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K-theory of Furstenberg Transformation Group C^*-algebras

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Abstract. This paper studies the K-theoretic invariants of the crossed product C^*-algebras associated with an important family of homeomorphisms of the tori $\mathbb{T}^n$ called *Furstenberg transformations*. Using the Pimsner–Voiculescu theorem, we prove that given n, the K-groups of those crossed products whose corresponding $n \times n$ integer matrices are unipotent of maximal degree always have the same rank a_n. We show using the theory developed here that a claim made in the literature about the torsion subgroups of these K-groups is false. Using the representation theory of the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, we show that, remarkably, a_n has a combinatorial significance. For example, every a_{2n+1} is just the number of ways that 0 can be represented as a sum of integers between $-n$ and n (with no repetitions). By adapting an argument of van Lint (in which he answered a question of Erdős), a simple explicit formula for the asymptotic behavior of the sequence $\{a_n\}$ is given. Finally, we describe the order structure of the K_0-groups of an important class of Furstenberg crossed products, obtaining their complete Elliott invariant using classification results of H. Lin and N. C. Phillips.

1 Introduction

Furstenberg transformations were introduced in [9] as the first examples of homeomorphisms of the tori that under some necessary and sufficient conditions are minimal and uniquely ergodic. In some sense, they generalize the irrational rotations on the circle. They also appear in certain applications of ergodic theory to number theory (*e.g.*, in Diophantine approximation [8]), and are sometimes called *skew product transformations* or *compound skew translations* of the tori. In this paper, the crossed products associated with Furstenberg transformations are called "Furstenberg transformation group C^*-algebras", and are denoted by $\mathscr{F}_{\theta,f}$. There have been several contributions to the computations of K-theoretic invariants for some examples of these C^*-algebras in the literature (see [16, 18, 25, 30, 39] to name a few). However, there has not been a more general study of such invariants for these C^*-algebras to the best of our knowledge.

Remark 1.1 In independent (unpublished) work [16], R. Ji studied the K-groups of the C^*-algebras $\mathscr{F}_{\theta,f}$ (denoted by $A_{F_{f,\theta}}$ there) associated with the descending affine Furstenberg transformations (denoted by $F_{f,\theta}$ in there) on the tori. He comments that "explicitly computing the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_K \mathbb{Z}$ [$A_{F_{f,\theta}}$ for $\theta = 0$] is still not an easy matter." Moreover, he gives no information about the ranks of the K-groups or the order structure of K_0 in general, which are studied in this paper. As we shall

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see in Remark 1.7, the claim that he makes about the form of the torsion subgroup of $K_*(\mathscr{F}_{\theta,f})$ is, unfortunately, not correct.

From the C^*-algebraic point view, when a Furstenberg transformation is minimal and uniquely ergodic, the associated transformation group C^*-algebra is simple and has a unique tracial state with a dense tracial range of the K_0-group in the real line. Because of this, these C^*-algebras fit well into the classification program of G. Elliott by finding their K-theoretic invariants. In fact, in the class of transformation group C^*-algebras of uniquely ergodic minimal homeomorphisms on infinite compact metric spaces, K-theory is a complete invariant. More precisely, suppose that X is an infinite compact metric space with finite covering dimension and $h\colon X \to X$ is a uniquely ergodic minimal homeomorphism, and put $A := \mathcal{C}(X) \rtimes_h \mathbb{Z}$. Let τ be the trace induced by the unique invariant probability measure. Then τ is the unique tracial state on A. Let $\tau_*\colon K_0(A) \to \mathbb{R}$ be the induced homomorphism on $K_0(A)$ and assume that $\tau_* K_0(A)$ is dense in $\mathbb{R}$. Then the 4-tuple

$$\big(K_0(A), K_0(A)_+, [1_A], K_1(A)\big)$$

is a complete algebraic invariant (called the *Elliott invariant* of A) [21, Corollary 4.8]. In this case, A has stable rank one, real rank zero, and tracial topological rank zero in the sense of H. Lin [19]. The order on $K_0(A)$ is also determined by the unique trace τ, in the sense that an element $x \in K_0(A)$ is positive if and only if either $x = 0$ or $\tau_*(x) > 0$ [22, 29]. This implies, in particular, that the torsion subgroup of $K_0(A)$ contributes nothing interesting to the order information. In other words, the order on $K_0(A)$ is determined by the order on the free part. We will study the order structure of $K_0(\mathscr{F}_{\theta,f})$ in Section 7.

In order to compute the K-groups of a crossed product of the form $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ in general, we make use of the algebraic properties of $K_*(\mathcal{C}(\mathbb{T}^n))$ in Section 2. More precisely, $K_*(\mathcal{C}(\mathbb{T}^n))$ is an exterior algebra over $\mathbb{Z}^n$ with a certain natural basis, and the induced automorphism α_* on $K_*(\mathcal{C}(\mathbb{T}^n))$ is in fact a ring automorphism, which makes computations much easier. In fact, Theorem 2.1 shows that the problem of finding the K-groups of the transformation group C^*-algebra of a homeomorphism of the n-torus is completely computable in the sense that one only needs to calculate the kernels and cokernels of a finite number of integer matrices. These K-groups are finitely generated with the same rank (see Corollary 2.2). In the special case of Anzai transformation group C^*-algebras $\mathscr{A}_{n,\theta}$ associated with Anzai transformations on the n-torus, we denote this common rank by a_n, which we will study in detail in this paper. It is proved in Theorem 5.1 that a_n is the common rank of the K-groups of a larger class of transformation group C^*-algebras, including the C^*-algebras associated with Furstenberg transformations on $\mathbb{T}^n$. We describe a_n as the constant term in a certain Laurent polynomial (Theorem 6.7). Then we study the combinatorial properties of the sequence $\{a_n\}$, which leads to a simple asymptotic formula.

To present the results and proofs of this paper we need some definitions about transformations on the tori and the corresponding C^*-crossed products. Throughout this paper, $\mathbb{T}^n$ denotes the n-dimensional torus with coordinates $(\zeta_1, \zeta_2, \ldots, \zeta_n)$.

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Definition 1.2 An *affine transformation* on $\mathbb{T}^n$ is given by

$$\beta(\zeta_1,\zeta_2,\dots,\zeta_n)=(e^{2\pi it_1}\zeta_1^{b_{11}}\cdots\zeta_n^{b_{1n}},e^{2\pi it_2}\zeta_1^{b_{21}}\cdots\zeta_n^{b_{2n}},\dots,e^{2\pi it_n}\zeta_1^{b_{n1}}\cdots\zeta_n^{b_{nn}}),$$

where $t:=(t_1,t_2,\dots,t_n)\in(\mathbb{R}/\mathbb{Z})^n$ and $\mathrm{B}:=[b_{ij}]_{n\times n}\in\mathrm{GL}(n,\mathbb{Z})$. We identify the pair (t,B) with β.

Note that any automorphism of $\mathbb{T}^n$ followed by a rotation can be expressed in such a fashion. The set of affine transformations on $\mathbb{T}^n$ form a group $\mathrm{Aff}(\mathbb{T}^n)$ that can be identified with the semidirect product $(\mathbb{R}/\mathbb{Z})^n\rtimes\mathrm{GL}(n,\mathbb{Z})$. More precisely, for two affine transformations $\beta=(t,\mathrm{B})$ and $\beta'=(t',\mathrm{B}')$ on $\mathbb{T}^n$, we have

$$\beta\circ\beta'=(t+\mathrm{B}t',\mathrm{BB}')\ \text{ and }\ \beta^{-1}=(-\mathrm{B}^{-1}t,\mathrm{B}^{-1}).$$

(In the expression $\mathrm{B}t$, t is a column vector, but for convenience we write it as a row vector.)

We remind the reader of an important fact before giving the next definition. Consider homotopy classes of continuous functions from $\mathbb{T}^n$ to $\mathbb{T}$. It is well known that in each class there is a unique "linear" function $(\zeta_1,\dots,\zeta_n)\mapsto\zeta_1^{b_1}\cdots\zeta_n^{b_n}$ for some $b_1,\dots,b_n\in\mathbb{Z}$. More precisely, every continuous function $f\colon\mathbb{T}^n\to\mathbb{T}$ can be written as

$$f(\zeta_1,\dots,\zeta_n)=e^{2\pi ig(\zeta_1,\dots,\zeta_n)}\zeta_1^{b_1}\cdots\zeta_n^{b_n},$$

for some continuous function $g\colon\mathbb{T}^n\to\mathbb{R}$ and unique integer exponents $b_1,\dots,b_n$. In particular, the cohomotopy group $\pi^1(\mathbb{T}^n)$ is isomorphic to $\mathbb{Z}^n$. Following [8, p. 35], we denote the exponent b_i, which is uniquely determined by the homotopy class of f, as $b_i=A_i[f]$.

Definition 1.3 We define the following transformations in accordance with [16].

