Proceedings of the Edinburgh Mathematical Society (2013) **56**, 469–500
DOI:10.1017/S0013091512000429

INVERSION OF COLOMBEAU GENERALIZED FUNCTIONS

EVELINA ERLACHER

Institute for Statistics and Mathematics,
Vienna University of Economics and Business, Augasse 2–6,
1090 Vienna, Austria (evelina.erlacher@wu.ac.at)

(Received 18 April 2011)

Abstract We introduce different notions of invertibility for generalized functions in the sense of Colombeau. Several necessary conditions for (left, right) invertibility are derived, giving rise to the concepts of compactly asymptotic injectivity and surjectivity. We analyse the extent to which these properties are also sufficient to guarantee the existence of a (left, right) inverse of a generalized function. Finally, we establish several Inverse Function Theorems in this setting and study the relation to their classical counterparts.

Keywords: Colombeau algebra; inverse generalized function; Inverse Function Theorem; local existence result

2010 *Mathematics subject classification:* Primary 46F30
Secondary 47J07

1. Introduction

In the theory of generalized functions, there is a growing need for an appropriate notion of invertibility, together with corresponding Inverse Function Theorems. Classical distribution theory does not allow us to address such concepts, since generalized functions are modelled as continuous linear functionals on certain spaces of test functions, so a sufficiently general notion of point values is absent in this setting. Algebras of generalized functions, on the other hand, model generalized functions as equivalence classes of nets of smooth functions (see, for example, [**5**, **6**, **14**, **26**, **27**]). Thus, they naturally lend themselves to a direct generalization of certain concepts from classical (smooth) analysis in a componentwise manner.

The notion of composition of generalized functions was first introduced in [**2**]. This was later extended in [**19**, **22**, **24**] to provide a functorial theory of manifold-valued generalized functions. This geometric approach to algebras of generalized functions soon required adequate notions of invertibility: as a main example, we note that Lie group analysis of differential equations in generalized functions considers group actions that automatically induce certain one-parameter groups in generalized functions, and hence furnish a first example of 'generalized diffeomorphisms' (see [**8**, **18**, **20**]). In particular, in [**18**, Definition 3.5] Konjik and Kunzinger introduce an ad hoc notion of rank for generalized functions and note that an appropriately general concept will require an Inverse Function Theorem in this setting. More generally, flows of generalized vector fields on differentiable manifolds are studied in [**23**]. Again, such flows provide examples of invertible generalized functions. The assumptions made in [**23**, Theorems 3.1, 3.3 and 3.5] are all global in nature. Keeping in mind the equivalence of the main classical existence theorems (flows of ordinary differential equations, the Inverse Function Theorem and Frobenius Theorem) this reflects the fact that we still do not have a truly local existence result.

An independent approach to the composition and inversion of generalized functions can be found in [**3**, **4**], in which Aragona *et al*. consider invertibility in terms of the ultrametric structure on the space of generalized numbers. Contrary to our approach, they do not obtain local results in the usual topology on Euclidean space.

As a third main source underlining the need for Inverse Function Theorems, we mention regularity theory in algebras of generalized functions. In [**16**, § 3.3] an explicit inversion in the Colombeau algebra is carried out in the course of solving linear Cauchy problems (see also [**17**]). More generally, inversion, of course, plays a central role in studying the generalized bicharacteristic flow (see [**12**,**13**]). In [**15**] generalized pullbacks of Colombeau functions are studied from a microlocal point of view. Here too the question of inversion of generalized functions is of central importance.

Concerning applications in mathematical physics, ‘discontinuous coordinate transforms’ had been applied successfully in general relativity as early as in 1968 (though on an informal level [**28**]). This approach was later embedded into the Colombeau picture and analysed further in [**21**]. Invertibility of the generalized coordinate transformation in this setting is based on a global invertibility result by Gale and Nikaido [**11**]. Again, at the time, no local invertibility result was available.

In summary, there is a strong need for a general local Inverse Function Theorem in the context of algebras of generalized functions. This paper aims at closing this conceptual gap. The results presented here, essentially, comprise [**9**, Chapter 3].

The (remarkable) fact that, despite many appearances in the literature (see above), the question of local invertibility of generalized functions has not yet been settled may be due to a large extent to the lack of a sensible notion of the range or image of a set under a generalized function.

Let us illustrate briefly the kind of problems one runs into when attempting to develop the notion of an inverse of a generalized function. For a classical function $f\colon U \to \mathbb{R}^n$ (where U is an open subset of $\mathbb{R}^n$), the notion of f being invertible is unambiguous: f is invertible if there exists $g\colon f(U) \to U$ with $g \circ f = \mathrm{id}_U$, $f \circ g = \mathrm{id}_{f(U)}$. Thus, in the purely set-theoretic setting, f is invertible if and only if it is injective. If f is, say, smooth, we may also require g to be smooth. Considering a generalized Colombeau function $u \in \mathcal{G}(U)^n$,* however, the presumptive meaning of the statement ‘u is invertible’ is by no means clear: even before starting to look for some generalized function v with $v \circ u = \mathrm{id}_U$, $u \circ v = \mathrm{id}_?$,

* In this expository section, we shall make free use of the basic notions of Colombeau theory; for details, see § 2 or [**14**].

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

we are confronted with the problem of what the image of U under u is supposed to be. In practice, the generalized function u is given by some representative $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon\colon U \to \mathbb{R}^n$ smooth. Yet the sets $u_\varepsilon(U)$, for different values of ε, might be vastly different from each other; they do not even need to 'converge' in any sense as $\varepsilon \to 0$.

Now, the most plausible remedy certainly would consist in requiring $u_\varepsilon(U) = u_{\varepsilon'}(U)$ for all ε, ε' (at least for some representative $(u_\varepsilon)_\varepsilon$). Yet this assumption turns out to be much too strong a restriction for important applications (see, for example, [**14**, § 5.3.2.] or [**10**]) and, moreover, it fails to lead to a well-defined notion of $u(U)$, since every $(u_\varepsilon + n_\varepsilon)_\varepsilon$ with $(n_\varepsilon)_\varepsilon \in \mathcal{N}(U)$ is also a representative of u. As the following example shows, we can have $(u_\varepsilon + n_\varepsilon)(U) = (u_{\varepsilon'} + n_{\varepsilon'})(U)$ for all ε, ε', yet $(u_\varepsilon + n_\varepsilon)(U)$ is different from $u_\varepsilon(U)$.

Example 1.1. Let U be the open interval $(-1, 1)$ and set $u_\varepsilon(x) := x$ for all $\varepsilon \in (0, 1]$. Then, for all ε, we have $u_\varepsilon(U) = (-1, 1)$. Choose $\tilde{n}_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ with $\tilde{n}'_\varepsilon \geqslant 0$, $\tilde{n}_\varepsilon(\pm 1) = \pm 1$ and $\tilde{n}_\varepsilon = 0$ on $[-(1-\varepsilon), 1-\varepsilon]$; set $n_\varepsilon := \tilde{n}_\varepsilon|_{(-1,1)}$. Then $(n_\varepsilon)_\varepsilon \in \mathcal{N}(U)$ and, for all ε, $(u_\varepsilon + n_\varepsilon)(U) = (-2, 2)$. Thus, $[(u_\varepsilon)_\varepsilon](U)$ is not well defined, even if only representatives $(\hat{u}_\varepsilon)_\varepsilon$ are considered with $\hat{u}_\varepsilon(U)$ being independent of ε.

The lower bound of the scope of an inversion theory for generalized functions should certainly be chosen so as to allow for the applications to general relativity envisaged in [**14**, § 5.3.2] and [**10**]. A glance at the third equation of [**14**, (5.45), p. 463] (involving the Heaviside function in a non-trivial way) reveals that a sensible inversion theory should be able to cope with, at least, generalized functions modelling jumps. As will become clear in the following sections, this modest looking requirement already imposes considerable technical intricacies upon us.

From the statements above it can be expected that the study of a simple one-dimensional example of a jump function would provide a good basis for motivating the main ideas behind the notion of invertibility introduced in § 3. Indeed, the following paradigmatic example will be reconsidered in several places throughout the paper, in order to illustrate quite a number of important features of the theory.

Example 1.2. Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$, with $U := (-\alpha, \alpha)$, $\alpha > 0$, be defined by $u_\varepsilon(x) := x + \arctan(x/\varepsilon)$. Then u models a function with a jump of height π at 0.

After providing prerequisites in § 2 (in particular, the appropriate notion of compactly bounded generalized functions will be introduced), we give definitions of left and right invertibility, invertibility and strict invertibility in § 3, followed by a discussion of the immediate implications. Motivated by several questions arising naturally when trying to invert a net of smooth functions, we find several necessary conditions for (left, right) invertibility (§ 4), giving rise to the concepts of compactly asymptotic injectivity and surjectivity. In § 5, we analyse the extent to which the properties 'ca-injective' and 'ca-surjective' defined in the preceding section are sufficient to guarantee the existence of a (left, right) inverse of a generalized function. Finally, in § 6, we prove some generalized inverse function theorems and study their relation to the classical Inverse Function Theorem.

2. Notation and preliminaries

Throughout this paper, $\mathcal{C}^k(U)$ and $\mathcal{D}'(U)$ respectively denote the spaces of k-times continuously differentiable functions ($k \in \mathbb{N}_0 \cup \{\infty\}$) and the spaces of distributions on U with values in $\mathbb{K}$, where $\mathbb{K}$ can be either $\mathbb{R}$ or $\mathbb{C}$ and U is a non-empty subset of $\mathbb{R}^n$ for some $n \in \mathbb{N}$. For subsets A, B of a topological space $(X, \mathcal{T})$, the relation $A \subset\subset B$ is shorthand for the statement that A is a compact subset of the interior of B.

Our main reference for the (special) Colombeau algebra $\mathcal{G}(U)$ on some open subset U of $\mathbb{R}^n$ is [**14**, Chapter 1]. Note that $\mathcal{G}(U)$ is denoted by $\mathcal{G}^s(U)$ in [**14**], and analogously for related spaces. We shall use notation and results from this main source freely in the following. This refers to $\mathcal{G}(U)$ itself, as well as to the embeddings $\sigma\colon \mathcal{C}^\infty(U) \to \mathcal{G}(U)$ and $\iota\colon \mathcal{D}'(U) \to \mathcal{G}(U)$. For the convenience of the reader, we explicitly recall the definition of the spaces of moderate and negligible functions on U, as well as the very definition of the Colombeau algebra.

Definition 2.1. Let U be an open subset of $\mathbb{R}^n$. Set

$$
\begin{aligned}
\mathcal{E}(U) &:= \mathcal{C}^\infty(U)^{(0,1]},\\
\mathcal{E}_M(U) &:= \Big\{ (u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \;\Big|\; \forall K \subset\subset U,\ \forall \alpha \in \mathbb{N}_0^n,\ \exists N \in \mathbb{N}\colon \\
&\qquad\qquad \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0 \Big\},\\
\mathcal{N}(U) &:= \Big\{ (u_\varepsilon)_\varepsilon \in \mathcal{E}(U) \;\Big|\; \forall K \subset\subset U,\ \forall \alpha \in \mathbb{N}_0^n,\ \forall m \in \mathbb{N}\colon \\
&\qquad\qquad \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0 \Big\}.
\end{aligned}
$$

Elements of $\mathcal{E}_M(U)$ and $\mathcal{N}(U)$ respectively are called *moderate* and *negligible functions*. $\mathcal{E}_M(U)$ is a subalgebra of $\mathcal{E}(U)$; $\mathcal{N}(U)$ is an ideal in $\mathcal{E}_M(U)$. The *special Colombeau algebra* on U is defined as

$$\mathcal{G}(U) := \mathcal{E}_M(U)/\mathcal{N}(U).$$

A generalized function on some open subset U of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is given as an m-tuple $(u_1, \dots, u_m) \in \mathcal{G}(U)^m$ of generalized functions $u_j \in \mathcal{G}(U)$, where $j = 1, \dots, m$ and the range space of u_j is $\mathbb{R}$. From now on, the term 'generalized function' will be used exclusively in this sense.

The composition $v \circ u$ of two arbitrary generalized functions is not defined, even if $u \in \mathcal{G}(U)^m$ and $v \in \mathcal{G}(\mathbb{R}^m)^l$. However, if a generalized function u is assumed to be 'compactly bounded' (c-bounded) into the domain of a generalized function v (see [**14**, Definitions 1.2.7, 3.2.49]), the composition $v \circ u$ is well defined. However, before giving the definition of c-boundedness, a certain inconsistency in [**14**] as to the precise meaning of 'c-boundedness from Ω into Ω'' for moderate nets $(u_\varepsilon)_\varepsilon$ has to be dealt with.

- On the one hand, viewing Ω and Ω' simply as open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, Definition 1.2.7 of [**14**] does not require that any u_ε actually maps Ω into Ω'; only the corresponding compactness condition is stipulated (Definition 2.2 (i)).

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

- Alternatively, viewing Ω and Ω' as smooth manifolds of dimensions n and m, respectively, in the natural way, Definition 3.2.44 of [**14**] can also be applied, requiring that, in addition, each u_ε actually maps Ω into Ω'.

Contrary to the statement of [**14**, Example 3.2.50 (i)] it seems not to be known, in general, whether Definitions 1.2.7 and 3.2.49 in [**14**] lead to the same notion of c-bounded generalized functions from Ω into Ω'. Since in the present paper we focus on range spaces in many places, we shall include the requirement $u_\varepsilon(\Omega) \subseteq \Omega'$ in our definition of c-boundedness. Moreover, this leaves the door open for an immediate generalization to the manifold setting. Therefore, we shall adopt Definitions 3.2.44 and 3.2.49 (rather than Definition 1.2.7) of [**14**] for our definition of c-boundedness.

Definition 2.2. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. An element $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(U,V)^{(0,1]}$ is called *compactly bounded (c-bounded)* if the following conditions are satisfied:

(i) for every $K \subset\subset U$ there exist $L \subset\subset V$ and $\varepsilon_0 \in (0,1]$ such that $u_\varepsilon(K) \subseteq L$ for all $\varepsilon \leqslant \varepsilon_0$;

(ii) for every $K \subset\subset U$ and every $\alpha \in \mathbb{N}_0^m$ there exists $N \in \mathbb{N}$ with

$$\sup_{x\in K} |\partial^\alpha u_\varepsilon^j(x)| = O(\varepsilon^{-N})$$

for all component functions u_ε^j, $j = 1, \dots, m$, of u_ε.

The collection of c-bounded elements of $\mathcal{C}^\infty(U,V)^{(0,1]}$ is denoted by $\mathcal{E}_M[U,V]$.

From $V \subseteq \mathbb{R}^m$, it is immediate that $\mathcal{E}_M[U,V]$ can be viewed as a subset of $\mathcal{E}_M(U)^m$ (cf. Definition 2.1). For $u, v \in \mathcal{E}_M[U,V]$ we define an equivalence relation $\sim$ by

$$u \sim v \Longleftrightarrow u - v \in \mathcal{N}(U)^m. \tag{2.1}$$

Note that in the special case at hand this definition reproduces Definition 3.2.46 of [**14**]. According to [**14**, Definition 3.2.49], we set

$$\mathcal{G}[U,V] := \mathcal{E}_M[U,V]/\sim \tag{2.2}$$

to obtain the space of *c-bounded generalized functions from U into V*.

Remark 2.3. By definition, $\mathcal{E}_M[U,V]$ consists of certain nets $(u_\varepsilon)_\varepsilon$ with $u_\varepsilon \in \mathcal{C}^\infty(U,V)$. It is often convenient to deal with the (larger) set $\tilde{\mathcal{E}}_M[U,V]$ of all $(u_\varepsilon)_\varepsilon \in \mathcal{C}^\infty(U,\mathbb{R}^m)^{(0,1]}$ (satisfying parts (i) and (ii) of Definition 2.2) such that $u_\varepsilon(U) \subseteq V$ is required only for $\varepsilon \leqslant \varepsilon_0$, for some $\varepsilon_0 \in (0,1]$ depending on $(u_\varepsilon)_\varepsilon$. It follows from (2.1) that $\mathcal{G}[U,V]$ is not changed if $\mathcal{E}_M[U,V]$ is replaced by $\tilde{\mathcal{E}}_M[U,V]$. We shall henceforth use this fact tacitly.

