r \in \operatorname{int}S .$$

Now, since f and g are differentiable, for sufficiently small t, $0 < t \in R_+$,

$$g(z_0+tp, \bar{z}_0+\overline{tp}) = g(z_0, \bar{z}_0) + t\nabla_z g(z_0, \bar{z}_0)p + t\nabla_{\bar{z}} g(z_0, \bar{z}_0)\bar{p} + o(t) =$$
$$= [1+t(q+\bar{q})]g(z_0, \bar{z}_0) + t(s+r) + o(t) \in S + \operatorname{int}S \subset S .$$

Also, noting (4),

$$\operatorname{Re}f(z_0+tp, \bar{z}_0+\overline{tp}) =$$
$$= \operatorname{Re}[f(z_0, \bar{z}_0)+t\nabla_z f(z_0, \bar{z}_0)p+t\nabla_{\bar{z}} f(z_0, \bar{z}_0)\bar{p}+o(t)] < \operatorname{Re}f(z_0, \bar{z}_0) .$$

This contradicts the assumption that z_0 is a local minimum of (P).

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Department of Mathematics,
University of Melbourne,
Parkville,
Victoria;

Department of Mathematics,
La Trobe University,
Bundoora,
Victoria.