(i) A *Furstenberg transformation* $\varphi_{\theta,f}$ on $\mathbb{T}^n$ is given by

$$\varphi_{\theta,f}^{-1}(\zeta_1,\zeta_2,\dots,\zeta_n)=\left(e^{2\pi i\theta}\zeta_1,f_1(\zeta_1)\zeta_2,f_2(\zeta_1,\zeta_2)\zeta_3,\dots,f_{n-1}(\zeta_1,\dots,\zeta_{n-1})\zeta_n\right),$$

where θ is a real number, each $f_i\colon\mathbb{T}^i\to\mathbb{T}$ is a continuous function with $A_i[f_i]\neq0$ for $i=1,\dots,n-1$, and $f=(f_1,\dots,f_{n-1})$.

(ii) An *affine Furstenberg transformation* α on $\mathbb{T}^n$ is given by

$$\alpha^{-1}(\zeta_1,\zeta_2,\dots,\zeta_n)=(e^{2\pi i\theta}\zeta_1,\zeta_1^{b_{12}}\zeta_2,\zeta_1^{b_{13}}\zeta_2^{b_{23}}\zeta_3,\dots,\zeta_1^{b_{1n}}\zeta_2^{b_{2n}}\cdots\zeta_{n-1}^{b_{n-1,n}}\zeta_n),$$

where θ is a real number, the exponents b_{ij} are integers, and $b_{i,i+1}\neq0$ for $i=1,\dots,n-1$.

(iii) An *ascending Furstenberg transformation* α on $\mathbb{T}^n$ is given by

$$\alpha^{-1}(\zeta_1,\zeta_2,\dots,\zeta_n)=(e^{2\pi i\theta}\zeta_1,\zeta_1^{k_1}\zeta_2,\zeta_2^{k_2}\zeta_3,\dots,\zeta_{n-1}^{k_{n-1}}\zeta_n),$$

where θ is a real number, the exponents k_i are nonzero integers, and $k_i\mid k_{i+1}$ for $i=1,\dots,n-2$.

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(iv) In (iii), if $k_i = 1$ for $i = 1, \dots, n-1$, the transformation is called an *Anzai transformation* $\sigma_{n,\theta}$ on $\mathbb{T}^n$. Thus it is given by

$$\sigma_{n,\theta}^{-1}(\zeta_1, \zeta_2, \dots, \zeta_n) = (e^{2\pi i\theta}\zeta_1, \zeta_1\zeta_2, \dots, \zeta_{n-1}\zeta_n),$$

where θ is a real number. We usually drop the indices n and θ and write only σ for more convenience.

Note that one can easily verify that $\varphi_{\theta,f}$ is a homeomorphism. Also in the above definition, we have converted "descending", which is used in [16, Definition 2.16], to "ascending", since the order of coordinates in that paper is opposite to ours.

For certain Furstenberg transformations on $\mathbb{T}^n$ we have the following theorem.

Theorem 1.4 ([9, 2.3]) *If θ is irrational, then $\varphi_{\theta,f}$ defines a minimal dynamical system on $\mathbb{T}^n$. If in addition, each f_i satisfies a uniform Lipschitz condition in ζ_i for $i = 1, \dots, n-1$, then $\varphi_{\theta,f}$ is a uniquely ergodic transformation and the unique invariant measure is the normalized Lebesgue measure on $\mathbb{T}^n$. In particular, every affine Furstenberg transformation defines a minimal and uniquely ergodic dynamical system if θ is irrational.*

As a conclusion, we have the following result for the Furstenberg transformation group C^*-algebra $\mathscr{F}_{\theta,f} := \mathcal{C}(\mathbb{T}^n) \rtimes_{\varphi_{\theta,f}} \mathbb{Z}$ as introduced in [16].

Corollary 1.5 *$\mathscr{F}_{\theta,f} = \mathcal{C}(\mathbb{T}^n) \rtimes_{\varphi_{\theta,f}} \mathbb{Z}$ is a simple C^*-algebra for irrational θ. If in addition, each f_i satisfies a uniform Lipschitz condition in ζ_i for $i = 1, \dots, n-1$, then $\mathscr{F}_{\theta,f}$ has a unique tracial state.*

Proof For the first part, the minimality of the action as stated in the preceding theorem implies the simplicity of $\mathscr{F}_{\theta,f}$ [4, 32]. For the second part, one can easily check that since θ is irrational, the action of $\mathbb{Z}$ on $\mathbb{T}^n$ generated by $\varphi_{\theta,f}$ is free. So there are no periodic points in $\mathbb{T}^n$. This and the unique ergodicity of $\varphi_{\theta,f}$ yield the result [38, Corollary 3.3.10, p. 91]. ∎

Remark 1.6 Using the preceding corollary and much like the proof of [33, Theorem 2.1], one can prove that for irrational θ, $\mathscr{F}_{\theta,f}$ is in fact the unique C^*-algebra generated by unitaries $U, V_1, \dots, V_n$ satisfying the commutator relations

$$\textbf{(CR)}_f \qquad [U, V_1] = e^{2\pi i\theta}, [U, V_2] = f_1(V_1), \dots, [U, V_n] = f_{n-1}(V_1, \dots, V_{n-1}),$$

where $[a, b] := aba^{-1}b^{-1}$, and all other pairs of operators from $U, V_1, \dots, V_n$ commute.

Remark 1.7 In [16, Proposition 2.17], Ji claims to have proved the following:

(∗) If $\varphi_{\theta,f}$ is an ascending Furstenberg transformation on $\mathbb{T}^n$ with the ascending sequence $\{k_1, k_2, \dots, k_{n-1}\}$, then the torsion subgroup of $K_*(\mathscr{F}_{\theta,f})$ is isomorphic to $\mathbb{Z}_{k_1} \oplus \mathbb{Z}_{k_2}^{(m_2)} \oplus \cdots \oplus \mathbb{Z}_{k_{n-1}}^{(m_{n-1})}$, where the group $\mathbb{Z}_{k_i}^{(m_i)}$ is the direct product of m_i copies of the cyclic group $\mathbb{Z}_{k_i} = \mathbb{Z}/k_i\mathbb{Z}$.

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From this claim one would immediately deduce that the K-groups of the C^*-algebra $\mathscr{A}_{n,\theta} := \mathcal{C}(\mathbb{T}^n) \rtimes_\sigma \mathbb{Z}$ generated by an Anzai transformation σ on $\mathbb{T}^n$ should be torsion-free. However, we will show that this is not true in general. This type of example first appears for $n = 6$, which seems already beyond hand calculation. (We admit that hand calculation would be the most convincing method to use; however, it is not practicable.) As the first counterexample we obtained by computer, we will see in Example 3.3 that $K_1(\mathscr{A}_{6,\theta}) \cong \mathbb{Z}^{13} \oplus \mathbb{Z}_2$. In fact, the error in the proof of $(*)$ is in [16, p. 29, l.2]; there it is "clearly" assumed that using a matrix S in $\mathrm{GL}(2^n, \mathbb{Z})$, one can delete all entries denoted by $\star$'s in $\mathsf{K}_* - \mathsf{I}$, where K_* is the $2^n \times 2^n$ integer matrix corresponding to $\mathscr{F}_{\theta,f}$ that acts on $K_*(\mathcal{C}(\mathbb{T}^n)) = \Lambda^*\mathbb{Z}^n$ with respect to a certain ordered basis. This error arose originally from the general form of the matrix K_* in [16, p. 27], which is not correct. Ji went on to use the torsion subgroup in $(*)$ as an invariant to classify the C^*-algebras generated by ascending transformations and matrix algebras over them [16, Theorem 3.6]. We do not know whether those classifications hold.

Question Do there exist two different ascending Furstenberg transformations with the same parameter θ and with isomorphic transformation group C^*-algebras?

It is worth mentioning that explicit calculations of K-groups in terms of the parameters involved are possible in low dimensions (of the tori) and answer the question raised above negatively. However, such calculations in terms of the given integer parameters (exponents) of the ascending Furstenberg transformation quickly become cumbersome and impossible in higher dimensions. We have used our computer codes for several numerical values of the parameters in higher dimensions, and we have not found any examples leading to the negative answer to this question. The torsion subgroups of the K-groups are usually larger than what Ji claimed in [16, Proposition 2.17]; however, it is likely that the torsion subgroups depend on the parameters involved in such a way that Ji's classification result is still true. We are currently investigating this problem.

This paper is organized as follows. In Section 2, we review the general approach of exterior algebras for finding K-groups of transformation group C^*-algebras of homeomorphisms of the tori. In Section 3, we apply this method to the important case of Anzai transformations and give the K-groups of their transformation group C^*-algebras based on the tori of dimension up to 12 in Table 1 using the computer codes we have developed for this particular purpose. In Section 4, we establish a Poincaré type of duality for the cokernels of integer matrices that leads to some interesting facts about the K-groups when the dimension of the underlying torus is odd. In Section 5, we focus on the rank a_n of the K-groups of Anzai transformations group C^*-algebras based on the n-torus, and we show that a_n is, in fact, the rank of the K-groups of a large class of transformation group C^*-algebras including those associated with Furstenberg transformations on the n-torus. In Section 6, we first uncover an interesting connection between studying a_n and the irreducible representations of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. This leads to a formula for a_n in terms of certain partitions of integers. Then we use this formula to show several interesting combinatorial properties of the sequence $\{a_n\}$. In Section 7, we study the order structure of

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the K_0-group of a class of simple Furstenberg transformation group C^*-algebras to make their Elliott invariants more accessible. Section 8 contains some applications of the results of this paper to our earlier work [33].