It is immediate from (2.1) that the inclusions $\mathcal{E}_M[U,V] \subseteq \tilde{\mathcal{E}}_M[U,V] \subseteq \mathcal{E}_M(U)^m$ induce an injective map $j_{U,V} : \mathcal{G}[U,V] \to \mathcal{G}(U)^m$. From now on, we shall view $\mathcal{G}[U,V]$ as a subset of $\mathcal{G}(U)^m$.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

In practice, we shall often have to discuss for a given $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^m$ (respectively, $u \in \mathcal{G}(U)^m$) whether $(u_\varepsilon)_\varepsilon$ (respectively, u) can be viewed as c-bounded from U into V for various open subsets V of $\mathbb{R}^m$. Therefore, we adopt the following terminology.

Definition 2.4. Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

(i) An element $(u_\varepsilon)_\varepsilon$ of $\mathcal{E}_M(U)^m$ is called *c-bounded from U into V* if, in fact, $(u_\varepsilon)_\varepsilon \in \tilde{\mathcal{E}}_M[U,V]$ (cf. Remark 2.3).

(ii) An element u of $\mathcal{G}(U)^m$ is called *c-bounded from U into V* if, in fact, $u \in \mathcal{G}[U,V]$.

Proposition 2.5. *Let $u \in \mathcal{G}(U)^m$ be c-bounded into V and let $v \in \mathcal{G}(V)$, with representatives $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$, respectively. Then the composition*

$$v \circ u := [(v_\varepsilon \circ u_\varepsilon)_\varepsilon]$$

is a well-defined generalized function in $\mathcal{G}(U)$.

Since, plainly, an invertible generalized function must be capable of being composed with its inverse, the notion of c-boundedness will play a crucial role in the following.

We call $\mathcal{K} := \mathcal{E}_M/\mathcal{N}$ the ring of *generalized numbers*, where

$$\mathcal{E}_M := \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \exists N \in \mathbb{N}\colon |r_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\},$$
$$\mathcal{N} := \{(r_\varepsilon)_\varepsilon \in \mathbb{K}^{(0,1]} \mid \forall m \in \mathbb{N}\colon |r_\varepsilon| = O(\varepsilon^{m}) \text{ as } \varepsilon \to 0\}.$$

For $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ and $x_0 \in U$ the *point value of u at x_0* is defined as the class of $(u_\varepsilon(x_0))_\varepsilon$ in $\mathcal{K}$. An element $r \in \mathcal{K}$ is called *strictly non-zero* if there exist a representative $(r_\varepsilon)_\varepsilon$ of r and an $N \in \mathbb{N}$ such that $|r_\varepsilon| \geqslant \varepsilon^N$ for $\varepsilon \to 0$. The (multiplicatively) invertible elements of $\mathcal{K}$ are exactly those that are strictly non-zero.

On

$$U_M := \{(x_\varepsilon)_\varepsilon \in U^{(0,1]} \mid \exists N \in \mathbb{N}\colon |x_\varepsilon| = O(\varepsilon^{-N}) \text{ as } \varepsilon \to 0\}$$

we introduce an equivalence relation by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow \forall m \in \mathbb{N}\colon |x_\varepsilon - y_\varepsilon| = O(\varepsilon^m) \text{ as } \varepsilon \to 0$$

and denote by $\tilde{U} := U_M/{\sim}$ the set of *generalized points*. The set of *compactly supported points* is

$$\tilde{U}_{\mathrm{c}} := \{\tilde{x} = [(\tilde{x}_\varepsilon)_\varepsilon] \in \tilde{U} \mid \exists K \subset\subset U,\ \exists \varepsilon_0 \in (0,1] \text{ such that } \forall \varepsilon \leqslant \varepsilon_0\colon x_\varepsilon \in K\}.$$

For $U = \mathbb{K}$ we have $\tilde{\mathbb{K}} = \mathcal{K}$. Thus, we have the canonical identification $\widetilde{\mathbb{K}^n} = \tilde{\mathbb{K}}^n = \mathcal{K}^n$.

Definition 2.6. Let $u \in \mathcal{G}(U)$ and $f \in \mathcal{C}^k(U)$ for $k \in \mathbb{N}_0 \cup \{\infty\}$. The generalized function u is called *$\mathcal{C}^k$-associated with f* (denoted by $u \approx_k f$) if, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant k$ and one (hence any) representative $(u_\varepsilon)_\varepsilon$ of u,

$$\partial^\alpha u_\varepsilon \to \partial^\alpha f$$

for $\varepsilon \to 0$ uniformly on compact subsets of U.

Proposition 2.7. *Let $f \in \mathcal{C}^k(U)$ for $k \in \mathbb{N}_0 \cup \{\infty\}$. Then $\iota(f)$ is $\mathcal{C}^k$-associated with f.*

For a proof of the preceding result, we refer the reader to [**9**, Proposition 2.39].

3. Invertibility of generalized functions

We start with a definition of invertibility of a generalized function on an open set.

Definition 3.1 (invertibility). Let U be an open subset of $\mathbb{R}^n$ and let $u \in \mathcal{G}(U)^n$. Let G be an open subset of U.

(i) u is called *left invertible on* G if there exist $v \in \mathcal{G}(V)^n$ with V an open subset of $\mathbb{R}^n$ and an open set $H \subseteq V$ such that $u|_G$ is c-bounded into H and $v \circ u|_G = \mathrm{id}_G$. Then v is called a *left inverse* of u on G.

In shorthand, u is left invertible (on G) with left inversion data $[G, V, v, H]$.

(ii) u is called *right invertible on* G if there exist $v \in \mathcal{G}(V)^n$ with V an open subset of $\mathbb{R}^n$ and an open set $H \subseteq V$ such that $v|_H$ is c-bounded into G and $u \circ v|_H = \mathrm{id}_H$. Then v is called a *right inverse* of u on G.

In shorthand, u is right invertible (on G) with right inversion data $[G, V, v, H]$.

(iii) u is called *invertible on* G if it is both left and right invertible on G with left inversion data $[G, V, v, H_\mathrm{l}]$ and right inversion data $[G, V, v, H_\mathrm{r}]$. Then v is called an *inverse* of u on G.

In shorthand, u is invertible (on G) with inversion data $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$.

(iv) u is called *strictly invertible on* G if it is invertible on G with inversion data $[G, V, v, H, H]$ for some open subset H of V. Then v is called a *strict inverse* of u on G.

In shorthand, u is strictly invertible (on G) with inversion data $[G, V, v, H]$.

Throughout this paper we shall also use the wordings 'u is invertible (on G) by $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$' or '$[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$ is an inverse of u (on G)'. If we do not specify a set on which a given $u \in \mathcal{G}(U)^n$ is invertible, we always refer to invertibility on U, i.e. on the whole of its domain. The same applies to the cases of 'left invertible', 'right invertible' and 'strictly invertible'.

Remark 3.2.

(i) Note that u need not be a c-bounded function on U. Only the restriction to the set G where it is composed with a left inverse must have this property.

(ii) The notion of invertibility of a generalized function u is stronger than the combination of left and right invertibility with respect to the same v and yet possibly different sets G_l and G_r (where 'l' and 'r' denote 'left' and 'right' respectively).

(iii) If a smooth function $f \colon U \to V$ (with U and V open subsets of $\mathbb{R}^n$) is classically invertible with smooth inverse $g \colon V \to U$, then, obviously, $\sigma(f) = \iota(f)$ is strictly invertible on U with inversion data $[U, V, \sigma(g), V]$.

Let us apply the above notions to a generalized function modelling a jump.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

Example 3.3. Recall the definition of the generalized function u from Example 1.2: a representative was given by $u_\varepsilon\colon (-\alpha,\alpha)\to\mathbb{R}$, $u_\varepsilon(x):=x+\arctan(x/\varepsilon)$. We are interested in inverting u 'around the jump', i.e. we want to find an inverse in the sense of Definition 3.1 (iii) on some open set $G\subseteq U$ containing 0, with respect to some dataset $[G,V,v,H_\mathrm{l},H_\mathrm{r}]$ 'as large as possible' (at least for the particular representative). For the sake of simplicity, however, we assume all sets to be intervals symmetric around 0.

It is important in the following discussion not to make use of the fact that, actually, both u_ε and u_ε^{-1} could be defined as diffeomorphisms of $\mathbb{R}$ onto itself. By strictly confining ourselves to $U=(-\alpha,\alpha)$ and $(u_\varepsilon(-\alpha),u_\varepsilon(\alpha))$ as the domains of u_ε and u_ε^{-1}, respectively, we model the general situation where the images of the u_ε (and hence, the domains of their classical inverses) do depend on ε.

For every ε, the function u_ε has a $\mathcal{C}^\infty$-inverse $v_\varepsilon\colon (u_\varepsilon(-\alpha),u_\varepsilon(\alpha))\to U$. In the following, we shall successively specify sets V, G, H_l and H_r, showing that, in fact, u is invertible in the sense of Definition 3.1 (iii).

To this end, first note that $u_\varepsilon(x)\nearrow x+\frac{1}{2}\pi$ for every $x>0$. Setting $x=\alpha$ and choosing $\beta\in(0,\alpha)$, we see that for small ε the set $u_\varepsilon(U)$ contains $(-(\beta+\frac{1}{2}\pi),\beta+\frac{1}{2}\pi)$. So $V:=(-(\beta+\frac{1}{2}\pi),\beta+\frac{1}{2}\pi)$ is a suitable choice for a common domain for all v_ε (ε sufficiently small). Note that choosing $\beta\geqslant\alpha$ would fail to give $V\subseteq u_\varepsilon(U)$, for any $\varepsilon\in(0,1]$.

Supposing G to be the interval $(-\alpha_1,\alpha_1)$, the condition $u_\varepsilon(G)\subseteq V$ forces us to choose $\alpha_1\leqslant\beta$, since it is only then that we have $u_\varepsilon(\alpha_1)\nearrow\alpha_1+\frac{1}{2}\pi\leqslant\beta+\frac{1}{2}\pi$.

Looking for a suitable H_l of the form $(-(\beta_\mathrm{l}+\frac{1}{2}\pi),\beta_\mathrm{l}+\frac{1}{2}\pi)$, observe that we need $u_\varepsilon(G)\subseteq H_\mathrm{l}\subseteq V$ for all $\varepsilon\leqslant\varepsilon_0$, i.e. $\sup_{\varepsilon\leqslant\varepsilon_0}u_\varepsilon(\alpha_1)=\alpha_1+\frac{1}{2}\pi\leqslant\beta_\mathrm{l}+\frac{1}{2}\pi\leqslant\beta+\frac{1}{2}\pi$, and thus we have to choose $\beta_\mathrm{l}\in[\alpha_1,\beta]$. The c-boundedness of u from G into H_l is a consequence of the strict monotonicity of all the u_ε.

In a final step, we have to specify $H_\mathrm{r}=(-(\beta_\mathrm{r}+\frac{1}{2}\pi),\beta_\mathrm{r}+\frac{1}{2}\pi)\subseteq V$ such that $v_\varepsilon(H_\mathrm{r})\subseteq G$. Here, analogously to the case of V, a limit argument necessitates the choice $\beta_\mathrm{r}\in(0,\alpha_1)$. Then, for small ε, we obtain the inclusions $H_\mathrm{r}=(-(\beta_\mathrm{r}+\frac{1}{2}\pi),\beta_\mathrm{r}+\frac{1}{2}\pi)\subseteq(u_\varepsilon(-\alpha_1),u_\varepsilon(\alpha_1))\subseteq V$, implying $v_\varepsilon(H_\mathrm{r})\subseteq(-\alpha_1,\alpha_1)=G$. Once more, the c-boundedness of v from H_r into G follows from the strict monotonicity of all the u_ε and v_ε.

Summing up, we have the following inequalities and corresponding inclusions:

$$0<\beta_\mathrm{r}<\alpha_1\leqslant\beta_\mathrm{l}\leqslant\beta<\alpha,$$
$$G\subsetneq U,$$
$$H_\mathrm{r}\subsetneq H_\mathrm{l}\subseteq V.$$

In the preceding example the set H_r is contained in H_l. The following proposition shows that this is no coincidence.

Proposition 3.4. *Letting $u\in\mathcal{G}(U)^n$ be invertible on G with inversion data $[G,V,v,H_\mathrm{l},H_\mathrm{r}]$, we have $H_\mathrm{r}\subseteq H_\mathrm{l}$.*

Proof. Let $x\in H_\mathrm{r}$. On the one hand, there exists ε_0 such that $u_\varepsilon\circ v_\varepsilon(x)$ is an element of some compact subset K of H_l for all $\varepsilon\leqslant\varepsilon_0$ by the c-boundedness of $v|_{H_\mathrm{r}}$ and $u|_G$. On

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

the other hand, $u_\varepsilon \circ v_\varepsilon(x) = x + n_\varepsilon(x) \to x$ as $\varepsilon \to 0$ for some $(n_\varepsilon)_\varepsilon \in \mathcal{N}(H_\mathrm{r})^n$. Since K is compact, the limit x is also an element of K and hence of H_l. □

From the definition of invertibility and the preceding proposition, we immediately obtain the following results.

Proposition 3.5.

(i) *If $u \in \mathcal{G}(U)^n$ is left (right) invertible on G with left (right) inversion data $[G, V, v, H]$, then v is right (left) invertible on H with right (left) inversion data $[H, U, u, G]$.*

(ii) *If $u \in \mathcal{G}(U)^n$ is invertible on G with inversion data $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$, then v is left invertible on H_r with left inversion data $[H_\mathrm{r}, U, u, G]$ and right invertible on H_l with right inversion data $[H_\mathrm{l}, U, u, G]$.*

(iii) *The inverse is unique in the following sense: if u is invertible on G with inversion data $[G, V^1, v^1, H_\mathrm{l}^1, H_\mathrm{r}^1]$ and $[G, V^2, v^2, H_\mathrm{l}^2, H_\mathrm{r}^2]$, then $v^1|_{H_\mathrm{r}} = v^2|_{H_\mathrm{r}}$, where $H_\mathrm{r} := H_\mathrm{r}^1 \cap H_\mathrm{r}^2$.*

(iv) *If $u \in \mathcal{G}(U)^n$ is strictly invertible on G with inversion data $[G, V, v, H]$, then v is strictly invertible on H with inversion data $[H, U, u, G]$.*

(v) *The strict inverse is unique in the following sense: if u is strictly invertible on G with inversion data $[G, V^1, v^1, H^1]$ and $[G, V^2, v^2, H^2]$, then $v^1|_H = v^2|_H$, where $H := H^1 \cap H^2$.*

For the remainder of this section let us discuss various aspects of the notions of invertibility introduced above. In classical inversion theory we are used to the fact that if a function is invertible (as a function) on some set G, this is still true for any subset of G. In the case of generalized functions, however, we have to be more careful: for some left invertible $u \in \mathcal{G}(U)^n$ with left inversion data $[G, V, v, H_\mathrm{l}]$ decreasing the size of G does not affect left invertibility. On the other hand, if u is right invertible with right inversion data $[G, V, v, H_\mathrm{r}]$, shrinking G may not be possible, even if H_r is shrunk as well. We illustrate this with the following example.

Example 3.6. Consider v from Example 3.3. By Proposition 3.5 (i), it is right invertible with right inversion data $[H_\mathrm{l}, U, u, G]$. (Observe the reversed roles of U and V, as well as G and H_l, compared with the notation in Definition 3.1 (ii).) Let H be an open subset of H_l. The generalized function v is right invertible on H, provided H contains the closed interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and G is shrunk accordingly (while still containing 0). If H fails to satisfy this condition, then no open subset G' of G is small enough that $(u_\varepsilon|_{G'})_\varepsilon$ is c-bounded into H.

This shows that right invertibility on some set is not a local property in the usual sense. However, in Example 3.6 it is 'local around the jump': the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ that has to be contained in H is exactly the 'gap' in the image of the jump function modelled by u.

The issue of shrinking H_l and H_r is settled by the symmetry between left and right invertibility (see Proposition 3.5 (i)).

For a left invertible $u \in \mathcal{G}(U)^n$ with left inversion data $[G, V, v, H_\mathrm{l}]$, enlarging G is not possible without further information on u, as is the case in classical theory. In contrast, let $[G, V, v, H_\mathrm{r}]$ be a right inverse of u. Replacing G by a larger set (that is still contained in U) poses no problem at all since $(v_\varepsilon|_{H_\mathrm{r}})_\varepsilon$ is c-bounded into any superset of G.

Again, the question of modifying H_l and H_r is answered by referring to Proposition 3.5 (i).

Summarizing, we conclude that for an invertible u with inversion data $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$, without further specific information, G may neither be enlarged nor shrunk; H_r can safely be made smaller, and H_l larger.

As to strict invertibility, there is no tolerance left for changing the size of either G or H.

These results reflect the fact that in the case of invertibility of u on G the set G has a double role: it has to be big enough as to allow $(v_\varepsilon|_{H_\mathrm{r}})_\varepsilon$ being c-bounded into it and at the same time it has to be small enough for the composition of $(u_\varepsilon|_G)_\varepsilon$ with $(v_\varepsilon|_{H_\mathrm{l}})_\varepsilon$ to still give the identity on $\mathcal{G}(G)^n$. So, the size of G has to be carefully balanced between the requirements of left and right invertibility.

At first sight, a convenient way to circumvent the difficulty of balancing the size of G might consist in allowing different sets G_l and G_r to obtain a notion of 'weak invertibility' involving datasets $[G_\mathrm{l}, G_\mathrm{r}, V, v, H_\mathrm{l}, H_\mathrm{r}]$. This choice, however, would make it difficult, if not impossible, to prove uniqueness of the inverse (cf. parts (iii) and (v) of Proposition 3.5).

Finally, the notion of strict invertibility is the one that comes closest to a generalized equivalent of classical invertibility. However, in most cases we are interested in, it will be too much to ask for, as may be demonstrated by the following example.

Example 3.7. Again, consider the function u from Example 3.3 modelling a jump. We attempt to find open sets G and H such that u is strictly invertible with strict inversion data $[G, V, v, H]$. Without loss of generality we may assume that G and H are open intervals.

In Example 3.6 we have already discussed that H must contain the closed interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Therefore, $H := (-(\gamma + \frac{1}{2}\pi), \gamma + \frac{1}{2}\pi)$ for some $\gamma > 0$. For $(v_\varepsilon|_H)_\varepsilon$ to be c-bounded into G, the set G has to contain the closed interval $[-\gamma, \gamma]$. Let

$$G := (-(\gamma + \delta), \gamma + \delta)$$

for some $\delta > 0$. For any $0 < \eta < \delta$ we eventually obtain $H = (-(\gamma + \frac{1}{2}\pi), \gamma + \frac{1}{2}\pi) \subseteq u_\varepsilon([-(\gamma + \eta), \gamma + \eta])$, thereby destroying any hope of c-boundedness into H. Thus, u is not strictly invertible on any open set G containing 0.

4. Necessary conditions for invertibility

We start with a few (heuristic) questions that arise when looking for an inverse of some given $u \in \mathcal{G}(U)^n$ by looking at a representative $(u_\varepsilon)_\varepsilon$ ε-wise.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

Question 4.1. If u_ε is not injective for every ε, is it possible for another representative of u to have this property?

Question 4.2. If every u_ε is injective on U, does there exist an open set that is contained in all the (possibly different!) domains of the inverses v_ε, so that we can indeed speak of a net of functions on some fixed domain V?

Question 4.3. Are all v_ε smooth? If yes, is $(v_\varepsilon|_V)_\varepsilon$ in $\mathcal{E}_M(V)^n$?

Concerning Question 4.1, we consider an example.

Example 4.4. Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ given by $u_\varepsilon(x) := \sin(x/\varepsilon)$. No matter how small we choose a subset of U, eventually u_ε becomes non-injective on this set. The discussion following Proposition 4.5 will show that this fact is sufficient to conclude that u is not left invertible.

Proposition 4.5. *Let U be an open subset of $\mathbb{R}^n$. Then for every representative $(u_\varepsilon)_\varepsilon$ of $\mathrm{id}_U \in \mathcal{G}(U)^n$ and for every compact subset K of U there exists some $\varepsilon_0 \in (0,1]$ such that $u_\varepsilon|_K$ is injective for all $\varepsilon \leqslant \varepsilon_0$.*

Proof. Let $K \subset\subset U$ and $m_\varepsilon := \mathrm{id}_U - u_\varepsilon \in \mathcal{N}(U)^n$. By [**14**, Lemma 3.2.47], for all ε there exists a constant $C_\varepsilon > 0$ such that

$$|m_\varepsilon(x) - m_\varepsilon(y)| \leqslant C_\varepsilon \cdot |x - y| \tag{4.1}$$

for all $x, y \in K$. C_ε can be chosen as $C_1 \cdot \sup_{z \in L}(|m_\varepsilon(z)| + \|\mathrm{D}m_\varepsilon(z)\|)$, where L is any fixed compact neighbourhood of K in U and C_1 depends only on L. Since m is negligible and because of the form of the C_ε, we may find some ε_0 such that $C_\varepsilon < \frac{1}{2}$ for all $\varepsilon \leqslant \varepsilon_0$. Then

$$|u_\varepsilon(x) - u_\varepsilon(y)| = |(x - y) - (m_\varepsilon(x) - m_\varepsilon(y))| \geqslant |x - y| - C_\varepsilon|x - y| \geqslant \tfrac{1}{2}|x - y|,$$

yielding the injectivity of u_ε on K for all $\varepsilon \leqslant \varepsilon_0$. □

If u is left invertible on G by $[G, V, v, H_l]$, then, for every representative $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v, the composition $(v_\varepsilon \circ u_\varepsilon|_G)_\varepsilon$ is a representative of the identity in $\mathcal{G}(G)^n$. Therefore, $v_\varepsilon \circ u_\varepsilon$, and consequently u_ε is injective on any compact subset of G for sufficiently small ε. In particular, this implies that the generalized function in Example 4.4 has no chance of being left invertible. This result motivates the following definition.

Definition 4.6. A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$ is called *compactly asymptotically injective (ca-injective)* if for every compact subset K of U there exists some $\varepsilon_0 \in (0,1]$ such that $u_\varepsilon|_K$ is injective for all $\varepsilon \leqslant \varepsilon_0$.

An element u of $\mathcal{G}(U)^n$ is called *compactly asymptotically injective (ca-injective)* if all representatives have this property.

Remark 4.7. Note that if one representative of a generalized function is ca-injective, this is not necessarily true for every other: consider $n_\varepsilon(x) := \mathrm{e}^{-1/\varepsilon}x$ and $\tilde{n}_\varepsilon(x) := 0$. $(n_\varepsilon)_\varepsilon$ is injective for all ε (even on $\mathbb{R}$), while $(\tilde{n}_\varepsilon)_\varepsilon$ is not. Yet they are both representatives of the same generalized function.

Using the terminology of Definition 4.6 we obtain the following proposition.

Proposition 4.8. *If $u \in \mathcal{G}(U)^n$ is left invertible, then u is ca-injective.*

To illustrate Question 4.2, we consider $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-1, 1)$ given by $u_\varepsilon(x) := \varepsilon x$. With ε decreasing, the domains of the inverses v_ε shrink to the singleton $\{0\}$, so that there is no common open domain on which to define the inverse net. One could say that, in a sense, the net $(u_\varepsilon)_\varepsilon$ lacks surjectivity, and hence a common domain for the inverses u_ε^{-1}. We shall see that this is sufficient to destroy any hope of u being invertible.

In the one-dimensional case, a simple condition guarantees a common domain for the inverses: let u_ε be injective on an open interval U in $\mathbb{R}$ for all ε. Suppose that two different points x and y in U (without loss of generality $x < y$) can be found such that $u_\varepsilon(x)$ and $u_\varepsilon(y)$ converge to different limits a and b (without loss of generality $a < b$). Then the Intermediate Value Theorem ensures that for all $\delta > 0$ there exists some ε_0 such that $[a + \delta, b - \delta] \subseteq u_\varepsilon((x, y))$ for all $\varepsilon \leqslant \varepsilon_0$.

The following theorem provides a generalization of the previous argument based on the Intermediate Value Theorem to the n-dimensional case. Roughly speaking, it establishes a kind of continuous dependence of connected parts $f(A)$ of the image set $f(U)$ on the function f.

Theorem 4.9. *Let U be an open subset of $\mathbb{R}^n$, let $f, g \in \mathcal{C}(U, \mathbb{R}^n)$ both be injective and let W be a connected open subset of $\mathbb{R}^n$ with $\bar{W} \subset\subset f(U)$. Choose an open ball $B_\delta(y)$ ($y \in W$, $\delta > 0$) inside W such that the closure of $W_\delta := W + B_\delta(0)$ is still a subset of $f(U)$. If, for $A := f^{-1}(\overline{W_\delta})$,*

$$\|g - f\|_{\infty,A} < \delta \tag{4.2}$$

holds, then

$$\bar{W} \subseteq g(A)^\circ.$$

Proof. By a theorem of Brouwer (e.g. [**25**, Theorem 7.12]), both $f(U)$ and $g(U)$ are open and f and g map U homeomorphically to $f(U)$ and $g(U)$, respectively. Clearly, $\bar{W}$ is the disjoint union of the three sets

$$\begin{aligned} G_1 &:= \bar{W} \cap g(A)^\circ, \\ G_2 &:= \bar{W} \cap \partial g(A), \\ G_3 &:= \bar{W} \cap \operatorname{ext} g(A). \end{aligned}$$

We shall show that $G_1 \neq \emptyset$ and $G_2 = \emptyset$. By the connectedness of $\bar{W}$, it follows that $\bar{W} = G_1$ (note that G_1 and G_3 are open in the relative topology of $\bar{W}$), that is

$$\bar{W} \subseteq g(A)^\circ.$$

$G_1 \neq \emptyset$: let $x := f^{-1}(y)$. Then x is an element of A. Since f and g are homeomorphisms and y is an element of the interior of $\overline{W_\delta}$, it follows that $x \in A^\circ$ and $g(x) \in g(A)^\circ$. By

$$|g(x) - y| = |g(x) - f(x)| \leqslant \|g - f\|_{\infty,A} < \delta,$$

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

we obtain

$$g(x) \in B_\delta(y) \cap g(A)^\circ \subseteq \bar{W} \cap g(A)^\circ.$$

$G_2 = \emptyset$: assume that there exists $a \in \bar{W} \cap \partial g(A)$. By $\partial g(A) = g(\partial A)$, the point $x := g^{-1}(a)$ is an element of ∂A. Moreover, $f(x) \in \partial f(A) = \partial \overline{W_\delta}$. On the one hand,

$$|a - f(x)| = |g(x) - f(x)| \leqslant \|g - f\|_{\infty,A} < \delta. \tag{4.3}$$

On the other hand, a being an element of $\bar{W}$, we obtain

$$|a - f(x)| \geqslant \operatorname{dist}(\bar{W}, \partial W_\delta) = \operatorname{dist}(W, W_\delta^c) = \delta,$$

which contradicts (4.3). Hence, $\bar{W} \cap \partial g(A) = \emptyset$. □

With respect to generalized functions, the above theorem implies the following corollary.

Corollary 4.10. *Let U be an open subset of $\mathbb{R}^n$. Then for every representative $(u_\varepsilon)_\varepsilon$ of $\mathrm{id}_U \in \mathcal{G}(U)^n$ and for every compact subset K of U there exist a compact subset L of U containing K and a suitable $\varepsilon_0 \in (0,1]$ such that $K \subseteq u_\varepsilon(L)$ for all $\varepsilon \leqslant \varepsilon_0$.*

Proof. Let K be a (non-empty) compact subset of U. Without loss of generality we may assume that U is connected (otherwise K can be written as a finite union of sets K_i, where each K_i is contained in only one connected component; L can then be defined as the finite union of compact sets L_i obtained for each K_i). Then there exists a non-empty connected open subset W of U with $\bar{W} \subset\subset U$ which contains K. Choose δ such that $B_\delta(y) \subseteq W$ for a suitable y and such that the closure of $W_{2\delta} := W + B_{2\delta}(0)$ is contained in U. By Proposition 4.5, u_ε is injective on $\overline{W_2\delta}$. Applying Theorem 4.9 to $W_{2\delta}$, $\mathrm{id}_{W_{2\delta}}$, $\underline{u_\varepsilon|_{W_{2\delta}}}$, W and δ in place of U, f, g, W and δ (ε sufficiently small) and setting $L := \overline{W + \beta_\delta(0)}$ concludes the proof. □

For right invertible $u \in \mathcal{G}(U)^n$ with right inversion data $[G, V, v, H_{\mathrm{r}}]$, Corollary 4.10 has the following meaning: for any representatives $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v, the composition $(u_\varepsilon \circ v_\varepsilon|_{H_{\mathrm{r}}})_\varepsilon$ is a representative of the identity in $\mathcal{G}(H_{\mathrm{r}})^n$. Therefore, for every compact subset K of H_{r} there exists a compact subset L of H_{r} with $K \subseteq L$ such that $K \subseteq u_\varepsilon \circ v_\varepsilon(L)$ for ε sufficiently small. Since $(v_\varepsilon|_{H_{\mathrm{r}}})_\varepsilon$ is c-bounded into G, there exists a compact subset L' of G such that $v_\varepsilon(L) \subseteq L'$ for small ε. This entails that $K \subseteq u_\varepsilon(L')$ for ε small enough. This observation motivates the next definition.

Definition 4.11. Let U and V be open subsets of $\mathbb{R}^n$. A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$ is called *compactly asymptotically surjective (ca-surjective) onto* V if for every compact subset K of V there exist a compact subset L of U and some $\varepsilon_0 \in (0,1]$ such that $K \subseteq u_\varepsilon(L)$ for all $\varepsilon \leqslant \varepsilon_0$.

An element u of $\mathcal{G}(U)^n$ is called *compactly asymptotically surjective (ca-surjective) onto* V if all representatives have this property.

Using the terminology of Definition 4.11 we obtain the following proposition.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

Proposition 4.12. *If $u \in \mathcal{G}(U)^n$ is right invertible on G with right inversion data $[G, V, v, H_{\mathrm{r}}]$, then u is ca-surjective onto H_{r}.*

Turning to Question 4.3, we consider $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(\mathbb{R})$ given by $u_\varepsilon(x) := x^3$. u_ε is invertible as a function on $\mathbb{R}$ for every ε, but the inverses are not smooth. As the following proposition will show, u cannot be inverted on any open set containing 0.

Proposition 4.13. *Let U be an open subset of $\mathbb{R}^n$, G an open subset of U and $u \in \mathcal{G}(U)^n$ left invertible on G with left inversion data $[G, V, v, H_{\mathrm{l}}]$. Then, for every representative $(u_\varepsilon)_\varepsilon$ of u and for every compact subset K of G, there exist $C > 0$, $N \in \mathbb{N}$ and $\varepsilon_0 \in (0, 1]$ such that*

$$\inf_{x \in K} |\det(\mathrm{D}u_\varepsilon(x))| \geqslant C\varepsilon^N \tag{4.4}$$

for all $\varepsilon \leqslant \varepsilon_0$. In particular, $\det(\mathrm{D}u(x))$ is strictly non-zero for all $x \in G$.