2 K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$

In this section we describe a general method to compute the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$, where α is an arbitrary homeomorphism of $\mathbb{T}^n$. (By abuse of notation, the automorphism α of $\mathcal{C}(\mathbb{T}^n)$ is defined by $\alpha(f) = f \circ \alpha^{-1}$ for $f \in \mathcal{C}(\mathbb{T}^n)$.) To do this, we will pay special attention to the algebraic structure of $K^*(\mathbb{T}^n)$ and how the induced automorphisms on it can be realized. Note that it is sufficient to consider the special case of "linear" homeomorphisms, since, as stated before Definition 1.3, every continuous function $f\colon \mathbb{T}^n \to \mathbb{T}$ is homotopic to a unique "linear" function $(\zeta_1, \dots, \zeta_n) \mapsto \zeta_1^{b_1} \cdots \zeta_n^{b_n}$ for some integer exponents $b_1, \dots, b_n$. Moreover, the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ depend (up to isomorphism) only on the homotopy class of α [3, Corollary 10.5.2].

It is well known that $K^*(\mathbb{T}^n)$ is a $\mathbb{Z}_2$-graded ring, and by the Künneth formula (see [2, Corollary 2.7.15] or [36, Theorem 4.1]), it is an exterior algebra (over $\mathbb{Z}$) on n generators, where the elements of even degree are in $K^0(\mathbb{T}^n)$, and those of odd degree are in $K^1(\mathbb{T}^n)$. The generators of this exterior algebra correspond to the generators of the dual group $\mathbb{Z}^n$ of $\mathbb{T}^n$ [37, p. 185]. Indeed, in this case the Chern character

$$\mathrm{ch}\colon K^*(\mathbb{T}^n) \longrightarrow \check{H}^*(\mathbb{T}^n, \mathbb{Q})$$

is integral and gives the Chern isomorphisms

$$\mathrm{ch}_0\colon K^0(\mathbb{T}^n) \longrightarrow \check{H}^{\mathrm{even}}(\mathbb{T}^n, \mathbb{Z}), \quad \mathrm{ch}_1\colon K^1(\mathbb{T}^n) \longrightarrow \check{H}^{\mathrm{odd}}(\mathbb{T}^n, \mathbb{Z}),$$

where $\check{H}^*(\mathbb{T}^n, \mathbb{Z}) \cong \Lambda^*_{\mathbb{Z}}(e_1, \dots, e_n)$ is the (Čech) cohomology ring of $\mathbb{T}^n$ under the cup product, and $\check{H}^k(\mathbb{T}^n, \mathbb{Z}) \cong \Lambda^k_{\mathbb{Z}}(e_1, \dots, e_n)$. On the other hand, $K^*(\mathbb{T}^n) \cong K_*(\mathcal{C}(\mathbb{T}^n))$. So by introducing $e_i := [z_i]_1$, *i.e.*, the class in $K_1(\mathcal{C}(\mathbb{T}^n))$ of the coordinate function $z_i\colon \mathbb{T}^n \to \mathbb{T}$ given by

$$z_i(\zeta_1, \dots, \zeta_n) = \zeta_i$$

as a unitary element of $\mathcal{C}(\mathbb{T}^n)$ for $i = 1, \dots, n$, we have the isomorphisms $K_*(\mathcal{C}(\mathbb{T}^n)) \cong \Lambda^*_{\mathbb{Z}}(e_1, \dots, e_n) \cong \Lambda^*\mathbb{Z}^n$, which respect the canonical embedding of $\mathbb{Z}^n$. Moreover, these isomorphisms are unique since only the identity automorphism of the ring $\Lambda^*\mathbb{Z}^n$ fixes each element of $\mathbb{Z}^n$.

Now we use the Pimsner–Voiculescu six term exact sequence [31] as the main tool for computing the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$. Let $\alpha_*(= K_*(\alpha))$ be the ring automorphism of $K_*(\mathcal{C}(\mathbb{T}^n))$ induced by α and let α_i be the restriction of α_* on $K_i(\mathcal{C}(\mathbb{T}^n))$ for $i = 0, 1$, and set $A := \mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$. Then we have the following exact sequence:

$$\begin{array}{ccccc} K_0(\mathcal{C}(\mathbb{T}^n)) & \xrightarrow{\alpha_0 - \mathrm{id}} & K_0(\mathcal{C}(\mathbb{T}^n)) & \xrightarrow{j_0} & K_0(A) \\ {\scriptstyle \exp}\uparrow & & & & \downarrow{\scriptstyle \partial} \\ K_1(A) & \xleftarrow{j_1} & K_1(\mathcal{C}(\mathbb{T}^n)) & \xleftarrow{\alpha_1 - \mathrm{id}} & K_1(\mathcal{C}(\mathbb{T}^n)). \end{array} \tag{2.1}$$

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Here, $j\colon \mathcal{C}(\mathbb{T}^n) \to A$ is the canonical embedding of $\mathcal{C}(\mathbb{T}^n)$ in A, $j_0 := K_0(j)$ and $j_1 := K_1(j)$. Also, from now on id denotes the identity function on each underlying set. As a result, we have the following short exact sequences

$$0 \longrightarrow \operatorname{coker}(\alpha_0 - \mathrm{id}) \longrightarrow K_0(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \longrightarrow \ker(\alpha_1 - \mathrm{id}) \longrightarrow 0, \tag{2.2}$$

$$0 \longrightarrow \operatorname{coker}(\alpha_1 - \mathrm{id}) \longrightarrow K_1(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \longrightarrow \ker(\alpha_0 - \mathrm{id}) \longrightarrow 0. \tag{2.3}$$

Since all the groups involved are abelian and finitely generated, and $\ker(\alpha_i - \mathrm{id})$ is torsion-free ($i = 0, 1$), these short exact sequences split (since projective $\mathbb{Z}$-modules are precisely the free abelian groups), and we have

$$K_0(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \operatorname{coker}(\alpha_0 - \mathrm{id}) \oplus \ker(\alpha_1 - \mathrm{id}), \tag{2.4}$$

$$K_1(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \operatorname{coker}(\alpha_1 - \mathrm{id}) \oplus \ker(\alpha_0 - \mathrm{id}). \tag{2.5}$$

So it suffices to determine the kernel and cokernel of $(\alpha_0 - \mathrm{id})$ and $(\alpha_1 - \mathrm{id})$ acting as endomorphisms on the finitely generated abelian groups $\Lambda_{\mathbb{Z}}^{\mathrm{even}}(e_1, \ldots, e_n) \cong \mathbb{Z}^{2^{n-1}}$ and $\Lambda_{\mathbb{Z}}^{\mathrm{odd}}(e_1, \ldots, e_n) \cong \mathbb{Z}^{2^{n-1}}$, respectively. Note that from the isomorphisms (2.4) and (2.5), the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ are finitely generated abelian groups. Now, since α_* is a ring homomorphism, it suffices to know the action of α_* on $e_1, \ldots, e_n$. In fact, for a general basis element $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}$ of $K_*(\mathcal{C}(\mathbb{T}^n)) \cong \Lambda_{\mathbb{Z}}^*(e_1, \ldots, e_n)$ we have

$$\alpha_*(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r}) = \alpha_*(e_{i_1}) \wedge \alpha_*(e_{i_2}) \wedge \cdots \wedge \alpha_*(e_{i_r}).$$

Thus, if we consider $\{e_1, \ldots, e_n\}$ as the canonical basis of $\mathbb{Z}^n$ and take $\widehat{\alpha} = \alpha_*|_{\mathbb{Z}^n}$, we have $\alpha_* = \wedge^*\widehat{\alpha} = \bigoplus_{r=1}^n \wedge^r\widehat{\alpha}$, $\alpha_0 = \wedge^{\mathrm{even}}\widehat{\alpha} = \bigoplus_{r\geq 0} \wedge^{2r}\widehat{\alpha}$ and $\alpha_1 = \wedge^{\mathrm{odd}}\widehat{\alpha} = \bigoplus_{r\geq 0} \wedge^{2r+1}\widehat{\alpha}$, where $\wedge^i\widehat{\alpha}$ is the i-th exterior power of $\widehat{\alpha}$, which acts on $\Lambda^i\mathbb{Z}^n$ for $i = 0, 1, \ldots, n$. Now, let $\alpha^{-1} = (f_1, \ldots, f_n)$ and $a_{ji} := A_j[f_i]$, or in other words, assume that f_i is homotopic to $z_1^{a_{1i}} \ldots z_n^{a_{ni}}\colon (\zeta_1, \ldots, \zeta_n) \mapsto \zeta_1^{a_{1i}} \ldots \zeta_n^{a_{ni}}$ for $i = 1, \ldots, n$. So we can write

$$\begin{aligned}\alpha_*(e_i) = \alpha_*[z_i]_1 = [\alpha(z_i)]_1 = [z_i \circ \alpha^{-1}]_1 = [f_i]_1 = [z_1^{a_{1i}} \ldots z_n^{a_{ni}}]_1 \\ = \sum_{j=1}^n a_{ji}[z_j]_1 = \sum_{j=1}^n a_{ji}e_j.\end{aligned}$$