Proof. Let $K \subset\subset G$ and $(v_\varepsilon)_\varepsilon$ be a representative of v. Differentiating the equality $v_\varepsilon \circ u_\varepsilon|_G = \mathrm{id}_G + n_\varepsilon$ and applying the determinant on both sides, we obtain, for sufficiently small ε (i.e. such that $\det(I + \mathrm{D}n_\varepsilon(x)) \geqslant \frac{1}{2}$),

$$|\det(\mathrm{D}u_\varepsilon(x))| \geqslant \frac{1}{2|\det(\mathrm{D}v_\varepsilon(u_\varepsilon(x)))|} \tag{4.5}$$

for all $x \in K$. By the c-boundedness of $(u_\varepsilon)_\varepsilon$ into H_{l}, there exists a compact subset L of H_{l} such that

$$\sup_{x \in K} |\det(\mathrm{D}v_\varepsilon(u_\varepsilon(x)))| \leqslant \sup_{y \in L} |\det(\mathrm{D}v_\varepsilon(y))| \leqslant C_1\varepsilon^{-N}$$

for suitable $N \in \mathbb{N}$ and $C_1 > 0$. Plugging this inequality into (4.5) yields the desired estimate. □

Definition 4.14. Let U be an open subset of $\mathbb{R}^n$. A moderate net $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)$ is called *strictly non-zero* if for every compact subset K of U there exist $C > 0$, a natural number N and some $\varepsilon_0 \in (0, 1]$ such that

$$\inf_{x \in K} |u_\varepsilon(x)| \geqslant C\varepsilon^N \tag{4.6}$$

for all $\varepsilon \leqslant \varepsilon_0$.

An element u of $\mathcal{G}(U)$ is called *strictly non-zero* if it possesses a representative with this property.

Clearly, if one representative satisfies (4.6), then so do all representatives. Using the terminology of Definition 4.14, Proposition 4.13 now reads as follows.

Proposition 4.15. *If $u \in \mathcal{G}(U)^n$ is left invertible, then $\det \circ \mathrm{D}u$ is strictly non-zero.*

5. Sufficient conditions for invertibility

We have determined three properties that are necessary for a given $u \in \mathcal{G}(U)^n$ to be invertible on some open subset G of U by $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$, namely ca-injectivity on G, ca-surjectivity of $u|_G$ onto H_r and $\det \circ \mathrm{D}u_\varepsilon$ being strictly non-zero on G. In this section we shall prove that these three conditions are also sufficient to guarantee at least local invertibility of a c-bounded u in the following sense.

Definition 5.1 (local invertibility). Let U be an open subset of $\mathbb{R}^n$ and let $u \in \mathcal{G}(U)^n$. We call u *locally (left, right) invertible* if for every point $z \in U$ there exists an open neighbourhood G of z in U such that u is (left, right) invertible on G.

Obviously, (left, right) invertibility on some open set implies local (left, right) invertibility on that very set but not vice versa.

Note that, contrary to the widespread usage of the term 'local' and the intuition based thereupon, for a generalized function u which is locally (left, right) invertible on some open set U, and some given $z \in U$, the neighbourhood G of z on which u is (left, right) invertible cannot, in general, be chosen either arbitrarily small or arbitrarily large. Local invertibility only guarantees the *existence* of such a neighbourhood, its (minimum and respectively maximum) size depending on the function u and the point z (cf. Example 3.6).

Our first aim in this section is to prove that compact asymptotic injectivity (ca-injectivity) of a c-bounded $u \in \mathcal{G}[U, \mathbb{R}^n]$, with $\det \circ \mathrm{D}u$ strictly non-zero, implies local left invertibility of u. To this end, some preliminaries are necessary.

Let $u \in \mathcal{G}(U)^n$ and assume that $(u_\varepsilon)_\varepsilon$ is a representative such that u_ε is injective with inverse $v_\varepsilon \colon u_\varepsilon(U) \to U$ for every ε. If we are interested only in left inverses of u, it is of no importance whether there is a common non-trivial open set *inside* of all $u_\varepsilon(U)$; rather, we need some open set *containing* all $u_\varepsilon(U)$ to serve as a common domain for the v_ε. Therefore, we somehow have to extend the functions v_ε (in a smooth way!) to a larger set without losing their property of being (left) inverse to the u_ε on some open subset G of U, independent of ε and possibly smaller than U. We shall do this by means of two-member partitions of unity $(p_\varepsilon, 1 - p_\varepsilon)$, where the plateau functions p_ε serve to retain the values of v_ε on some $K_\varepsilon \subset\subset u_\varepsilon(U)$. The following proposition ensures the existence of moderate nets of suitable plateau functions.

Proposition 5.2. *Let* U_ε *(for* $\varepsilon \in (0, \varepsilon_0]$*) be an open subset of* $\mathbb{R}^n$ *and* K_ε *compact in* U_ε *such that* $(\mathrm{dist}(K_\varepsilon, U_\varepsilon^\mathrm{c}))_\varepsilon$ *is strictly non-zero. Let* U *be another open subset of* $\mathbb{R}^n$ *such that* $U_\varepsilon \subseteq U$ *for all* ε*. Then there exists a net* $(p_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)$ *of plateau functions such that* $p_\varepsilon|_{K_\varepsilon} = 1$ *and* $\operatorname{supp} p_\varepsilon \subset\subset U_\varepsilon$ *for* ε *sufficiently small.*

Proof. Since $\mathrm{dist}(K_\varepsilon, U_\varepsilon^\mathrm{c}) \geqslant C\varepsilon^N$ for some $N \in \mathbb{N}$ and $C > 0$, we can choose η_ε with $C\varepsilon^{N+1} \leqslant \eta_\varepsilon < \mathrm{dist}(K_\varepsilon, U_\varepsilon^\mathrm{c})$ such that every n-dimensional cube with side length η_ε having non-empty intersection with K_ε is contained in U_ε for ε sufficiently small. Construct plateau functions $q_\varepsilon \colon U_\varepsilon \to [0, 1]$ as in the proof of [**7**, Chapter I, § 2] for U_ε and K_ε using grid size η_ε. Let $p_\varepsilon \in \mathcal{D}(U)$ be the smooth extension by 0 of q_ε to U. Then,

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

conforming to the proof of [**7**, Chapter I, § 2], the plateau function p_ε is given by

$$p_\varepsilon = \biggl(\sum_{j\in J_\varepsilon} \varphi_j^\varepsilon\biggr)\bigg|_U,$$

where, for any $j = (j_1, \dots, j_n) \in \mathbb{Z}^n$, φ_j^ε maps from $\mathbb{R}^n$ to $\mathbb{R}$ and is given by

$$\varphi_j^\varepsilon(x_1, \dots, x_n) = \prod_{i=1}^n h\biggl(\frac{2x_i}{\eta_\varepsilon} - j_i\biggr),$$

the mapping $h\colon \mathbb{R} \to \mathbb{R}$ has support $[-1, 1]$ and $J_\varepsilon := \{j \in \mathbb{Z}^n \mid \operatorname{supp}\varphi_j^\varepsilon \cap K_\varepsilon \neq \emptyset\}$. The function h has compact support and, by our choice of $(\eta_\varepsilon)_\varepsilon$, the net $(\varphi_j^\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)$. Let $\varphi_0^\varepsilon := \varphi_{(0,\dots,0)}^\varepsilon$. Since any φ_j^ε can be written as the composition of the translation $x \mapsto x - \frac{1}{2}\eta_\varepsilon j$ and φ_0^ε, all moderateness estimates are the same as those for $(\varphi_0^\varepsilon)_\varepsilon$. For $J_\varepsilon^x := \{j \in J_\varepsilon \mid x \in (\operatorname{supp}\varphi_j^\varepsilon)^\circ\}$ for $x \in \mathbb{R}^n$ we have $|J_\varepsilon^x| \leqslant 2^n$ by the definition of the φ_j^ε. For arbitrary $\alpha \in \mathbb{N}_0^n$ it now follows by the above that

$$\begin{aligned} |\partial^\alpha p_\varepsilon(x)| &= \biggl|\sum_{j\in J_\varepsilon^x} \partial^\alpha \varphi_j^\varepsilon(x)\biggr| \\ &\leqslant \sum_{j\in J_\varepsilon^x} \sup_{y\in\operatorname{supp}\varphi_j^\varepsilon} |\partial^\alpha \varphi_j^\varepsilon(y)| \\ &\leqslant \sum_{j\in J_\varepsilon^x} C_1\varepsilon^{-N_1} \\ &\leqslant 2^n C_1 \varepsilon^{-N_1} = C_2\varepsilon^{-N_1} \end{aligned}$$

for $x \in U$, thereby concluding the proof of the proposition. □

Now, we turn to the question of moderateness. It turns out that if $(\det \circ \mathrm{D}u_\varepsilon)_\varepsilon$ is strictly non-zero, this is already sufficient to guarantee the desired result. The next proposition consists of two parts. Roughly speaking, the first part deals with the 'disposition to moderateness' of the inverse net $(v_\varepsilon)_\varepsilon$, while in the second part we take care of the smooth and moderate extension of the v_ε.

We introduce the following terminology: a function $\tilde{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ is called a (K, y_0)-*extension of* $f\colon U \to \mathbb{R}^m$ (where $U \subseteq \mathbb{R}^n$ open, $K \subset\subset U$ and $y_0 \in \mathbb{R}^m$) if $\tilde{f}|_K = f|_K$ and $\tilde{f}(x) = y_0$ for all $x \in \mathbb{R}^n \setminus U$.

Proposition 5.3. *Let U be an open subset of $\mathbb{R}^n$ containing open subsets W_ε for $\varepsilon \in (0, \varepsilon_0]$ such that $W_\varepsilon \subseteq K$ for some $K \subset\subset U$. Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$. For all ε let u_ε be injective on W_ε with inverse $v_\varepsilon\colon V_\varepsilon \to W_\varepsilon$, where $V_\varepsilon := u_\varepsilon(W_\varepsilon)$. Suppose that*

$$\inf_{x\in W_\varepsilon} |\det(\mathrm{D}u_\varepsilon(x))| \geqslant C_1\varepsilon^{N_1} \tag{5.1}$$

for some $C_1 > 0$ and $N_1 \in \mathbb{N}_0$ and for all $\varepsilon \leqslant \varepsilon_0$. Then the following hold.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

(1) *The inverses v_ε are smooth, and for all $\alpha \in \mathbb{N}_0^n$ there exist $C > 0$, $N \in \mathbb{N}$ and $\varepsilon_1 \in (0, \varepsilon_0]$ such that for all $\varepsilon \leqslant \varepsilon_1$ the estimate*

$$\sup_{y \in V_\varepsilon} |\partial^\alpha v_\varepsilon(y)| \leqslant C\varepsilon^{-N} \tag{5.2}$$

holds. In particular, if there exists a non-empty open subset V of $\mathbb{R}^n$ such that $V \subseteq \bigcap_{\varepsilon \in (0,\varepsilon_0]} V_\varepsilon$, then $(v_\varepsilon|_V)_\varepsilon$ is in $\mathcal{E}_M(V)^n$ and uniformly bounded (the latter following from the inclusion $W_\varepsilon \subseteq K$).

(2) *Let $K_\varepsilon \subset\subset V_\varepsilon$ for $\varepsilon \in (0, \varepsilon_0]$ and $[(\tilde{x}_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}_c^n$ such that $\tilde{x}_\varepsilon \in L \subset\subset \mathbb{R}^n$ for all $\varepsilon \leqslant \varepsilon_0$. If there exist a constant $C_2 > 0$ and a natural number N_2 such that*

$$\operatorname{dist}(K_\varepsilon, V_\varepsilon^c) \geqslant C_2 \varepsilon^{N_2} \tag{5.3}$$

for all $\varepsilon \leqslant \varepsilon_0$, then there exist smooth $(K_\varepsilon, \tilde{x}_\varepsilon)$-extensions $\tilde{v}_\varepsilon$ of v_ε such that $(\tilde{v}_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. Furthermore, the net $(\tilde{v}_\varepsilon)_\varepsilon$ is uniformly bounded. In particular, $(\tilde{v}_\varepsilon)_\varepsilon$ is c-bounded into any open subset of $\mathbb{R}^n$ containing the convex hull of $K \cup L$.

Proof. (1) We have only to prove (5.2). For $\alpha = 0$ we have $\operatorname{im} v_\varepsilon = W_\varepsilon \subseteq K \subset\subset U$. The first partial derivatives of the ith component $v_\varepsilon^{(i)}$ of v_ε at $y \in V_\varepsilon$ can be written as the product of the inverse of $\det(\mathrm{D}u_\varepsilon(v_\varepsilon(y)))$ and a polynomial in the first derivatives of the components of u_ε evaluated at $v_\varepsilon(y)$. By $\operatorname{im} v_\varepsilon \subseteq K$ for all $\varepsilon \leqslant \varepsilon_0$ and the moderateness of $(u_\varepsilon)_\varepsilon$, we see that the first partial derivatives of $v_\varepsilon^{(i)}$ do not grow faster than some inverse power of ε. By induction, we also obtain the desired estimates for higher partial derivatives of $v_\varepsilon^{(i)}$, thus concluding the proof of the first claim of the proposition.

(2) From Proposition 5.2 it follows that there exists a net $(p_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)$ of plateau functions such that $p_\varepsilon|_{K_\varepsilon} = 1$ and $\operatorname{supp} p_\varepsilon \subset\subset V_\varepsilon$ for all ε. Let $\tilde{v}_\varepsilon \colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\tilde{v}_\varepsilon(x) := \begin{cases} p_\varepsilon(x)v_\varepsilon(x) + (1 - p_\varepsilon(x))\tilde{x}_\varepsilon, & x \in V_\varepsilon, \\ \tilde{x}_\varepsilon, & \text{otherwise.} \end{cases}$$

By construction, all $\tilde{v}_\varepsilon$ are smooth, $\tilde{v}_\varepsilon|_{K_\varepsilon} = v_\varepsilon|_{K_\varepsilon}$ and $\tilde{v}_\varepsilon(x) = \tilde{x}_\varepsilon$ for all $x \in \mathbb{R}^n \setminus V_\varepsilon$. To prove moderateness we consider $\partial^\alpha \tilde{v}_\varepsilon^{(i)}$ ($\alpha \in \mathbb{N}_0^n$; $\tilde{v}_\varepsilon^{(i)}$ denotes the ith component of $\tilde{v}_\varepsilon$) on some $K \subset\subset \mathbb{R}^n$. On $K \cap V_\varepsilon$ the derivative of $\tilde{v}_\varepsilon^{(i)}$ of order α can be written as a polynomial in $x_\varepsilon^{(i)}$ and the derivatives of p_ε and v_ε^i. By the moderateness of $(p_\varepsilon)_\varepsilon$, the boundedness of $(\tilde{x}_\varepsilon)_\varepsilon$ and inequality (5.2), it follows that $\partial^\alpha \tilde{v}_\varepsilon^{(i)}$ is bounded on $K \cap V_\varepsilon$ by some inverse power of ε. Since on $K \setminus V_\varepsilon$ all derivatives of $\tilde{v}_\varepsilon^{(i)}$ are zero or at least constant (where all occurring values are contained in a compact set), corresponding estimates also hold for all $x \in K$. The uniform boundedness of $(\tilde{v}_\varepsilon^{(i)})_\varepsilon$ is a direct consequence of $\operatorname{im} v_\varepsilon \subseteq K$ and $\tilde{x}_\varepsilon \in L$. □

If in the above proposition the W_ε are equal to some open W ($\subseteq K$) for all ε and the compact sets K_ε are the images of a fixed compact subset of W under u_ε, condition (5.3) in the second part is automatically satisfied.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

Proposition 5.4. *Let U be an open subset of $\mathbb{R}^n$ and let W be a (non-empty) open subset of U with $\bar{W} \subset\subset U$ and $(u_\varepsilon)_\varepsilon \in \mathcal{E}_M(U)^n$. For all $\varepsilon \in (0, \varepsilon_0]$ let u_ε be injective on W with inverse $v_\varepsilon \colon u_\varepsilon(W) \to W$. Let $[(\tilde{x}_\varepsilon)_\varepsilon] \in \tilde{\mathbb{R}}^n_c$ with $\tilde{x}_\varepsilon \in K' \subset\subset \mathbb{R}^n$ for all $\varepsilon \leqslant \varepsilon_0$, let K be a compact subset of W and $K_\varepsilon := u_\varepsilon(K)$. If*

$$\inf_{x \in W} |\det(\mathrm{D}u_\varepsilon(x))| \geqslant C_1 \varepsilon^{N_1}$$

for some $C_1 > 0$, $N_1 \in \mathbb{N}_0$ and for all $\varepsilon \leqslant \varepsilon_0$, then all v_ε are smooth and there exist $(K_\varepsilon, \tilde{x}_\varepsilon)$-extensions $\tilde{v}_\varepsilon$ of v_ε such that $(\tilde{v}_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. Furthermore, the net $(\tilde{v}_\varepsilon)_\varepsilon$ is uniformly bounded. In particular, $(\tilde{v}_\varepsilon)_\varepsilon$ is c-bounded into any open subset of $\mathbb{R}^n$ that contains the convex hull of $\bar{W} \cup K'$.

Proof. Set $V_\varepsilon := u_\varepsilon(W)$. All we have to do is to show that

$$\operatorname{dist}(K_\varepsilon, V_\varepsilon^{\mathrm{c}}) \geqslant C\varepsilon^N$$

for some $C > 0$, a natural number N and sufficiently small ε. Applying Proposition 5.3 (2) then yields the desired result.

By a theorem of Brouwer (e.g. [**25**, Theorem 7.12]), V_ε is open in $\mathbb{R}^n$ and u_ε maps W homeomorphically to V_ε. Choose $y_{1\varepsilon} \in \partial K_\varepsilon$ and $y_{2\varepsilon} \in \partial V_\varepsilon$ such that $\operatorname{dist}(K_\varepsilon, V_\varepsilon^{\mathrm{c}}) = \operatorname{dist}(\partial K_\varepsilon, \partial V_\varepsilon) = |y_{1\varepsilon} - y_{2\varepsilon}|$. Set $\eta := \operatorname{dist}(K, W^{\mathrm{c}}) > 0$ and let $L := K + \overline{B_{\eta/2}(0)}$. Then L is a compact subset of W and $L_\varepsilon := u_\varepsilon(L)$ is a compact subset of V_ε. Set $\delta_\varepsilon := \operatorname{dist}(L_\varepsilon, V_\varepsilon^{\mathrm{c}}) > 0$. Since, by construction, $K_\varepsilon \subseteq L_\varepsilon^\circ$, we have

$$\delta_\varepsilon \leqslant \operatorname{dist}(K_\varepsilon, V_\varepsilon^{\mathrm{c}}) = |y_{1\varepsilon} - y_{2\varepsilon}|.$$

Choose some $\tilde{y}_{2\varepsilon}$ on the open line segment between $y_{1\varepsilon}$ and $y_{2\varepsilon}$ with

$$|\tilde{y}_{2\varepsilon} - y_{2\varepsilon}| < \delta_\varepsilon.$$

Since $y_{2\varepsilon} \in \partial V_\varepsilon$ and $\operatorname{dist}(L_\varepsilon, V_\varepsilon^{\mathrm{c}}) = \delta_\varepsilon$, it follows that $\tilde{y}_{2\varepsilon} \notin L_\varepsilon$. The set $\overline{y_{1\varepsilon} y_{2\varepsilon}} \setminus \{y_{2\varepsilon}\}$ is contained in V_ε ($\overline{y_{1\varepsilon} y_{2\varepsilon}}$ denoting the line segment $\{\lambda y_{1\varepsilon} + (1-\lambda) y_{2\varepsilon} \mid 0 \leqslant \lambda \leqslant 1\}$; for a proof see [**9**, Lemma 3.33]), and hence $\tilde{y}_{2\varepsilon} \in V_\varepsilon \setminus L_\varepsilon$. Let $x_{1\varepsilon} \in K$ and $\tilde{x}_{2\varepsilon} \in W \setminus L$ such that $u_\varepsilon(x_{1\varepsilon}) = y_{1\varepsilon}$ and $u_\varepsilon(\tilde{x}_{2\varepsilon}) = \tilde{y}_{2\varepsilon}$, respectively. Then,

$$\operatorname{dist}(K_\varepsilon, V_\varepsilon^{\mathrm{c}}) = |y_{1\varepsilon} - y_{2\varepsilon}| \geqslant |y_{1\varepsilon} - \tilde{y}_{2\varepsilon}| = |u_\varepsilon(x_{1\varepsilon}) - u_\varepsilon(\tilde{x}_{2\varepsilon})|. \tag{5.4}$$

Since $\operatorname{dist}(K, L^{\mathrm{c}}) = \operatorname{dist}(K, (K + B_{\eta/2}(0))^{\mathrm{c}}) = \frac{1}{2}\eta$ and $\tilde{x}_{2\varepsilon} \in W \setminus L \subseteq L^{\mathrm{c}}$, we have

$$|\tilde{x}_{2\varepsilon} - x_{1\varepsilon}| \geqslant \operatorname{dist}(\tilde{x}_{2\varepsilon}, K) \geqslant \tfrac{1}{2}\eta.$$

By the Mean Value Theorem (note that $\overline{y_{1\varepsilon} \tilde{y}_{2\varepsilon}} \subseteq V_\varepsilon$), we obtain

$$\begin{aligned} \tfrac{1}{2}\eta &\leqslant |x_{1\varepsilon} - \tilde{x}_{2\varepsilon}| \\ &= |v_\varepsilon(u_\varepsilon(x_{1\varepsilon})) - v_\varepsilon(u_\varepsilon(\tilde{x}_{2\varepsilon}))| \\ &\leqslant \sup_{y \in V_\varepsilon} \|\mathrm{D}v_\varepsilon(y)\| |u_\varepsilon(x_{1\varepsilon}) - u_\varepsilon(\tilde{x}_{2\varepsilon})|. \end{aligned} \tag{5.5}$$

By Proposition 5.3, there exist $N \in \mathbb{N}$ and $C' > 0$, both independent of ε, and some $\varepsilon_1 \in (0, \varepsilon_0]$ such that

$$\sup_{y \in V_\varepsilon} \|\mathrm{D}v_\varepsilon(y)\| \leqslant C'\varepsilon^{-N}$$

for all $\varepsilon \leqslant \varepsilon_1$. Together with (5.5), this entails

$$|u_\varepsilon(x_{1\varepsilon}) - u_\varepsilon(\tilde{x}_{2\varepsilon})| \geqslant C\varepsilon^N$$

for $C_2 := \eta/2C'$ and $\varepsilon \leqslant \varepsilon_1$ and we are done. □

Now, it is easy to prove the following theorem.

Theorem 5.5. *Let U be an open subset of $\mathbb{R}^n$ and $u \in \mathcal{G}[U, \mathbb{R}^n]$. If u is ca-injective and* $\det \circ \mathrm{D}u$ *is strictly non-zero, then u is left invertible on any open subset W of U with $\bar{W} \subset\subset U$.*

Proof. Let W and W' be two open subsets of U with $\bar{W} \subset\subset W' \subseteq \overline{W'} \subset\subset U$. By the ca-injectivity of $u = [(u_\varepsilon)_\varepsilon]$, there exists $\varepsilon_0 \in (0, 1]$ such that $u_\varepsilon|_{\overline{W'}}$ is injective for all $\varepsilon \leqslant \varepsilon_0$. Let $v_\varepsilon \colon u_\varepsilon(W') \to W'$ be the inverse of $u_\varepsilon|_{W'}$. Now apply Proposition 5.4 to U, W', $(u_\varepsilon)_\varepsilon$, $(v_\varepsilon)_\varepsilon$, $0 \in \{0\}$, $\bar{W}$ and $K_\varepsilon := u_\varepsilon(\bar{W})$. By the c-boundedness of u, there exists a compact set $K \subseteq \mathbb{R}^n$ such that $u_\varepsilon(\bar{W}) \subseteq K$ for sufficiently small ε. We obtain that u is left invertible on W by $[W, \mathbb{R}^n, \tilde{v} := [(\tilde{v}_\varepsilon)_\varepsilon], H_1]$, where $\tilde{v}_\varepsilon$ is a smooth $(K_\varepsilon, 0)$-extension of v_ε and H_1 can be any open subset of $\mathbb{R}^n$ that contains K. □

Note that to construct the left inverse in Theorem 5.5 we used only *one* representative that is ca-injective. However, by the discussion following Proposition 4.5, we know that for left invertible generalized functions *all* representatives have this property. Hence, Theorem 5.5 immediately yields the following corollary.

Corollary 5.6. *Let U be an open subset of $\mathbb{R}^n$, let $u \in \mathcal{G}[U, \mathbb{R}^n]$ and let* $\det \circ \mathrm{D}u$ *be strictly non-zero. If one representative of u is ca-injective, then all representatives have this property.*

At this point the question arises of whether we may prove a theorem with respect to ca-surjectivity and right invertibility corresponding to Theorem 5.5. A quick glance at the results from which Theorem 5.5 was derived shows that matters turn out to be more complex in the case of such a 'dual' statement: given ca-injectivity of $(u_\varepsilon)_\varepsilon$ we have set-theoretic inverses $(v_\varepsilon)_\varepsilon$ on suitable open sets. These can be lifted to the level of moderate c-bounded nets by Proposition 5.4, yielding a left inverse for $[(u_\varepsilon)_\varepsilon]$. Dually, given ca-surjectivity of $(u_\varepsilon)_\varepsilon$, we fail when trying to imitate this argument since we do not even obtain continuous right inverses, in general.

However, we can show that local invertibility follows from the combination of ca-injectivity and ca-surjectivity and the assumption that $\det \circ \mathrm{D}u$ is strictly non-zero.

Theorem 5.7. *Let U and H be open subsets of $\mathbb{R}^n$ and $u \in \mathcal{G}[U, \mathbb{R}^n]$. If u is ca-injective and ca-surjective onto H and if* $\det \circ \mathrm{D}u$ *is strictly non-zero, then u is locally invertible on U.*

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

More precisely, for every $z \in U$ *and every open subset* H_r *of* H *with* $\overline{H_\mathrm{r}} \subset\subset H$ *there exist an open neighbourhood* G *of* z *with* $\bar{G} \subset\subset U$*, an open relatively compact subset* H_l *of* $\mathbb{R}^n$ *containing* $\overline{H_\mathrm{r}}$*, and some* $v \in \mathcal{G}(\mathbb{R}^n)^n$ *such that* u *is invertible on* G *with inversion data* $[G, \mathbb{R}^n, v, H_\mathrm{l}, H_\mathrm{r}]$*. The set* G *can be chosen to contain any given set* $M \subset\subset U$*. Furthermore, there exist representatives* $(u_\varepsilon)_\varepsilon$ *of* u *and* $(v_\varepsilon)_\varepsilon$ *of* v *such that* $v_\varepsilon \circ u_\varepsilon|_G = \mathrm{id}_G$ *and* $u_\varepsilon \circ v_\varepsilon|_{H_\mathrm{r}} = \mathrm{id}_{H_\mathrm{r}}$ *for sufficiently small* ε*.*

Proof. Let $z \in U$, let $(u_\varepsilon)_\varepsilon$ be a representative of u and let H_r be an open subset of H with $\overline{H_\mathrm{r}} \subset\subset H$. Let $\delta > 0$ such that $\overline{(H_\mathrm{r})_\delta} \subset\subset H$ for $(H_\mathrm{r})_\delta := H_\mathrm{r} + B_\delta(0)$. By the ca-surjectivity of u onto H, there exists a compact subset K of U with $\overline{(H_\mathrm{r})_\delta} \subseteq u_\varepsilon(K)$ for ε sufficiently small. Choose a compact subset L of U with $K \cup \{z\} \cup M \subset\subset L^\circ$ for some given $M \subset\subset U$. Set $G := L^\circ$. Then $\overline{H_\mathrm{r}} \subseteq u_\varepsilon(G)$ for small ε. Let $\eta > 0$ such that the closure of $G_\eta := G + B_\eta(0)$ is a compact subset of U. From the ca-injectivity of u, it follows that u_ε is invertible (as a function) on G_η by, say, $w_\varepsilon \colon u_\varepsilon(G_\eta) \to G_\eta$ for ε small enough. Proposition 5.4 now yields the existence of smooth $(u_\varepsilon(\bar{G}), y)$-extensions v_ε of w_ε (for $y \in G_\eta$ fixed arbitrarily) such that $(v_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)^n$. Thus, $v_\varepsilon \circ u_\varepsilon|_G = \mathrm{id}_G$ and $u_\varepsilon \circ v_\varepsilon|_{H_\mathrm{r}} = \mathrm{id}_{H_\mathrm{r}}$. Since $H_\mathrm{r} \subseteq u_\varepsilon(K) \subseteq u_\varepsilon(G)$, we have $v_\varepsilon(H_\mathrm{r}) = w_\varepsilon(H_\mathrm{r}) \subseteq K \subset\subset L^\circ = G$. Hence, $v_\varepsilon|_{H_\mathrm{r}}$ is c-bounded into G. By the c-boundedness of u, we can find a compact subset K' of $\mathbb{R}^n$ such that $u_\varepsilon(\bar{G}) \subseteq K'$ for ε small. Finally, let H_l be an open relatively compact subset of $\mathbb{R}^n$ containing K'. Then $\overline{H_\mathrm{r}} \subseteq u_\varepsilon(G) \subseteq H_\mathrm{l}$ and, thus, u is invertible on G with inversion data $[G, \mathbb{R}^n, v, H_\mathrm{l}, H_\mathrm{r}]$. □

Remark 5.8.

(1) By the preceding theorem, we do not obtain an inverse of u on arbitrarily small open subsets of U (as was the case in Theorem 5.5). On the contrary, the size of the neighbourhood G of $z \in U$ depends on H_r. This does not constitute a deficiency of our proof, it rather originates from the necessity of proving the c-boundedness of $v|_{H_\mathrm{r}}$ into G. As was discussed earlier, G cannot be forced smaller, in general, by shrinking H_r (cf. Example 3.6).

(2) In the proof of Theorem 5.7 we construct, given some representative of u, a net of smooth (classically) inverse functions v_ε. This means we can find smooth inverse functions for *any* given representative of u. However, the sets G and H_l depend on the chosen representative.

Finally, we demonstrate to what extent for an invertible u with inverse v there exist representatives $(u_\varepsilon)_\varepsilon$ of u and $(v_\varepsilon)_\varepsilon$ of v such that the compositions $v_\varepsilon \circ u_\varepsilon$ and $u_\varepsilon \circ v_\varepsilon$ classically are the identity (on suitable sets).