Therefore $\widehat{\alpha}$ acts on $\mathbb{Z}^n$ via the corresponding integer matrix

$$\mathrm{A} := [a_{ij}]_{n\times n} \in \mathrm{GL}(n, \mathbb{Z}),$$

α_* acts on $\Lambda^*\mathbb{Z}^n$ via $\wedge^*\mathrm{A}$, and we have the isomorphisms

$$\begin{aligned}K_0(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) &\cong \operatorname{coker}(\alpha_0 - \mathrm{id}) \oplus \ker(\alpha_1 - \mathrm{id}) \\ &= \operatorname{coker}(\oplus_{r\geq 0} \wedge^{2r} \widehat{\alpha} - \mathrm{id}) \oplus \ker(\oplus_{r\geq 0} \wedge^{2r+1} \widehat{\alpha} - \mathrm{id}),\end{aligned}$$

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so we can write

$$K_0(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \bigoplus_{r\geq 0} \left[\operatorname{coker}(\wedge^{2r}\widehat{\alpha} - \mathrm{id}) \oplus \ker(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id})\right],$$

and similarly

$$K_1(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \bigoplus_{r\geq 0} \left[\operatorname{coker}(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id}) \oplus \ker(\wedge^{2r}\widehat{\alpha} - \mathrm{id})\right].$$

We summarize the arguments discussed above in the following theorem.

Theorem 2.1 *Let α be a homeomorphism of $\mathbb{T}^n$ and $\widehat{\alpha} \in \mathrm{Aut}(\mathbb{Z}^n)$ be the restriction of α_* to $\mathbb{Z}^n$ (as above). Then $\alpha_* = \wedge^*\widehat{\alpha} = \bigoplus_{r=1}^n \wedge^r\widehat{\alpha}$ on $K^*(\mathbb{T}^n) = \Lambda^*\mathbb{Z}^n$ and*

$$K_0(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \bigoplus_{r\geq 0} \left[\operatorname{coker}(\wedge^{2r}\widehat{\alpha} - \mathrm{id}) \oplus \ker(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id})\right],$$

$$K_1(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong \bigoplus_{r\geq 0} \left[\operatorname{coker}(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id}) \oplus \ker(\wedge^{2r}\widehat{\alpha} - \mathrm{id})\right].$$

Therefore in order to compute the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$, we must find the kernel and cokernel of $\wedge^r\widehat{\alpha} - \mathrm{id}$ as an endomorphism of $\Lambda^r\mathbb{Z}^n$ for $r = 0, 1, \ldots, n$. Note that the matrix of $\wedge^r\widehat{\alpha} - \mathrm{id}$ with respect to the canonical basis $\{e_{i_1} \wedge \cdots \wedge e_{i_r} | 1 \leq i_1 < \cdots < i_r \leq n\}$ with lexicographic order is $\mathrm{A}_{n,r} := \wedge^r\mathrm{A} - \mathrm{I}_{\binom{n}{r}}$, which is an integer matrix of order $\binom{n}{r}$ (I_k is the identity matrix of order k - we often omit k whenever it is clear). So by computing the kernel and cokernel of $\mathrm{A}_{n,r}$ for $r = 0, 1, \ldots, n$ with appropriate tools (such as the Smith normal form), one can determine the K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$. The author has written some codes in Maple to handle such computations.

Corollary 2.2 *The K-groups of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ are finitely generated abelian groups with the same rank. Moreover, this common rank equals*

$$\operatorname{rank}\ker(\wedge^*\widehat{\alpha} - \mathrm{id}) = \sum_{r=0}^{n} \operatorname{rank}\ker(\wedge^r\widehat{\alpha} - \mathrm{id}).$$

Proof Use the previous theorem and note that for any $\widehat{\varphi} \in \mathrm{End}(\mathbb{Z}^n)$ one has $\operatorname{rank}\ker\widehat{\varphi} = \operatorname{rank}\operatorname{coker}\widehat{\varphi}$ by the Smith normal form theorem (see, for example, [27, p. 26]). ∎

Corollary 2.3 *If α, β are homeomorphisms of $\mathbb{T}^n$, whose corresponding integer matrices $\mathrm{A}, \mathrm{B} \in \mathrm{GL}(n, \mathbb{Z})$ are similar over $\mathbb{Z}$, then*

$$K_j(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong K_j(\mathcal{C}(\mathbb{T}^n) \rtimes_\beta \mathbb{Z}), \quad (j = 1, 2).$$

Proof The assumption obviously implies that the automorphisms $\widehat{\alpha}$ and $\widehat{\beta}$ are conjugate in $\mathrm{Aut}(\mathbb{Z}^n)$. This together with an easy application of the identity $\wedge^r(\widehat{\phi} \circ \widehat{\psi}) = (\wedge^r\widehat{\phi}) \circ (\wedge^r\widehat{\psi})$ (see, for example, [11, (5.20)]) imply that $\wedge^r\widehat{\alpha}$ and $\wedge^r\widehat{\beta}$ (and therefore $\wedge^r\widehat{\alpha} - \mathrm{id}$ and $\wedge^r\widehat{\beta} - \mathrm{id}$) are conjugate in $\mathrm{Aut}(\Lambda^r\mathbb{Z}^n)$ for $r = 0, 1, \ldots, n$. The result follows now from Theorem 2.1. ∎

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3 Anzai Transformation Group C^*-algebras $\mathscr{A}_{n,\theta}$

The simplest case of a Furstenberg transformation on an n-torus is an Anzai transformation σ, which was defined in Definition 1.3(iv). To study the K-groups of Anzai transformation group C^*-algebras $\mathscr{A}_{n,\theta} = \mathcal{C}(\mathbb{T}^n) \rtimes_\sigma \mathbb{Z}$ using methods of the previous section, we will first need the "linearized" form of the corresponding affine homeomorphism σ^{-1}:

$$(\zeta_1, \zeta_2, \ldots, \zeta_n) \longmapsto (\zeta_1, \zeta_1\zeta_2, \ldots, \zeta_{n-1}\zeta_n).$$

So $\widehat{\sigma}(e_i) = e_{i-1} + e_i$ for $i = 1, \ldots, n$ ($e_0 := 0$). The matrix with respect to the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$ that corresponds to $\widehat{\sigma}$ is the full Jordan block

$$\mathrm{S}_n := \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & & 0 & 1 & 1 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{n\times n}.$$

The following examples illustrate the methods described in the previous section for computing the K-groups of Anzai transformation group C^*-algebras $\mathscr{A}_{n,\theta}$.

Example 3.1 We compute the K-groups of $\mathscr{A}_{3,\theta}$, which were computed in [39] by another method (the C^*-algebra was denoted by $A_\theta^{5,5}$ there). In fact, the Chern character and noncommutative geometry were used in [39] to compute the kernel and cokernel of $\sigma_i - \mathrm{id}$ for $i = 0, 1$. However, we compute the kernel and cokernel of $\mathrm{S}_{3,r} := \wedge^r \mathrm{S}_3 - \mathrm{I}_{\binom{3}{r}}$ for $r = 0, 1, 2, 3$, where

$$\mathrm{S}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$r = 0$: $\mathrm{S}_{3,0} = \wedge^0 \mathrm{S}_3 - \mathrm{I}_1 = [0]$. So $\ker \mathrm{S}_{3,0} = \mathbb{Z}$ and $\operatorname{coker} \mathrm{S}_{3,0} = \mathbb{Z}/\langle 0\rangle \cong \mathbb{Z}$.

$r = 1$:

$$\mathrm{S}_{3,1} = \wedge^1 \mathrm{S}_3 - \mathrm{I}_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So

$$\ker \mathrm{S}_{3,1} = \{(x, y, z) \in \mathbb{Z}^3 \mid y = z = 0\} = (\mathbb{Z}, 0, 0) \cong \mathbb{Z},$$
$$\operatorname{coker} \mathrm{S}_{3,1} = \mathbb{Z}^3/\mathrm{S}_{3,1}\mathbb{Z}^3 = \mathbb{Z}^3/\langle e_1, e_2\rangle \cong \mathbb{Z}.$$

$r = 2$:

$$\mathrm{S}_{3,2} = \wedge^2 \mathrm{S}_3 - \mathrm{I}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

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So

$$\ker \mathrm{S}_{3,2} = \{(x, y, z) \in \mathbb{Z}^3 \mid y + z = z = 0\} = (\mathbb{Z}, 0, 0) \cong \mathbb{Z},$$

$$\operatorname{coker}\ \mathrm{S}_{3,2} = \mathbb{Z}^3/\mathrm{S}_{3,2}\mathbb{Z}^3 = \mathbb{Z}^3/\langle e_1, e_2\rangle \cong \mathbb{Z}.$$

$r = 3$: $\mathrm{S}_{3,3} = \wedge^3\mathrm{S}_3 - \mathrm{I}_1 = [0]$. So $\ker \mathrm{S}_{3,3} = \mathbb{Z}$ and $\operatorname{coker}\ \mathrm{S}_{3,3} = \mathbb{Z}/\langle 0\rangle \cong \mathbb{Z}$.

Now, using Theorem 2.1 we have

$$K_0(A_\theta^{5,5}) = K_0(\mathscr{A}_{3,\theta}) \cong (\operatorname{coker} \mathrm{S}_{3,0} \oplus \operatorname{coker} \mathrm{S}_{3,2}) \oplus (\ker \mathrm{S}_{3,1} \oplus \ker \mathrm{S}_{3,3})$$
$$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^4,$$

$$K_1(A_\theta^{5,5}) = K_1(\mathscr{A}_{3,\theta}) \cong (\operatorname{coker}\ \mathrm{S}_{3,1} \oplus \operatorname{coker} \mathrm{S}_{3,3}) \oplus (\ker \mathrm{S}_{3,0} \oplus \ker \mathrm{S}_{3,2})$$
$$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}^4.$$

Notation 3.2 We let

$$a_n := \operatorname{rank} K_0(\mathscr{A}_{n,\theta}) = \operatorname{rank} K_1(\mathscr{A}_{n,\theta}),$$

$$a_{n,r} := \operatorname{rank} \ker(\wedge^r \mathrm{S}_n - \mathrm{I})$$

for $r = 0, 1, \ldots, n$. From Corollary 2.2 we have

$$a_n = \operatorname{rank} \ker(\wedge^* \mathrm{S}_n - \mathrm{I}) = \sum_{r=0}^{n} a_{n,r}.$$

Example 3.3 Using the methods described in Section 2, we have obtained the K-groups of $\mathscr{A}_{n,\theta}$ by computer for $1 \le n \le 12$. The cases $n = 1, 2, 3$ have been calculated in the literature already: $\mathscr{A}_{1,\theta} = A_\theta$ in [34], $\mathscr{A}_{2,\theta} = A_\theta^4$ in [25], and $\mathscr{A}_{3,\theta} = A_\theta^{5,5}$ in [39]. However, there are no explicit computations for the higher dimensional cases starting with $\mathscr{A}_{4,\theta} = A_\theta^{6,10}$ as in [24], since hand calculations of kernels and cokernels of the maps quickly become impossible. Using the Maple codes we have developed we can find the kernel and cokernel of

$$\mathrm{S}_{n,r} := \wedge^r \mathrm{S}_n - \mathrm{I}_{\binom{n}{r}}$$

for $r = 0, 1, \ldots, n$ and $n = 1, \ldots, 12$ by means of the Smith normal form theorem (see, for example, [27, p. 26]), and therefore we can compute the K-groups. The results are illustrated in Table 1, where $\mathbb{Z}_k^{(m)}$ denotes the direct sum of m copies of the cyclic group $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

Due to computational limitations, we do not currently have any results for $n > 12$, except for the sequence of ranks $\{a_n\}$, which we will study in detail in Sections 5 and 6. We will show the importance of this sequence in Section 5. Briefly, a_n is the common rank of the K-groups of a certain family of C^*-algebras including Furstenberg transformation group C^*-algebras $\mathscr{F}_{\theta,f}$ based on $\mathbb{T}^n$. Also, we will prove that $\{a_n\}$ is a strictly increasing sequence (see Proposition 6.5). On the other hand, it seems that the K-groups of $\mathscr{A}_{n,\theta}$ have torsion in general. The first example is $K_1(\mathscr{A}_{6,\theta})$; this is due to the fact that $\operatorname{coker}\ \mathrm{S}_{6,3} = \operatorname{coker}(\wedge^3\mathrm{S}_6 - \mathrm{I}_{20}) \cong \mathbb{Z}^3 \oplus \mathbb{Z}_2$ (see Remark 1.7). Also, it is seen that the K_0- and K_1-groups are isomorphic for odd values of n in Table 1. In fact, this is true for more general cases (see Theorem 4.3).

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Table 1: The K-groups of $\mathscr{A}_{n,\theta}$ for $1 \leq n \leq 12$

n	$K_0(\mathscr{A}_{n,\theta})$	$K_1(\mathscr{A}_{n,\theta})$	a_n
1	$\mathbb{Z}^2$	$\mathbb{Z}^2$	2
2	$\mathbb{Z}^3$	$\mathbb{Z}^3$	3
3	$\mathbb{Z}^4$	$\mathbb{Z}^4$	4
4	$\mathbb{Z}^6$	$\mathbb{Z}^6$	6
5	$\mathbb{Z}^8$	$\mathbb{Z}^8$	8
6	$\mathbb{Z}^{13}$	$\mathbb{Z}^{13} \oplus \mathbb{Z}_2$	13
7	$\mathbb{Z}^{20}$	$\mathbb{Z}^{20}$	20
8	$\mathbb{Z}^{32} \oplus \mathbb{Z}_8^{(2)}$	$\mathbb{Z}^{32} \oplus \mathbb{Z}_{18}^{(2)}$	32
9	$\mathbb{Z}^{52} \oplus \mathbb{Z}_3^{(2)} \oplus \mathbb{Z}_9^{(2)}$	$\mathbb{Z}^{52} \oplus \mathbb{Z}_3^{(2)} \oplus \mathbb{Z}_9^{(2)}$	52
10	$\mathbb{Z}^{90} \oplus \mathbb{Z}_{55}^{(4)}$	$\mathbb{Z}^{90} \oplus \mathbb{Z}_{11}^{(2)} \oplus \mathbb{Z}_{99} \oplus \mathbb{Z}_{198} \oplus \mathbb{Z}_{2574}$	90
11	$\mathbb{Z}^{152} \oplus \mathbb{Z}_{11}^{(12)} \oplus \mathbb{Z}_{143}^{(4)} \oplus \mathbb{Z}_{286}^{(2)}$	$\mathbb{Z}^{152} \oplus \mathbb{Z}_{11}^{(12)} \oplus \mathbb{Z}_{143}^{(4)} \oplus \mathbb{Z}_{286}^{(2)}$	152
12	$\mathbb{Z}^{268} \oplus \mathbb{Z}_{13}^{(14)} \oplus \mathbb{Z}_{26}^{(4)} \oplus \mathbb{Z}_{1716}^{(4)} \oplus \mathbb{Z}_{3432}^{(2)} \oplus \mathbb{Z}_{58344}^{(2)}$	$\mathbb{Z}^{268} \oplus \mathbb{Z}_{13}^{(4)} \oplus \mathbb{Z}_{26}^{(4)} \oplus \mathbb{Z}_{286}^{(6)} \oplus \mathbb{Z}_{4862}^{(2)} \oplus \mathbb{Z}_{68068}^{(2)}$	268

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4 A Poincaré Type of Duality

As stated in Theorem 2.1, the K-groups of a transformation group C^*-algebras of the form $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ are completely determined by the corresponding homomorphism $\widehat{\alpha} \in \mathrm{Aut}(\mathbb{Z}^n)$ and its exterior powers. From a computational point of view, we only need the cokernels of the maps involved, since we know that for any endomorphism ϱ on $\mathbb{Z}^m$, $\ker \varrho \cong \mathrm{coker}\, \varrho \,/\, \mathrm{tor}(\mathrm{coker}\, \varrho)$, where $\mathrm{tor}(G)$ denotes the torsion subgroup of the finitely generated abelian group G. When $\det \widehat{\alpha} = 1$, we do not even need to compute all the cokernels. This is due to the following proposition, which establishes a Poincaré type of duality between cokernels of certain integer matrices.

Two endomorphisms $\widehat{\alpha}, \widehat{\beta} \in \mathrm{End}(\mathbb{Z}^m)$ are said to be equivalent over $\mathbb{Z}$ (and write $\widehat{\alpha}$ equiv $\widehat{\beta}$) if there exist $\widehat{u}, \widehat{v} \in \mathrm{Aut}(\mathbb{Z}^m)$ such that $\widehat{u} \circ \widehat{\alpha} \circ \widehat{v} = \widehat{\beta}$. Similarly, if A and B are integer $m \times m$ matrices, A is equivalent to B if there exist $\mathrm{U}, \mathrm{V} \in \mathrm{GL}(m, \mathbb{Z})$ such that $\mathrm{UAV} = \mathrm{B}$. We recall that $\widehat{\alpha}$ equiv $\widehat{\beta}$ if and only if $\mathrm{coker}\, \widehat{\alpha} \cong \mathrm{coker}\, \widehat{\beta}$ if and only if $\widehat{\alpha}$ and $\widehat{\beta}$ have the same Smith normal form. Also, A equiv B if and only if B is obtainable from A by a finite number of elementary operations. An elementary operation on an integer matrix is one of the following types: interchanging two rows (or two columns), adding an integer multiple of one row (or column) to another, and multiplying a row (or column) by -1 [27].