Theorem 5.9. *Let* U *be an open subset of* $\mathbb{R}^n$, G *an open subset of* U *and* $u \in \mathcal{G}(U)^n$ *invertible on* G *with inversion data* $[G, V, v, H_\mathrm{l}, H_\mathrm{r}]$*. For every representative* $(u_\varepsilon)_\varepsilon$ *of* u *and for every open subset* W *of* H_r *with* $\bar{W} \subset\subset H_\mathrm{r}$ *the following hold: there exist an open subset* G' *of* G *with* $\overline{G'} \subset\subset G$ *and a moderate net of functions* $(w_\varepsilon)_\varepsilon \in \mathcal{E}_M(\mathbb{R}^n)^n$ *such that* $w_\varepsilon \circ u_\varepsilon|_{G'} = \mathrm{id}_{G'}$ *and* $u_\varepsilon \circ w_\varepsilon|_W = \mathrm{id}_W$ *for sufficiently small* ε*. Moreover,* u *is invertible on* G' *by* $[G', \mathbb{R}^n, w := [(w_\varepsilon)_\varepsilon], H_\mathrm{l}, W]$ *and* $w|_W = v|_W$ *in* $\mathcal{G}(W)^n$*. The set* G' *can be chosen to contain any given* $M \subset\subset G$*.*

Proof. Let $(u_\varepsilon)_\varepsilon$ be a representative of u, W an open subset of H_{r} with $\bar{W} \subset\subset H_{\mathrm{r}}$, M a compact subset of G and $z \in M$. Let $\delta > 0$ such that $\overline{W_\delta} \subset\subset H_{\mathrm{r}}$, where $W_\delta := W + B_\delta(0)$. By Propositions 4.8 and 4.12, we know that $(u_\varepsilon)_\varepsilon$ is ca-injective on G and ca-surjective on G onto H_{r}. Furthermore, Proposition 4.13 says that $\det \circ \mathrm{D}u$ is strictly non-zero on G. Then it follows from Theorem 5.7 (applied to G, H_{r} and W in place of U, H and H_{r}) that there exist an open neighbourhood G' of z in G with $M \subseteq G' \subseteq \overline{G'} \subset\subset G$ and some $w \in \mathcal{G}(\mathbb{R}^n)^n$ such that u is invertible on G' with inversion data $[G', \mathbb{R}^n, w, H_{\mathrm{l}}, W]$. Furthermore, by Remark 5.8 (2), there exists a representative $(w_\varepsilon)_\varepsilon$ of w such that $w_\varepsilon \circ u_\varepsilon|_{G'} = \mathrm{id}_{G'}$ and $u_\varepsilon \circ w_\varepsilon|_W = \mathrm{id}_W$ for ε sufficiently small. The equality $w|_W = v|_W$ in $\mathcal{G}(W)^n$ follows from Proposition 3.5 (iii). □

6. Generalized Inverse Function Theorems

The classical Inverse Function Theorem says that, solely from the invertibility of the derivative at a point x_0 of a given function f, we may deduce that on a suitable neighbourhood of x_0 the function itself is $\mathcal{C}^1$-invertible. Conversely, by the chain rule, if f is $\mathcal{C}^1$-invertible on some open set W, then its derivative is invertible at every $x \in W$. In analogy to the latter statement we proved in § 4 that for every generalized function $u \in \mathcal{G}(U)^n$ invertible on G the determinant of the derivative is strictly non-zero at all points of G. Contrary to the classical case, however, this latter property at only one point is not sufficient to imply invertibility of u on some neighbourhood. Certainly, it provides ε-wise smooth inverses of a representative, but it says nothing about the sizes of the neighbourhoods on which those inverses are defined. In the following series of examples, we consider generalized functions defined on open subsets of $\mathbb{R}$ and examine their derivative at 0 and their (non-)invertibility behaviour on certain neighbourhoods of 0.

Example 6.1. Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ be defined by $u_\varepsilon(x) := \varepsilon \sin x$. The derivative at 0 is $\mathrm{D}u_\varepsilon(0) = \varepsilon$, i.e. $\det \circ \mathrm{D}u(0)$ is strictly non-zero. Nevertheless, u is not invertible on any neighbourhood of 0 since it is not ca-surjective on $(-\alpha, \alpha)$ onto any open subset of $\mathbb{R}$.

Even if we demand that $\mathrm{D}u_\varepsilon(x_0)$ grows as $1/\varepsilon$, or at least is bounded away from 0, the situation does not get better.

Example 6.2. Consider $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ given by $u_\varepsilon(x) := \varepsilon \sin(x/\varepsilon)$. The derivative at 0 is $\mathrm{D}u_\varepsilon(0) = 1$ for all ε. Again, u is not invertible on any neighbourhood of 0 since it is not ca-surjective on $(-\alpha, \alpha)$ onto any open subset of $\mathbb{R}$.

Example 6.3. Let $u := [(u_\varepsilon)_\varepsilon] \in \mathcal{G}(U)$ with $U := (-\alpha, \alpha)$ for $\alpha > 0$ be given by $u_\varepsilon(x) := \varepsilon \sin(x/\varepsilon^2)$. This time the derivative at 0 is $\mathrm{D}u_\varepsilon(0) = 1/\varepsilon$, i.e. growing as $\varepsilon \to 0$. But still u is not invertible on any neighbourhood of 0, for the same reasons as before.

To stabilize the sizes of the sets on which the functions u_ε and their inverses are defined, it seems inevitable that we must impose conditions on u and/or its derivative even on some neighbourhood of x_0.

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

A crucial tool in the theory of inversion of generalized functions is the following theorem, which can be considered as a refined version of the classical Inverse Function Theorem. It allows precise control over numerical constants and, in particular, over the minimum size of the domain of the inverse function (see [**1**] for a similar result).

Theorem 6.4 (Inverse Function Theorem). *Let X and Y be Banach spaces and U an open subset of X. Let $f \in \mathcal{C}^k(U,Y)$ for $k \in \mathbb{N} \cup \{\infty\}$ and $x_0 \in U$. If $\mathrm{D}f(x_0)$ is invertible in the space of bounded linear operators from X to Y, then there exist open neighbourhoods W of x_0 in U and V of $y_0 := f(x_0)$ and a function $g \in \mathcal{C}^k(V,W)$ such that g is the inverse of $f|_W$.*

More precisely, let $a := \|\mathrm{D}f(x_0)^{-1}\|$. Let $b > 0$ with $ab < 1$ and $r > 0$ with $\overline{B_r(x_0)} \subseteq U$ such that

$$\|\mathrm{D}f(x_0) - \mathrm{D}f(x)\| \leqslant b \tag{6.1}$$

for all $x \in B_r(x_0)$. Setting $c := a/(1-ab)$, the following hold:

(1) *$|x_1 - x_2| \leqslant c|f(x_1) - f(x_2)|$ for all $x_1, x_2 \in \overline{B_r(x_0)}$;*

(2) *$\mathrm{D}f(x)$ is invertible and $\|\mathrm{D}f(x)^{-1}\| \leqslant c$ for all $x \in \overline{B_r(x_0)}$;*

(3) *$V := f(B_r(x_0))$ is open;*

(4) *$f|_W : W \to V$ is a $\mathcal{C}^k$-diffeomorphism for $W := B_r(x_0)$;*

(5) *$\overline{B_{r/c}(y_0)} \subseteq f(\overline{B_r(x_0)})$ and $B_{r/c}(y_0) \subseteq f(B_r(x_0))$.*

The proof of Theorem 6.4 follows the classical pattern. It is based on applying Banach's Fixed Point Theorem to the function $g^y : \overline{B_r(x_0)} \to Y$ defined by

$$\begin{aligned} g^y(x) &:= x + \mathrm{D}f(x_0)^{-1}(y - f(x)) \\ &= \mathrm{D}f(x_0)^{-1}(y) + \mathrm{D}f(x_0)^{-1}(\mathrm{D}f(x_0)(x) - f(x)). \end{aligned}$$

For $y \in \overline{B_{r/c}(y_0)}$ the function g^y maps $\overline{B_r(x_0)}$ into $\overline{B_r(x_0)}$. For a detailed proof, we refer the reader to [**9**, Theorem 1.3].

Remark 6.5. Given U, f, k, x_0, a and b as in Theorem 6.4 then, by continuity of $\mathrm{D}f$, there always exists $r > 0$ satisfying (6.1). Furthermore, note that all statements in Theorem 6.4 remain true if only $\|\mathrm{D}f(x_0)^{-1}\| \leqslant a$ is assumed to hold, and b and r are chosen accordingly.

A by-product of the proof of the preceding version of the Inverse Function Theorem (see [**9**, Theorem 1.3]) is stated as the following proposition. We shall use it in the proof of a generalized inverse function theorem.

Proposition 6.6. *In the situation of Theorem 6.4 the following also holds: let $0 < \beta < 1$ and $y_1 \in \mathbb{R}^n$ be such that*

$$|y_0 - y_1| \leqslant (1-\beta)\frac{r}{c}.$$

Then g^y maps $\overline{B_r(x_0)}$ into $\overline{B_r(x_0)}$ for all $y \in \overline{B_{\beta r/c}(y_1)}$, and $B_{\beta r/c}(y_1) \subseteq f(B_r(x_0))$.

The quickest way to obtain an Inverse Function Theorem for generalized functions u (with representative $(u_\varepsilon)_\varepsilon$) consists of assuming that the estimates of Theorem 6.4 hold uniformly in ε for all u_ε. This generalization, however, is not capable of handling situations such as jumps (cf. Example 3.3), which definitely have to be included within the scope of any generalized inverse function theorem due to their appearance in applications (see [**10**, **14**, **21**]). In order to obtain a more flexible result we allow the constants a and b from Theorem 6.4 to depend on ε.

Generalizing [**14**, Definition 1.2.69] to $\mathbb{R}^n$, we shall write $\tilde{x} \approx y$ ($\tilde{x} \in \tilde{\mathbb{R}}^n$, $y \in \mathbb{R}^n$) to express that y is the shadow of $\tilde{x}$, i.e. that for one (hence any) representative $(\tilde{x}_\varepsilon)_\varepsilon$ of $\tilde{x}$ the net $(\tilde{x}_\varepsilon)_\varepsilon$ converges to y as $\varepsilon \to 0$.

Theorem 6.7. *Let U be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}[U, \mathbb{R}^n]$ and $x_0 \in U$. Let $y_0 \in \mathbb{R}^n$, $r > 0$, a_ε, $b_\varepsilon > 0$, $d > 0$, $N \in \mathbb{N}_0$ and $\varepsilon_1 \in (0, 1]$ satisfy the following conditions:*

(i) $u(x_0) \approx y_0$;

(ii) $\overline{B_r(x_0)} \subseteq U$;

(iii) $a_\varepsilon b_\varepsilon + d\varepsilon^N \leqslant 1$ *for all* $\varepsilon \leqslant \varepsilon_1$;

(iv) $s := \sup\{a_\varepsilon | 0 < \varepsilon \leqslant \varepsilon_1\}$ *is finite.*

If there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that for all $\varepsilon \leqslant \varepsilon_1$,

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \leqslant a_\varepsilon \varepsilon^N$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \leqslant b_\varepsilon \varepsilon^{-N}$ *for all* $x \in \overline{B_r(x_0)}$,

then u is invertible on $B_{\alpha r}(x_0)$ with inversion data

$$[B_{\alpha r}(x_0), \mathbb{R}^n, v, H_1, B_{\beta(d/s)\gamma r}(y_0)],$$

where α and β are arbitrary in $(0, 1)$, γ is arbitrary in $(0, \alpha)$ and $H_1 \subseteq \mathbb{R}^n$ is an arbitrary open set containing $\overline{\bigcup_{\varepsilon \leqslant \varepsilon_2} u_\varepsilon(B_{\alpha r}(x_0))}$ for some suitable $\varepsilon_2 \leqslant \varepsilon_1$.

Furthermore, $v(y_0) \approx x_0$ and and $B_{\beta(d/s)\gamma r}(y_0) \subseteq u_\varepsilon(B_{\gamma r}(x_0))$ for all $\varepsilon \leqslant \varepsilon_2$. Also, there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r}(x_0))} = u_\varepsilon|_{B_{\alpha r}(x_0)}{}^{-1}$$

for all $\varepsilon \leqslant \varepsilon_2$.

Proof. Without loss of generality we assume $x_0 = 0$ (otherwise, replace U by $U - x_0$ and $u_\varepsilon(x)$ by $u_\varepsilon(x + x_0)$) and $y_0 = 0$ (otherwise consider $u_\varepsilon(x) - y_0$); therefore, we have $u_\varepsilon(0) \approx 0$.

Let $\varepsilon \leqslant \varepsilon_1$. Substituting a by $a_\varepsilon \varepsilon^N$ and b by $b_\varepsilon \varepsilon^{-N}$ in Theorem 6.4 shows that (by Remark 6.5) u_ε is smoothly invertible on $B_r(0)$. Let $w_\varepsilon \colon V_\varepsilon \to B_r(0)$ denote the smooth inverse of $u_\varepsilon|_{B_r(0)}$, where $V_\varepsilon := u_\varepsilon(B_r(0))$ is open in $\mathbb{R}^n$. By (iii),

$$\frac{a_\varepsilon \varepsilon^N}{1 - a_\varepsilon b_\varepsilon} \leqslant \frac{a_\varepsilon \varepsilon^N}{d\varepsilon^N} \leqslant \frac{s}{d}$$

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

holds. Therefore, $a_\varepsilon \varepsilon^N/(1-a_\varepsilon b_\varepsilon)$ being the value corresponding to c in Theorem 6.4, we obtain

$$\|\mathrm{D}u_\varepsilon(x)^{-1}\| \leqslant \frac{s}{d}$$

for all $x \in \overline{B_r(0)}$. By Hadamard's Inequality, it follows that

$$|\det(\mathrm{D}u_\varepsilon(x))| = \left|\frac{1}{\det(\mathrm{D}u_\varepsilon(x)^{-1})}\right| \geqslant \frac{d^n}{Cs^n} \tag{6.2}$$

for some constant $C > 0$ and for all $x \in \overline{B_r(0)}$. Now let $\alpha \in (0,1)$ and $K_\varepsilon := u_\varepsilon(\overline{B_{\alpha r}(0)})$. From (6.2), it immediately follows by Proposition 5.4 that there exist $(K_\varepsilon, 0)$-extensions v_ε of w_ε such that $(v_\varepsilon)_\varepsilon$ is in $\mathcal{E}_M(\mathbb{R}^n)^n$. In particular, $v_\varepsilon \circ u_\varepsilon|_{B_{\alpha r}(0)} = \mathrm{id}_{B_{\alpha r}(0)}$. Now let $\beta \in (0,1)$ and $\gamma \in (0,\alpha)$. Since $u_\varepsilon(0)$ converges to 0 for $\varepsilon \to 0$, there exists $\varepsilon_2 \leqslant \varepsilon_1$ such that

$$|u_\varepsilon(0)| \leqslant (1-\beta)\frac{d}{s}\gamma r$$

for all $\varepsilon \leqslant \varepsilon_2$. Thus, by Proposition 6.6, $B_{\beta(d/s)\gamma r}(0) \subseteq u_\varepsilon(B_{\gamma r}(0))$ for all $\varepsilon \leqslant \varepsilon_2$. From now on, we always let $\varepsilon \leqslant \varepsilon_2$. Since $u_\varepsilon(B_{\gamma r}(0)) \subseteq K_\varepsilon$, we have

$$u_\varepsilon \circ v_\varepsilon|_{B_{\beta(d/s)\gamma r}(0)} = \mathrm{id}_{B_{\beta(d/s)\gamma r}(0)}\,.$$

Moreover, $(v_\varepsilon|_{B_{\beta(d/s)\gamma r}(0)})_\varepsilon$ is c-bounded into $B_{\alpha r}(0)$, since

$$v_\varepsilon(B_{\beta(d/s)\gamma r}(0)) \subseteq \overline{B_{\gamma r}(0)} \subseteq B_{\alpha r}(0).$$

Furthermore, $(u_\varepsilon|_{B_{\alpha r}(0)})_\varepsilon$ is c-bounded into any open set $H_1 \subseteq \mathbb{R}^n$ that contains

$$\overline{\bigcup_{\varepsilon \leqslant \varepsilon_2} u_\varepsilon(B_{\alpha r}(0))},$$

since u is c-bounded into $\mathbb{R}^n$.

Finally, applying Theorem 6.4 (1) and due to the fact that

$$v_\varepsilon|_{B_{\beta(d/s)\gamma r}(0)} = w_\varepsilon|_{B_{\beta(d/s)\gamma r}(0)},$$

we get

$$|v_\varepsilon(0)| = |v_\varepsilon(0) - v_\varepsilon(u_\varepsilon(0))| \leqslant \frac{s}{d}|0 - u_\varepsilon(0)|.$$

Since $u_\varepsilon(0) \to 0$, this also shows that $v_\varepsilon(0) \to 0$ as $\varepsilon \to 0$. □

Note that Theorem 6.7 (2) implies that $(\det(\mathrm{D}u_\varepsilon(x_0)))_\varepsilon$ is strictly non-zero. Theorem 6.7 (iii) guarantees that Banach's Fixed Point Theorem can be applied (implicitly via Theorem 6.4).