Proposition 4.1 (Poincaré duality) *Let* $\widehat{\alpha} \in \mathrm{SL}(n, \mathbb{Z})$ (i.e., $\det \widehat{\alpha} = 1$). *Then* $\wedge^r \widehat{\alpha} - \mathrm{id}$ *and* $\wedge^{n-r} \widehat{\alpha} - \mathrm{id}$ *are equivalent as endomorphisms of* $\Lambda^r \mathbb{Z}^n = \Lambda^{n-r} \mathbb{Z}^n = \mathbb{Z}^{\binom{n}{r}}$. *Equivalently,* $\mathrm{coker}(\wedge^r \widehat{\alpha} - \mathrm{id})$ *is isomorphic to* $\mathrm{coker}(\wedge^{n-r} \widehat{\alpha} - \mathrm{id})$ *for* $r = 0, 1, \ldots, n$.

Proof We prove the equivalence of the endomorphisms for their corresponding integer matrices with respect to a certain basis. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be a basis for $\mathbb{Z}^n$ and set $S = \{1, 2, \ldots, n\}$. For $I = \{i_1, \ldots, i_r\} \subset S$ with $1 \leq i_1 < \cdots < i_r \leq n$, put $e_I = e_{i_1} \wedge \cdots \wedge e_{i_r} \in \Lambda^r \mathbb{Z}^n$. Then $\mathcal{E}_r := \{e_I \mid I \subset S\,,\ |I| = r\}$ is a basis for $\Lambda^r \mathbb{Z}^n$. Let $\omega := e_1 \wedge \cdots \wedge e_n$, which generates $\Lambda^n \mathbb{Z}^n$. We have $\wedge^0 \widehat{\alpha} - \mathrm{id} = 0$, and $\wedge^n \widehat{\alpha}(\omega) = \widehat{\alpha}(e_1) \wedge \cdots \wedge \widehat{\alpha}(e_n) = (\det \widehat{\alpha})(e_1 \wedge \cdots \wedge e_n) = \omega$, so $\wedge^n \widehat{\alpha} - \mathrm{id} = 0$. Now, fix an $r \in \{1, \ldots, n-1\}$. For an arbitrary subset $I \subset S$ with $|I| = r$, take $J = \mathcal{E} \setminus I = \{j_1, \ldots, j_{n-r}\}$, so $|J| = n - r$. Then $e_I \wedge e_J = (\mathrm{sgn}\, \mu)\, \omega$, in which $\mu \in S_n$ is the permutation that converts $(1, 2, \ldots, n)$ to $(i_1, \ldots, i_r, j_1, \ldots, j_{n-r})$. It is easily seen that $\mu = \mu_1 \ldots \mu_r$, where μ_k is the permutation that takes i_k from its position in $(1, 2, \ldots, n)$ to its new position in $(i_1, \ldots, i_r, j_1, \ldots, j_{n-r})$. One can see that μ_k is the combination of $i_k - (r - k + 1)$ number of transpositions $(k = 1, \ldots, r)$. Thus

$$\mathrm{sgn}\, \mu = \prod_{k=1}^{r} (-1)^{i_k - (r-k+1)} = (-1)^{\ell(I) - \frac{r(r+1)}{2}},$$

where $\ell(I) := \sum_{k=1}^{r} i_k$. Now take $m = \binom{n}{r} = \binom{n}{n-r}$ and let $\mathcal{E}_r = \{e_{I_1}, \ldots, e_{I_m}\}$ be a basis for $\Lambda^r \mathbb{Z}^n$. Write $\mathcal{E}_{n-r} = \{e_{J_1}, \ldots, e_{J_m}\}$ as the basis for $\Lambda^{n-r} \mathbb{Z}^n$ such that $J_k = \mathcal{E} \setminus I_k$ for $k = 1, \ldots, m$. From the above argument one can write

$$e_{I_i} \wedge e_{J_j} = (-1)^{\ell(I_i) - \frac{r(r+1)}{2}} \delta_{ij} \omega,$$

since if $i \neq j$, then $I_i \cap J_j \neq \varnothing$ and $e_{I_i} \wedge e_{J_j} = 0$. Let $\mathrm{A} = [a_{ij}]_{m \times m}$ and $\mathrm{B} = [b_{ij}]_{m \times m}$ be the corresponding integer matrices of $\wedge^r \widehat{\alpha}$ and $\wedge^{n-r} \widehat{\alpha}$ with respect to $\mathcal{E}_r$ and $\mathcal{E}_{n-r}$,

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respectively. So $\wedge^r\widehat{\alpha}(e_{I_i}) = \sum_{p=1}^m a_{pi}e_{I_p}$ and $\wedge^{n-r}\widehat{\alpha}(e_{J_j}) = \sum_{q=1}^m b_{qj}e_{J_q}$. What we want to show is that $\mathrm{A} - \mathrm{I}$ is equivalent to $\mathrm{B} - \mathrm{I}$. We have

$$\begin{aligned}\wedge^n\widehat{\alpha}(e_{I_i} \wedge e_{J_j}) &= (-1)^{\ell(I_i)-\frac{r(r+1)}{2}}\delta_{ij}\omega = \wedge^r\widehat{\alpha}(e_{I_i}) \bigwedge \wedge^{n-r}\widehat{\alpha}(e_{J_j}) \\ &= \sum_{p,q=1}^m a_{pi}b_{qj}(-1)^{\ell(I_p)-\frac{r(r+1)}{2}}\delta_{pq}\omega.\end{aligned}$$

Therefore one obtains

$$\sum_{k=1}^m (-1)^{\ell(I_k)-\ell(I_i)}a_{ki}b_{kj} = \delta_{ij}. \tag{4.1}$$

Now, if we set

$$c_{ij} := (-1)^{\ell(I_j)-\ell(I_i)}a_{ji} \quad \text{and} \quad \mathrm{C} := [c_{ij}]_{m\times m},$$

then $c_{ij} - \delta_{ij} = (-1)^{\ell(I_j)-\ell(I_i)}(a_{ji} - \delta_{ji})$. Therefore $\mathrm{C} - \mathrm{I}$ is obtained from $\mathrm{A} - \mathrm{I}$ by exchanging rows (and columns) and occasionally multiplying some rows (and columns) by -1. This means that $\mathrm{C} - \mathrm{I}$ is equivalent to $\mathrm{A} - \mathrm{I}$. On the other hand, equation (4.1) says that $\mathrm{CB} = \mathrm{I}$. So $\mathrm{C} - \mathrm{I} = \mathrm{C}(\mathrm{B} - \mathrm{I})(-\mathrm{I})$, and $\mathrm{B} - \mathrm{I}$ is also equivalent to $\mathrm{C} - \mathrm{I}$. Consequently, $\mathrm{A} - \mathrm{I}$ is equivalent to $\mathrm{B} - \mathrm{I}$. ∎

Corollary 4.2 *If* $\det\widehat{\alpha} = 1$, *then* $\operatorname{rank}\ker(\wedge^r\widehat{\alpha} - \mathrm{id}) = \operatorname{rank}\ker(\wedge^{n-r}\widehat{\alpha} - \mathrm{id})$. *In particular, using Notation 3.2, we have* $a_{n,r} = a_{n,n-r}$ *for* $r = 0, 1, \ldots, n$.

We are now ready to apply our Poincaré duality to a the following K-theoretic result.

Theorem 4.3 *Let* $A := \mathcal{C}(\mathbb{T}^{2m-1}) \rtimes_\alpha \mathbb{Z}$ *be such that the corresponding homomorphism* $\widehat{\alpha}$ *satisfies* $\det\widehat{\alpha} = 1$. *Then* $K_0(A) \cong K_1(A)$ *as abelian groups, and the (common) rank of the K-groups of A is an even number. In particular, for every Furstenberg transformation group C*-algebra* $\mathscr{F}_{\theta,f}$ *based on an odd-dimensional torus* (e.g., $\mathscr{A}_{2m-1,\theta}$), *one has* $K_0(\mathscr{F}_{\theta,f}) \cong K_1(\mathscr{F}_{\theta,f})$.

Proof Combining Theorem 2.1 and Proposition 4.1, one obtains

$$K_0(A) \cong K_1(A) \cong \bigoplus_{k=0}^{m-1}\left[\operatorname{coker}(\wedge^k\widehat{\alpha} - \mathrm{id}) \oplus \ker(\wedge^k\widehat{\alpha} - \mathrm{id})\right].$$

As a result, the rank of the K-groups of A is an even number, since the ranks of the cokernel and kernel of an endomorphism coincide. Note that for $\mathscr{F}_{\theta,f}$ the corresponding integer matrix of $\widehat{\alpha}$ is an upper triangular matrix with 1's on the diagonal. Thus $\det\widehat{\alpha} = 1$. ∎

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5 The Rank a_n of the K-groups of $\mathscr{A}_{n,\theta}$

In this section, we study some general properties of a_n, the (common) rank of the K-groups of Anzai transformation group C^*-algebras based on $\mathbb{T}^n$. We specify a family of C^*-algebras whose ranks of K-groups are given by the same sequence $\{a_n\}$. As an application, we characterize the rank of the K-groups of Furstenberg transformation group C^*-algebras $\mathscr{F}_{\theta,f}$. This will, in particular, provide some applications in Section 8 to the classification of simple infinite dimensional quotients of the Heisenberg-type group C^*-algebras $C^*(\mathfrak{D}_n)$, which were studied in an earlier work [33].

We compare the ranks of the K-groups of a class of C^*-algebras of the form $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ in the following theorem, which shows that the rank a_n of the K-groups of $\mathscr{A}_{n,\theta}$ is somehow generic. We remind the reader that a square matrix A is called *nilpotent* (respectively, *unipotent*) if $A^n = 0$ (respectively, $(A - I)^n = 0$), for some positive integer n, and the least such n is called the *degree of* A, denoted by deg(A).

Theorem 5.1 *Let $A = \mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$, in which α is a homeomorphism of $\mathbb{T}^n$, whose corresponding integer matrix $A \in \mathrm{GL}(n,\mathbb{Z})$ is unipotent of maximal degree* (i.e., $\deg(A) = n$). *Then*

$$\operatorname{rank} K_0(A) = \operatorname{rank} K_1(A) = a_n = \operatorname{rank} K_0(\mathscr{A}_{n,\theta}) = \operatorname{rank} K_1(\mathscr{A}_{n,\theta}).$$

In particular, the rank of the K-groups of any Furstenberg transformation group C^*-algebra $\mathscr{F}_{\theta,f} = \mathcal{C}(\mathbb{T}^n) \rtimes_{\varphi_{\theta,f}} \mathbb{Z}$ is equal to the rank of the K-groups of $\mathscr{A}_{n,\theta}$, namely, to a_n.

Proof Let $\widehat{\alpha}$ denote the restriction of α_* to $\mathbb{Z}^n$ and A be the corresponding matrix of $\widehat{\alpha}$ acting on $\mathbb{Z}^n$. Also, let S_n be the corresponding matrix for $\mathscr{A}_{n,\theta}$ as denoted in Section 3. Since A is unipotent of maximal degree by assumption, and S_n is unipotent of maximal degree too, the matrices A and S_n are similar over $\mathbb{C}$. In fact, the Jordan normal form of $A - I$ is precisely $S_n - I$. On the other hand, we know by Corollary 2.2 that the rank of the K-groups of A is equal to $\operatorname{rank}\ker(\wedge^* A - I)$. Note that by the Smith normal form theorem, $\operatorname{rank}\ker(\wedge^* A - I) = \dim_{\mathbb{C}} \ker(\wedge^* A - I)$. The similarity of A and S_n implies the similarity of $\wedge^* A - I$ and $\wedge^* S_n - I$ as matrices acting on $\Lambda^*\mathbb{C}^n$. So $\dim_{\mathbb{C}} \ker(\wedge^* A - I) = \dim_{\mathbb{C}} \ker(\wedge^* S_n - I) = a_n$, which yields the result.

For the second part, note that the corresponding integer matrix of a Furstenberg transformation $\varphi_{\theta,f}$ on $\mathbb{T}^n$ is of the form

$$\begin{pmatrix} 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & 1 & b_{23} & & \vdots \\ 0 & \ddots & \ddots & \ddots & b_{n-2,n} \\ \vdots & & 0 & 1 & b_{n-1,n} \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{n\times n}, \tag{5.1}$$

which is unipotent of maximal degree since $b_{i,i+1} \neq 0$ for $i = 1, \ldots, n-1$ (see Definition 1.3). Now, the proof of the first part yields the result. ∎

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Remark 5.2 In the preceding theorem, the basis for $\mathbb{Z}^n$ for the matrices involved is $\{e_1, \ldots, e_n\}$, where $e_i := [z_i]_1$ as introduced at the beginning of Section 2. It is interesting to know that if $\widehat{\alpha}$ is an arbitrary unipotent automorphism of $\mathbb{Z}^n$, then there is a basis for $\mathbb{Z}^n$ with respect to which the integer matrix A of $\widehat{\alpha}$ is of the form (5.1) (but not necessarily with $b_{i,i+1} \neq 0$ for $i = 1, \ldots, n-1$, unless $\widehat{\alpha}$ is of maximal degree) [12, Theorems 16 and 18]. The unipotency of $\widehat{\alpha}$ has also important effects on the dynamics of the generated flow on $\mathbb{T}^n$. For example, if α is an affine transformation on $\mathbb{T}^n$ and $\widehat{\alpha}$ is unipotent, then the dynamical system $(\mathbb{T}^n, \alpha)$ has quasi-discrete spectrum [12, Theorem 19]. More generally, let $\alpha = (t, \mathrm{A})$ be an affine transformation on $\mathbb{T}^n$ and take $Z_p(\mathrm{A}) = \ker(\mathrm{A}^p - \mathrm{id}) \subset \mathbb{Z}^n$ for $p \in \mathbb{N}$ and consider the following conditions:

(a) $Z_1(\mathrm{A}) = Z_p(\mathrm{A}), \forall p \in \mathbb{N}$;
(b) t is rationally independent over $Z_1(\mathrm{A})$, *i.e.*, if $k = (k_1, \ldots, k_n) \in Z_1(\mathrm{A})$ is such that $\langle t, k\rangle := \sum_{j=1}^n t_j k_j$ is a rational number, then $k = 0$;
(c) $Z_1(\mathrm{A}) \neq \{0\}$;
(d) A is unipotent.

Then $(\mathbb{T}^n, \alpha)$ is ergodic with respect to Haar measure if and only if α satisfies conditions (a) and (b) [12]. Moreover, if α satisfies conditions (a) through (d), then the dynamical system $(\mathbb{T}^n, \alpha)$ is minimal, uniquely ergodic with respect to Haar measure, and has quasi-discrete spectrum. Conversely, any minimal transformation on $\mathbb{T}^n$ with topologically quasi-discrete spectrum is conjugate to an affine transformation that must satisfy conditions (a) through (d) [13]. The C^*-algebras corresponding to such actions are therefore simple and have a unique tracial state.

6 Combinatorial Properties of the Sequence $\{a_n\}$

As mentioned before, one of our main goals is to describe a_n, the rank of the K-groups of $\mathscr{A}_{n,\theta}$. Since $a_n = \sum_{r=0}^n a_{n,r}$, it makes sense to first study $a_{n,r}$. So we begin by finding some combinatorial properties of $a_{n,r}$, which is the rank of $\ker(\wedge^r\widehat{\sigma} - \mathrm{id})$ for $r = 0, 1, \ldots, n$, where $\widehat{\sigma}$ is the automorphism of $\mathbb{Z}^n$ corresponding to the Anzai transformation σ on $\mathbb{T}^n$ and is represented by the integer matrix S_n as in the beginning of Section 3. In fact, we will show that $a_{n,r}$ equals the number of partitions of $[r(n+1)/2]$ to r distinct positive integers not greater than n. To do this, we will use properties of the irreducible representations of the simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

6.1 Connections with the Representation Theory of $\mathfrak{sl}(2, \mathbb{C})$

The automorphism $\widehat{\sigma}$ is realized through its action on the basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$, where $e_i := [z_i]$ for $i = 1, \ldots, n$ as in Section 2, and we have $\widehat{\sigma}(e_i) = e_i + e_{i-1}$ with $e_0 := 0$. Therefore, introducing a new endomorphism of $\mathbb{Z}^n$ by $\widehat{\varphi} := \widehat{\sigma} - \mathrm{id}$, we will get $\widehat{\varphi}(e_i) = e_{i-1}$. This is precisely a relation that may be recognized as part of the data of the canonical representation π_n of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ on a complex vector space V with basis $\{e_1, \ldots, e_n\}$. More precisely, if we still let $\widehat{\varphi}$ denote the natural linear extension of $\widehat{\varphi}$ on V and let $\{h := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\}$ be the natural basis for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$, then we observe that $\widehat{\varphi} = \pi_n(f)$, where

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the representation π_n is defined by

(a) $\pi_n(h)e_i = (2i - n - 1)e_i$;
(b) $\pi_n(e)e_i = i(n - i)e_{i+1}$, $(e_{n+1} := 0)$;
(c) $\pi_n(f)e_i = e_{i-1}$, $(e_0 := 0)$.

It is well known that every n-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ is equivalent to π_n (see, for example, [15, Section II.7]); in particular, the image of h under any finite dimensional irreducible representation is a diagonalizable operator.

The following lemma asserts that $a_{n,r}$ can be realized as the nullity of a certain linear mapping that is induced by $\widehat{\varphi}$ on the r-th exterior space $\Lambda^r V$. This is, in fact, a change of view from the level of Lie groups to the level of Lie algebras, which will prove very useful in that we can invoke the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ for computing $a_{n,r}$.