The convergence of $(u_\varepsilon(x_0))_\varepsilon$ to some y_0 in Theorem 6.7 ensures that the images of the open ball $B_r(x_0)$ under the u_ε are not scattered wildly all over $\mathbb{R}^n$ but stay centred around y_0. One may suspect that this condition is stronger than necessary. As a matter of fact, the theorem still holds true if $u_\varepsilon(x_0)$ stays close enough to y_0 in the following

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

sense: in the proof, convergence of $(u_\varepsilon(x_0))_\varepsilon$ is needed in one place only, namely to obtain ε_2 such that

$$|u_\varepsilon(x_0) - y_0| \leqslant (1-\beta)\frac{d}{s}\gamma r \tag{6.3}$$

holds for all $\varepsilon \leqslant \varepsilon_2$. Hence, u is invertible even if the convergence condition is weakened to (6.3).

The next proposition shows that the conditions of Theorem 6.7 are, in fact, independent of the choice of the representative.

Proposition 6.8. *If one representative of $u \in \mathcal{G}[U, \mathbb{R}^n]$ satisfies the conditions of Theorem 6.7, then every representative does.*

Proof. The demonstration consists of a series of rather technical estimates. Essentially, the proof establishes that if $(u_\varepsilon)_\varepsilon$ satisfies the conditions of the theorem with x_0, y_0, r, a_ε, b_ε, d, N, ε_1 and s, then another given representative $(\bar{u}_\varepsilon)_\varepsilon$ of u satisfies them with x_0, y_0, r, $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{d}$, N, $\bar{\varepsilon}_1$ and $\bar{s}$, where $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{d}$, $\bar{\varepsilon}_1$ and $\bar{s}$ can be chosen to satisfy $\bar{a}_\varepsilon \geqslant a_\varepsilon$ and $\bar{b}_\varepsilon \geqslant b_\varepsilon$ (for all $\varepsilon \leqslant \bar{\varepsilon}_1$), $\bar{d} \leqslant d$, $\bar{\varepsilon}_1 \leqslant \varepsilon_1$ and $\bar{s} \geqslant s$. Then $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ for $\varepsilon \to 0$, and $\bar{d} \nearrow d$ and $\bar{s} \searrow s$ hold for $\bar{\varepsilon}_1 \to 0$. For full details of the proof we refer the reader to [**9**, Proposition 3.47]. □

The following example shows that the inversion issue of the jump function of Example 3.3 is settled affirmatively by Theorem 6.7.

Example 6.9. Let $u \in \mathcal{G}((-\alpha, \alpha))$ (for $\alpha > 0$) be the generalized function modelling a jump with $u_\varepsilon(x) = x + \arctan(x/\varepsilon)$ as a representative. We found in Example 3.3 that u is invertible on an open neighbourhood of 0. Indeed, $(u_\varepsilon)_\varepsilon$ satisfies all conditions of Theorem 6.7 with $x_0 = 0$, $y_0 = 0$, $r \in (0, \alpha)$, $a_\varepsilon = 1/(\varepsilon + 1)$ (then $s = 1$), $b_\varepsilon = 1$, $d \in (0, \frac{1}{2}]$, $N = 1$ and $\varepsilon_1 = 1$.

The next example emphasizes the role of Theorem 6.7 (iii): if this condition is violated, we cannot expect u to be invertible.

Example 6.10. Recall u from Example 4.4: a representative of u was given by $u_\varepsilon \colon (-\alpha, \alpha) \to \mathbb{R}$, $u_\varepsilon(x) = \sin(x/\varepsilon)$. Let $x_0 = 0$. Then $y_0 = 0$. No matter how small we choose r or ε_1, we always end up with $a_\varepsilon = 1$, $b_\varepsilon = 2$ and $N = 1$. Since the product of a_ε and b_ε is already greater than 1, no $d > 0$ can be found that is consistent with condition Theorem 6.7 (iii). This is not surprising since we have already noted that $(u_\varepsilon)_\varepsilon$ is not ca-injective on any neighbourhood of 0 and, thus, u cannot be left invertible.

Despite the lack of left invertibility there is still hope for u from Example 6.10 to be right invertible since $(u_\varepsilon)_\varepsilon$ at least is ca-surjective onto $(-1, 1)$. Therefore, a theorem yielding right invertibility of generalized functions similar to u from Example 6.10, assuming properties of u similar to those of Theorem 6.7, might be desirable.

Theorem 6.11. *Let U be an open subset of $\mathbb{R}^n$, $u \in \mathcal{G}(U)^n$ and $x_0 \in U$. Let $y_0 \in \mathbb{R}^n$, $r > 0$, a_ε, $b_\varepsilon > 0$, $d > 0$ and $\varepsilon_1 \in (0, 1]$ satisfy*

(i) $u(x_0) \approx y_0$,

(ii) $\overline{B_r(x_0)} \subseteq U$,

(iii) $a_\varepsilon(b_\varepsilon + d) \leqslant 1$ *for all* $\varepsilon \leqslant \varepsilon_1$,

and $N \in \mathbb{N}$. If there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that, for all $\varepsilon \leqslant \varepsilon_1$,

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \leqslant a_\varepsilon \varepsilon^N$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \leqslant b_\varepsilon \varepsilon^{-N}$ *for all* $x \in \overline{B_{r\varepsilon^N}(x_0)}$,

then u is right invertible on $B_{\alpha r \varepsilon_2^N}(x_0)$ with right inversion data

$$[B_{\alpha r \varepsilon_2^N}(x_0), \mathbb{R}^n, v, B_{\beta d \gamma r}(y_0)],$$

where α and β are arbitrary in $(0, 1)$, and γ is arbitrary in $(0, \alpha)$ for some suitable $\varepsilon_2 \leqslant \varepsilon_1$.

Furthermore, $v(y_0) \approx x_0$ and $B_{\beta d \gamma r}(y_0) \subseteq u_\varepsilon(B_{\gamma r \varepsilon^N}(x_0))$ for all $\varepsilon \leqslant \varepsilon_2$. Also, there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r \varepsilon^N}(x_0))} = u_\varepsilon|_{B_{\alpha r \varepsilon^N}(x_0)}{}^{-1}$$

for all $\varepsilon \leqslant \varepsilon_2$.

Proof. The main difference from Theorem 6.7 is the fact that the size of the ball where u_ε is injective is shrinking with ε. Consequently, no left inverse can be found without further conditions (cf. Example 6.10). To prove the theorem just use (iii) instead of Theorem 6.7 (iii) to obtain an estimate for $a_\varepsilon \varepsilon^N/(1 - a_\varepsilon b_\varepsilon)$ and replace s/d by ε^N/d and r by $r\varepsilon^N$ in the proof of Theorem 6.7, while omitting the part concerning the left inverse. □

Note that we do not require u to be c-bounded into $\mathbb{R}^n$. This is due to the fact that the c-boundedness of u is only necessary when composing with a left inverse, whereas the aim of the theorem is to produce a right inverse. Moreover, Theorem 6.11 (iii) has a shape different from its equivalent in Theorem 6.7, corresponding to the difference in the estimates due to the replacement of r by $r\varepsilon^N$. Note that Theorem 6.11 (3) is weaker than Theorem 6.7 (3) and that Theorem 6.11 (iii) implies Theorem 6.7 (iv). The actual shape of Theorem 6.11 (iii) seems to be incomparable to the corresponding Theorem 6.7 (iii); it reflects the necessity of the proof to employ Theorem 6.11 (3). Finally, the convergence condition can again be exchanged for

$$|u_\varepsilon(x_0) - y_0| \leqslant (1 - \beta) d \gamma r$$

for all $\varepsilon \leqslant \varepsilon_1$.

Example 6.12. Checking $u_\varepsilon(x) := \sin(x/\varepsilon)$ for the conditions of Theorem 6.11, we easily see that $(u_\varepsilon)_\varepsilon$ satisfies all the requirements with respect to $x_0 = 0$, $y_0 = 0$, $r \in (0, \frac{1}{2}\pi)$, $a_\varepsilon = 1$, $b_\varepsilon = 1 - \cos r$, $d \in (0, 1 - \cos r)$, $N = 1$ and $\varepsilon_1 = 1$. Therefore, $u = [(u_\varepsilon)_\varepsilon]$ is right invertible on a suitable neighbourhood of 0.

Again, the conditions of Theorem 6.11 hold true independently of the choice of the representative.

Proposition 6.13. *If one representative of $u \in \mathcal{G}(U)^n$ satisfies all the conditions of Theorem 6.11, then every representative does.*

Proof. As before, we refer the reader to [**9**] for the demonstration in full detail (see [**9**, Proposition 3.53]). The proof establishes that if $(u_\varepsilon)_\varepsilon$ satisfies the conditions of the theorem with x_0, y_0, r, a_ε, b_ε, d, N and ε_1, then another given representative $(\bar{u}_\varepsilon)_\varepsilon$ of u satisfies them with x_0, y_0, r, $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{d}$, N and $\bar{\varepsilon}_1$, where $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{d}$ and $\bar{\varepsilon}_1$ can be chosen to satisfy $\bar{a}_\varepsilon \geqslant a_\varepsilon$ and $\bar{b}_\varepsilon \geqslant b_\varepsilon$ (for all $\varepsilon \leqslant \bar{\varepsilon}_1$), $\bar{d} \leqslant d$ and $\bar{\varepsilon}_1 \leqslant \varepsilon_1$. Then $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ hold for $\varepsilon \to 0$, and $\bar{d} \nearrow d$ holds for $\bar{\varepsilon}_1 \to 0$. □

Now that we have been successful in proving a 'right inverse function theorem' the question arises whether a modification with respect to 'only left invertible' is also possible. Typically, the generalized functions being only left invertible are ca-injective on a fixed set but the interior of the intersection of the images of this set under u_ε is empty. In addition, we know that the inverse of any right invertible function is left invertible (see Proposition 3.5 (i)). So let us examine the following example.

Example 6.14. Consider $v \in \mathcal{G}((-1, 1))$, which has $v_\varepsilon(x) := \varepsilon \arcsin x$ as a representative. This v is a right inverse to the function u we studied in Examples 6.10 and 6.12. Since $\mathrm{D}v_\varepsilon(0)$ is the reciprocal value of $\mathrm{D}u_\varepsilon(0)$, it is not surprising to discover that $(v_\varepsilon)_\varepsilon$ satisfies estimates similar to conditions (2) and (3) of Theorem 6.11 with the sign of N reversed.

Indeed, reversing the sign of N in conditions (2) and (3) of Theorem 6.7 leads to sufficient conditions for left invertibility.

Theorem 6.15. *Let U be an open subset of $\mathbb{R}^n$, let $u \in \mathcal{G}[U, \mathbb{R}^n]$ and let $x_0 \in U$. Let $r > 0$, a_ε, $b_\varepsilon > 0$, $d > 0$, $N \in \mathbb{N}_0$ and $\varepsilon_1 \in (0, 1]$ satisfy the following conditions:*

(i) $\overline{B_r(x_0)} \subseteq U$;

(ii) $a_\varepsilon b_\varepsilon + d\varepsilon^N \leqslant 1$ *for all* $\varepsilon \leqslant \varepsilon_1$;

(iii) $s := \sup\{a_\varepsilon | 0 < \varepsilon \leqslant \varepsilon_1\}$ *is finite.*

If there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that, for all $\varepsilon \leqslant \varepsilon_1$,

(1) $\det(\mathrm{D}u_\varepsilon(x_0)) \neq 0$,

(2) $\|\mathrm{D}u_\varepsilon(x_0)^{-1}\| \leqslant a_\varepsilon \varepsilon^{-N}$,

(3) $\|\mathrm{D}u_\varepsilon(x_0) - \mathrm{D}u_\varepsilon(x)\| \leqslant b_\varepsilon \varepsilon^N$ *for all* $x \in \overline{B_r(x_0)}$,

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

then u is left invertible on $B_{\alpha r}(x_0)$ with left inversion data

$$[B_{\alpha r}(x_0), \mathbb{R}^n, v, H_l],$$

where α is arbitrary in $(0,1)$ and $H_l \subseteq \mathbb{R}^n$ is an arbitrary open set containing

$$\overline{\bigcup_{\varepsilon \leqslant \varepsilon_1} u_\varepsilon(B_{\alpha r}(x_0))}.$$

Furthermore, there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that

$$v_\varepsilon|_{u_\varepsilon(B_{\alpha r}(x_0))} = u_\varepsilon|_{B_{\alpha r}(x_0)}{}^{-1}$$

for all $\varepsilon \leqslant \varepsilon_2$ for some suitable $\varepsilon_2 \leqslant \varepsilon_1$.

Proof. To prove the theorem we just use (ii), as Theorem 6.7 (iii) is used to obtain an estimate for $a_\varepsilon \varepsilon^{-N}/(1 - a_\varepsilon b_\varepsilon)$, and replace $a_\varepsilon \varepsilon^N$ by $a_\varepsilon \varepsilon^{-N}$, $b_\varepsilon \varepsilon^{-N}$ by $b_\varepsilon \varepsilon^N$ and d by ε^{2N} in the proof of Theorem 6.7, while omitting the part introducing the constant β and the part concerning the convergence of $v_\varepsilon(0)$ to 0. □

The preceding theorem lacks the convergence condition on $(u_\varepsilon(x_0))_\varepsilon$ corresponding to Theorem 6.7 (i), since for the construction of a left inverse we do not care if the intersection of the images under u_ε still contains a non-empty open set.

Example 6.16. Let v be the generalized function from Example 6.14. Then $(v_\varepsilon)_\varepsilon$ satisfies the conditions of Theorem 6.15 with respect to $x_0 = 0$, $r \in (0, \frac{\sqrt{3}}{2})$, $a_\varepsilon = 1$, $b_\varepsilon = 1/(\sqrt{1 - r^2}) - 1 < 1$, $d \in (0, 2 - 1/\sqrt{1 - r^2}]$, $N = 1$ and $\varepsilon_1 = 1$.

Once more we have independence of the choice of the representative in Theorem 6.15.

Proposition 6.17. *If one representative of $u \in \mathcal{G}[U, \mathbb{R}^n]$ satisfies the conditions of Theorem 6.15, then every representative does.*

Proof. Again, we refer the reader to [**9**] for the demonstration in full detail (see [**9**, Proposition 3.57]). The proof establishes that if $(u_\varepsilon)_\varepsilon$ satisfies the conditions of the theorem with x_0, r, a_ε, b_ε, d, N, ε_1 and s, then another given representative $(\bar{u}_\varepsilon)_\varepsilon$ of u satisfies them with x_0, r, $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{d}$, N, $\bar{\varepsilon}_1$ and $\bar{s}$, where $\bar{a}_\varepsilon$, $\bar{b}_\varepsilon$, $\bar{d}$, $\bar{\varepsilon}_1$ and $\bar{s}$ can be chosen to satisfy $\bar{a}_\varepsilon \geqslant a_\varepsilon$ and $\bar{b}_\varepsilon \geqslant b_\varepsilon$ (for all $\varepsilon \leqslant \bar{\varepsilon}_1$), $\bar{d} \leqslant d$, $\bar{\varepsilon}_1 \leqslant \varepsilon_1$ and $\bar{s} \geqslant s$. Then $|a_\varepsilon - \bar{a}_\varepsilon| \to 0$ and $|b_\varepsilon - \bar{b}_\varepsilon| \to 0$ hold for $\varepsilon \to 0$ and $\bar{d} \nearrow d$, and $\bar{s} \searrow s$ hold for $\bar{\varepsilon}_1 \to 0$. □

Finally, we take a look at the relation between the classical Inverse Function Theorem (Theorem 6.4) and the generalized inverse function theorem (Theorem 6.7). On the $\mathcal{C}^\infty$-level we saw in Remark 3.2 (iii) that if a smooth function $f \colon U \to V$ (with U and V open subsets of $\mathbb{R}^n$) is classically $\mathcal{C}^\infty$-invertible on a neighbourhood W of some point $x_0 \in U$ with smooth inverse g, then, obviously, $\sigma(f) = \iota(f)$ is strictly invertible on W with inversion data $[W, f(W), \sigma(g), f(W)]$. But what is the situation if f is not $\mathcal{C}^\infty$, i.e. if we cannot use the trivial embedding σ? In the following, we shall show that our notion of invertibility and Theorem 6.7 are consistent with the classical Inverse Function Theorem

https://doi.org/10.1017/S0013091512000429 Published online by Cambridge University Press

(Theorem 6.4), the latter taken for the special case $X = Y = \mathbb{R}^n$ and f a $\mathcal{C}^1$-function. In the proof we shall use the following proposition (for the proof of the proposition itself refer to [**9**, Proposition 3.60]).

Proposition 6.18. *Let U be an open subset of $\mathbb{R}^n$, V an open subset of $\mathbb{R}^m$, $f \in \mathcal{C}(U,V)$ and $f_\varepsilon \in \mathcal{C}(U,\mathbb{R}^m)$ for $\varepsilon \in (0,\varepsilon_0]$. Assume that $(f_\varepsilon)_\varepsilon$ converges to f uniformly on compact subsets of U as $\varepsilon \to 0$. If g is a continuous function on V, then $(g \circ f_\varepsilon|_K)_\varepsilon$ converges uniformly to $g \circ f|_K$ for all compact sets K in U.*

We now establish the relation between the classical (local) inverse of a continuously differentiable function f and the (generalized) inverse of $\iota(f)$. In doing so, we first present the essential statement in a rather concise, yet sloppy, way, followed by more elaborate and precise versions displaying more of the technical details.

Theorem 6.19. *Let U be an open subset of $\mathbb{R}^n$, let $x_0 \in U$ and let $f \in \mathcal{C}^1(U,\mathbb{R}^n)$ with $\det(\mathrm{D}f(x_0)) \neq 0$. Then the following hold.*

(1) *$\iota(f) \in \mathcal{G}(U)^n$ satisfies the conditions of Theorem 6.7 around x_0, and therefore is invertible on some neighbourhood of x_0.*

(2) *Let g be the classical inverse of f around $f(x_0)$ and let $v \in \mathcal{G}(\mathbb{R}^n)^n$ be the inverse of $\iota(f)$ obtained by Theorem 6.7. Then, for every representative $(v_\varepsilon)_\varepsilon$ of v, the nets $(v_\varepsilon)_\varepsilon$ and $(\mathrm{D}v_\varepsilon)_\varepsilon$ converge to g and $\mathrm{D}g$, respectively, uniformly on compact subsets of some suitable open neighbourhood of $f(x_0)$.*

More precisely, we have the following.

(i) *If f is (classically) invertible around x_0 with constants $y_0 := f(x_0)$, r, a, b, $W := B_r(x_0)$ and $V := f(B_r(x_0))$ as given in Theorem 6.4, then any representative of $\iota(f)$ satisfies the conditions of Theorem 6.7 with x_0, y_0, r, $a_\varepsilon := a + \delta$, $b_\varepsilon := b + \delta$ (where $\delta > 0$ is chosen such that $(a+\delta)(b+\delta) < 1$), $d := 1 - (a+\delta)(b+\delta)$, $N := 0$ and some suitable ε_1.*

(ii) *Let $g \in \mathcal{C}^1(V,W)$ be the inverse of $f|_W$ around $f(x_0) \in V$ given by the (classical) Inverse Function Theorem 6.4 and let $v \in \mathcal{G}(\mathbb{R}^n)^n$ be the inverse of $\iota(f)$ obtained by Theorem 6.7 with inversion data $[G,\mathbb{R}^n,v,H_\mathrm{l},H_\mathrm{r}]$. Assume that for both g and v the relevant constants are given as in (i).*

Then $G \subsetneq W$, $H_\mathrm{r} \subsetneq V$ and, for every representative $(v_\varepsilon)_\varepsilon$ of v, the nets $(v_\varepsilon)_\varepsilon$ and $(\mathrm{D}v_\varepsilon)_\varepsilon$ converge to g and $\mathrm{D}g$, respectively, uniformly on compact subsets of $H_\mathrm{r} \cap V = H_\mathrm{r}$.

Proof. (i) We check the conditions of Theorem 6.7 for a representative $(f_\varepsilon)_\varepsilon$ of $\iota(f)$. Obviously, conditions (ii)–(iv) of Theorem 6.7 (with $s := a + \delta$) hold for all $\varepsilon \in (0,1]$. By Proposition 2.7, $\iota(f)$ is $\mathcal{C}^1$-associated with f. Therefore, $f_\varepsilon(x_0) \to f(x_0) = y_0$ and $\det(\mathrm{D}f_\varepsilon(x_0)) \to \det(\mathrm{D}f(x_0)) \neq 0$, showing that conditions (i) and (1) of Theorem 6.7 (for sufficiently small ε, say $\varepsilon \leqslant \varepsilon_0$) are satisfied. Since $\mathrm{D}f_\varepsilon(x) \to \mathrm{D}f(x)$ for all $x \in \overline{B_r(x_0)}$, we can find $\varepsilon_1 \leqslant \varepsilon_0$ such that $\|\mathrm{D}f_\varepsilon(x_0)\| \leqslant a + \delta$ and

$$\begin{aligned}\|\mathrm{D}f_\varepsilon(x_0) - \mathrm{D}f_\varepsilon(x)\| &\leqslant \|\mathrm{D}f_\varepsilon(x_0) - \mathrm{D}f(x_0)\| + \|\mathrm{D}f(x_0) - \mathrm{D}f(x)\| + \|\mathrm{D}f(x) - \mathrm{D}f_\varepsilon(x)\| \\ &\leqslant \tfrac{1}{2}\delta + b + \tfrac{1}{2}\delta \\ &= b + \delta\end{aligned}$$

for all $x \in \overline{B_r(x_0)}$ and $\varepsilon \leqslant \varepsilon_1$, yielding conditions (2) and (3) of Theorem 6.7, as required.

(ii) Since $G = B_{\alpha r}(x_0)$ with $\alpha \in (0, 1)$, the inclusion $W = B_r(x_0) \subsetneq B_{\alpha r}(x_0) = G$ holds. According to Theorem 6.4, $B_{r/c}(y_0) \subseteq f(B_r(x_0)) = V$ with $c = a/(1 - ab)$. Theorem 6.7 yields that $H_\mathrm{r} = B_{\beta(d/s)\gamma r}(y_0)$ with $s = a + \delta$, $\beta \in (0, 1)$ and $\gamma \in (0, \alpha) \subseteq (0, 1)$. From

$$\frac{r}{c} = \frac{1 - ab}{a} r > \beta \frac{1 - (a + \delta)(b + \delta)}{a + \delta} \gamma r = \beta \frac{d}{s} \gamma r$$

it follows that $H_\mathrm{r} = B_{\beta(d/s)\gamma r}(y_0) \subsetneq B_{r/c}(y_0) \subseteq f(B_r(x_0)) = V$.

Next, we prove the uniform convergence of $(v_\varepsilon)_\varepsilon$ to g on compact subsets of H_r. By Theorem 6.7, there exists a representative $(v_\varepsilon)_\varepsilon$ of v such that $f_\varepsilon \circ v_\varepsilon|_{H_\mathrm{r}} = \mathrm{id}_{H_\mathrm{r}}$. Observing that $v_\varepsilon(H_\mathrm{r}) \subseteq \bar{G}$, we obtain

$$\begin{aligned}\sup_{x \in H_\mathrm{r}} |f \circ v_\varepsilon(x) - x| &= \sup_{x \in H_\mathrm{r}} |f(v_\varepsilon(x)) - f_\varepsilon(v_\varepsilon(x))| \\ &\leqslant \sup_{y \in \bar{G}} |f(y) - f_\varepsilon(y)|.\end{aligned}$$

By Proposition 2.7, the right-hand side converges uniformly to 0 for $\varepsilon \to 0$ and, hence, so does the left-hand side. Applying Proposition 6.18 to $\mathrm{id}_{H_\mathrm{r}}$, $(f \circ v_\varepsilon)_\varepsilon$ and g yields that $(v_\varepsilon)_\varepsilon$ converges to g uniformly on compact subsets of H_r.

Finally, we prove the uniform convergence of the derivatives on compact sets. By $v_\varepsilon(H_\mathrm{r}) \subseteq \bar{G}$,

$$\begin{aligned}\sup_{x \in H_\mathrm{r}} \|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - I\| &= \sup_{x \in H_\mathrm{r}} \|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - \mathrm{D}u_\varepsilon(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x)\| \\ &\leqslant \sup_{z \in \bar{G}} \|\mathrm{D}f(z) - \mathrm{D}u_\varepsilon(z)\| \|\mathrm{D}u_\varepsilon(z)^{-1}\|\end{aligned}$$

holds. As shown in the proof of Theorem 6.7, $(\mathrm{D}u_\varepsilon(\cdot)^{-1})_\varepsilon$ is uniformly bounded on $\bar{G}$ with respect to ε. By Proposition 2.7, $(\mathrm{D}u_\varepsilon)_\varepsilon$ converges to $\mathrm{D}f$ uniformly on the compact set $\bar{G}$ for $\varepsilon \to 0$. Hence,

$$\sup_{x \in H_\mathrm{r}} \|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - I\| \to 0 \quad \text{uniformly as } \varepsilon \to 0. \tag{6.4}$$

Applying Proposition 6.18 to g, $(v_\varepsilon)_\varepsilon$ and $\mathrm{D}f$, we obtain that

$$\sup_{x \in L} \|\mathrm{D}f(g(x)) - \mathrm{D}f(v_\varepsilon(x))\| \to 0 \quad \text{uniformly as } \varepsilon \to 0 \tag{6.5}$$

for all compact subsets L of H_{r}. Let $K \subset\subset H_{\mathrm{r}}$ and $x \in K$. Then

$$\begin{aligned}
&\|\mathrm{D}v_\varepsilon(x) - \mathrm{D}g(x)\| \\
&\quad = \|\mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - \mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}g(x)\| \\
&\quad \leqslant \|\mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - \mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(g(x)) \circ \mathrm{D}g(x)\| \\
&\qquad + \|\mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(g(x)) \circ \mathrm{D}g(x) - \mathrm{D}f(v_\varepsilon(x))^{-1} \circ \mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}g(x)\| \\
&\quad \leqslant \|\mathrm{D}f(v_\varepsilon(x))^{-1}\|(\|\mathrm{D}f(v_\varepsilon(x)) \circ \mathrm{D}v_\varepsilon(x) - I\| + \|\mathrm{D}f(g(x)) - \mathrm{D}f(v_\varepsilon(x))\|\|\mathrm{D}g(x)\|)
\end{aligned}$$

holds. $\mathrm{D}f(v_\varepsilon(\cdot))^{-1}$ (by $v_\varepsilon(K) \subseteq \bar{G}$) and $\mathrm{D}g$ are bounded on K, independently of ε. By (6.4) and (6.5), the two expressions in the parentheses converge to 0 uniformly on K as $\varepsilon \to 0$, thereby concluding the proof. □

Acknowledgements. The author thanks Michael Grosser for his valuable input and support during all the work on the author's thesis and this paper. This work was supported by Project P16742 and START Project Y-237 of the Austrian Science Fund.

References

1. R. Abraham, J. E. Marsden and T. Raţiu, *Manifolds, tensor analysis, and applications* (Addison-Wesley, Reading, MA, 1983).
2. J. Aragona and H. A. Biagioni, Intrinsic definition of the Colombeau algebra of generalized functions, *Analysis Math.* **17** (1991), 75–132.
3. J. Aragona, R. Fernandez and S. O. Juriaans, A discontinuous Colombeau differential calculus, *Monatsh. Math.* **144** (2005), 13–29.
4. J. Aragona, R. Fernandez, S. O. Juriaans and M. Oberguggenberger, Differential calculus and integration of generalized functions over membranes, *Monatsh. Math.* **166** (2012), 1–18.
5. J.-F. Colombeau, *New generalized functions and multiplication of distributions*, North-Holland Mathematics Studies, Volume 84 (North-Holland, Amsterdam, 1984).
6. J.-F. Colombeau, *Elementary introduction to new generalized functions*, North-Holland Mathematics Studies, Volume 113 (North-Holland, Amsterdam, 1985).
7. G. De Rham, *Differentiable manifolds* (Springer, 1984).
8. N. Djapić, M. Kunzinger and S. Pilipović, Symmetry group analysis of weak solutions, *Proc. Lond. Math. Soc.* **84** (2002), 686–710.
9. E. Erlacher, Local existence results in algebras of generalised functions, PhD thesis, University of Vienna (2007; available at www.mat.univie.ac.at/~diana/uploads/publication47.pdf).
10. E. Erlacher and M. Grosser, Inversion of a 'discontinuous coordinate transformation' in general relativity, *Applic. Analysis* **90** (2011), 1707–1728.
11. D. Gale and H. Nikaidô, The Jacobian matrix and global univalence of mappings, *Math. Annalen* **159** (1965), 81–93.
12. C. Garetto and G. Hörmann, On duality theory and pseudodifferential techniques for Colombeau algebras: generalized delta functionals, kernels and wave front sets, *Bull. Classe Sci. Math. Nat.* **31** (2006), 115–136.
13. C. Garetto, G. Hörmann and M. Oberguggenberger, Generalized oscillatory integrals and Fourier integral operators, *Proc. Edinb. Math. Soc.* **52** (2009), 351–386.
14. M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer, *Geometric theory of generalized functions*, Mathematics and Its Applications, Volume 537 (Kluwer Academic, Dordrecht, 2001).

15. S. Haller, Microlocal analysis of generalized pullbacks of Colombeau functions, *Acta Appl. Math.* **105** (2009), 83–109.
16. G. Hörmann, Hölder–Zygmund regularity in algebras of generalized functions, *Z. Analysis Anwend.* **23** (2004), 139–165.
17. G. Hörmann and M. V. de Hoop, Microlocal analysis and global solutions of some hyperbolic equations with discontinuous coefficients, *Acta Appl. Math.* **67** (2001), 173–224.
18. S. Konjik and M. Kunzinger, Generalized group actions in a global setting, *J. Math. Analysis Applic.* **322** (2006), 420–436.
19. M. Kunzinger, Generalized functions valued in a smooth manifold, *Monatsh. Math.* **137** (2002), 31–49.
20. M. Kunzinger and M. Oberguggenberger, Group analysis of differential equations and generalized functions, *SIAM J. Math. Analysis* **31** (2000), 1192–1213.
21. M. Kunzinger and R. Steinbauer, A note on the Penrose junction conditions, *Class. Quant. Grav.* **16** (1999), 1255–1264.
22. M. Kunzinger and R. Steinbauer, Generalized pseudo-Riemannian geometry, *Trans. Am. Math. Soc.* **354** (2002), 4179–4199.
23. M. Kunzinger, M. Oberguggenberger, R. Steinbauer and J. Vickers, Generalized flows and singular ODEs on differentiable manifolds, *Acta Appl. Math.* **80** (2004), 221–241.
24. M. Kunzinger, R. Steinbauer and J. Vickers, Intrinsic characterization of manifold-valued generalized functions, *Proc. Lond. Math. Soc.* **87** (2003), 451–470.
25. I. Madsen and J. Tornehave, *From calculus to cohomology* (Cambridge University Press, 1997).
26. M. Nedeljkov, S. Pilipović and D. Scarpalézos, *The linear theory of colombeau generalized functions*, Pitman Research Notes in Mathematics, Volume 385 (Longman, Harlow, 1998).
27. M. Oberguggenberger, *Multiplication of distributions and applications to partial differential equations*, Pitman Research Notes in Mathematics, Volume 259 (Longman, Harlow, 1992).
28. R. Penrose, Twistor quantization and curved space-time, *Int. J. Theor. Phys.* **1** (1968), 61–99.