Lemma 6.1 *Let $\widehat{\varphi}\colon V \to V$ be defined as above, and let $\mathcal{D}^*\widehat{\varphi}$ be the derivation induced by $\widehat{\varphi}$ on the exterior algebra $\Lambda^* V$ by the relations*

$$\mathcal{D}^*\widehat{\varphi}(v_1 \wedge \cdots \wedge v_r) = \sum_{i=1}^{r} v_1 \wedge \cdots \wedge \widehat{\varphi}(v_i) \wedge \cdots \wedge v_r,$$

$\mathcal{D}^*\widehat{\varphi}(1) = 0$, and $\mathcal{D}^*\widehat{\varphi}(v) := \widehat{\varphi}(v)$. Then $\wedge^r\widehat{\sigma} - \mathrm{id}$ and $\mathcal{D}^r\widehat{\varphi}$ are similar over $\mathbb{C}$, where $\mathcal{D}^r\widehat{\varphi}$ is the restriction of $\mathcal{D}^*\widehat{\varphi}$ to $\Lambda^r V$. In particular, we have

$$a_{n,r} = \operatorname{rank}\ker(\wedge^r\widehat{\sigma} - \mathrm{id}) = \dim\ker \mathcal{D}^r\widehat{\varphi}.$$

Proof The first general observation to make is that $\mathcal{D}^r$, viewed as a linear mapping, is in fact a representation of the Lie algebra $\mathfrak{gl}(V)$ on $\Lambda^r V$ (*cf.* [11, (5.25)]), and is the derivative at the identity transformation of the the representation $\wedge^r$ of the Lie group $\mathrm{GL}(V)$ on $\Lambda^r V$. Consequently, the exponential map intertwines the two representations in the sense that we should have $\wedge^r \exp(\widehat{\phi}) = \exp(\mathcal{D}^r\widehat{\phi})$ for all $\widehat{\phi} \in \mathfrak{gl}(V)$. This last equality may also be established directly. It suffices to see that the function $s\colon \mathbb{R} \to \mathrm{GL}(\Lambda^r V)$ defined by $s(t) = \wedge^r \exp(t\widehat{\phi})$ is indeed the 1-parameter subgroup generated by $\mathcal{D}^r\widehat{\phi}$ (*i.e.*, s is a group homomorphism and $\dot{s}(0) = \mathcal{D}^r\widehat{\phi}$), hence $\exp(\mathcal{D}^r\widehat{\phi}) = s(1) = \wedge^* \exp(\widehat{\phi})$.

Now, since $\widehat{\varphi}$ is nilpotent, it is easy to see that $\exp(\widehat{\varphi}) - \mathrm{id} = \widehat{\varphi}\omega$, where ω is a unipotent linear mapping that commutes with $\widehat{\varphi}$. In particular, $\exp(\widehat{\varphi}) - \mathrm{id}$ is nilpotent of the same degree as $\widehat{\varphi}$, and therefore they share the same Jordan form over $\mathbb{C}$, which implies $\exp(\widehat{\varphi}) - \mathrm{id} \sim \widehat{\varphi}$ and $\widehat{\sigma} = \widehat{\varphi} + \mathrm{id} \sim \exp(\widehat{\varphi})$. Moreover, a straightforward calculation shows that $\mathcal{D}^r\widehat{\varphi}$ is also nilpotent; for example, $(\widehat{\varphi})^t = 0$ implies $(\mathcal{D}^r\widehat{\varphi})^{rt} = 0$. So for the same reason we have $\exp(\mathcal{D}^r\widehat{\varphi}) - \mathrm{id} \sim \mathcal{D}^r\widehat{\varphi}$. Therefore,

$$\wedge^r\widehat{\sigma} - \mathrm{id} \sim \wedge^r \exp(\widehat{\varphi}) - \mathrm{id} = \exp(\mathcal{D}^r\widehat{\varphi}) - \mathrm{id} \sim \mathcal{D}^r\widehat{\varphi}.$$

In particular, $\operatorname{rank}\ker(\wedge^r\widehat{\sigma} - \mathrm{id}) = \dim\ker \mathcal{D}^r\widehat{\varphi}$. ∎

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Notation 6.2 Let n, k, r be positive integers. Then $P(n, r, k)$ denotes the number of partitions of k to r distinct positive integers not greater than n. In other words,

$$P(n, r, k) = \#\big\{ (i_1, \ldots, i_r) \mid i_1 + \cdots + i_r = k, 1 \le i_1 < \cdots < i_r \le n \big\}.$$

By convention, we set $P(n, 0, 0) = 1$ and $P(n, r, 0) = P(n, 0, k) = 0$ for $r, k \ge 1$.

We are now ready to state the main theorem of this section.

Theorem 6.3 *With the above notation, $a_{n,r} = P(n, r, [r(n+1)/2])$, where $[x]$ denotes the greatest integer not greater than x. In particular,*

$$a_n = \sum_{r=0}^{n} P\Big(n, r, \Big[\frac{r(n+1)}{2}\Big]\Big).$$

Proof Let $\pi_n \colon \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V)$ be the canonical representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ on the n-dimensional complex vector space V, and extend π_n to $\pi_n^r \colon \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(\Lambda^r V)$ with $\pi_n^1 = \pi_n$. More precisely, for every $X \in \mathfrak{sl}(2, \mathbb{C})$ define

$$\pi_n^r(X)(v_1 \wedge \cdots \wedge v_r) = (\pi_n(X)v_1) \wedge v_2 \wedge \cdots \wedge v_r + \cdots + v_1 \wedge \cdots \wedge v_{r-1} \wedge (\pi_n(X)v_r).$$

In particular, $\mathcal{D}^r\widehat{\varphi} = \pi_n^r(f)$, and $a_{n,r}$ is the nullity of $\pi_n^r(f)$ by Lemma 6.1. Following Weyl's complete reducibility theorem, since the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is semisimple the representation π_n^r has to be completely reducible. This means we should have a decomposition $\Lambda^r V = \oplus_{p=1}^N W_p$, where W_p's are some π_n^r-invariant irreducible subspaces of $\Lambda^r V$. Moreover, by a corollary given in [15, Section 7.2], the number N of the summands in such a direct sum decomposition is equal to $\dim E_0 + \dim E_1$, where $E_\mu = \{\mathbf{v} \in \Lambda^r V \mid \pi_n^r(h)\,\mathbf{v} = \mu\,\mathbf{v}\}$. On the other hand, we claim that the number N is equal to the nullity of $\pi_r^n(f)$. In fact, since $\pi_r^n|_{W_p}$ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on W_p, it is equivalent to the canonical representation $\pi_{\dim W_p}$ of $\mathfrak{sl}(2, \mathbb{C})$ on W_p. But the image of f in every canonical representation has a 1-dimensional kernel. So the nullity of $\pi_r^n(f)$ counts the number of W_p's. Therefore,

$$a_{n,r} = \dim\ker \pi_n^r(f) = \dim E_0 + \dim E_1.$$

To compute the last two terms, note that the equality $\pi_n(h)e_i = (2i - n - 1)e_i$ implies

$$\pi_n^r(h)(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \big(2(i_1 + \cdots + i_r) - r(n+1)\big)\, e_{i_1} \wedge \cdots \wedge e_{i_r}.$$

So $\pi_n^r(h)$ is diagonal; for even $r(n+1)$ we have

$$E_1 = \{0\} \quad \text{and} \quad \dim E_0 = P(n, r, r(n+1)/2),$$

and for odd $r(n+1)$ we have

$$E_0 = \{0\} \quad \text{and} \quad \dim E_1 = P(n, r, r(n+1)/2 - 1).$$

To summarize, we have established the following equalities:

$$a_{n,r} = \dim\ker \mathcal{D}^r\widehat{\varphi} = \dim\ker \pi_n^r(f) = N = \dim E_0 + \dim E_1 = P\big(n, r, [r(n+1)/2]\big).$$

The desired formula for a_n is immediate now by writing $a_n = \sum_{r=0}^n a_{n,r}$. ∎

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Using the previous theorem, we can prove that $\{a_n\}$ is a strictly increasing sequence. We need a lemma first.

Lemma 6.4 *$P(n+1, r, k+s) \geq P(n, r, k)$ for $s = 0, 1, \ldots, r$.*

Proof For $s = 0$, the proof is clear. Now, let $1 \leq s \leq r$ and suppose that $(j_1, \ldots, j_r)$ is a partition of k such that $1 \leq j_1 < \cdots < j_r \leq n$. Define $i_q := j_q$ for $1 \leq q \leq r - s$ and $i_q := j_q + 1$ for $r - s + 1 \leq q \leq r$. Then $(i_1, \ldots, i_r)$ is a partition of $k + s$ and $1 \leq i_1 < \cdots < i_r \leq n + 1$. Thus $P(n+1, r, k+s) \geq P(n, r, k)$. ∎

Proposition 6.5 *$\{a_n\}$ is a strictly increasing sequence.*

Proof First, note that $a_{n,0} = a_{n,n} = P(n, 0, 0) = P(n, n, n(n+1)/2) = 1$, and from the previous theorem we have $a_n = \sum_{r=0}^{n} P(n, r, [r(n+1)/2])$. Fix $m \in \mathbb{N}$, and get

$$a_{2m+1} = 1 + \sum_{r=0}^{m} P(2m+1, 2r, 2rm+2r) + \sum_{r=0}^{m-1} P(2m+1, 2r+1, 2rm+2r+m+1),$$

$$\begin{aligned} a_{2m} &= \sum_{r=0}^{m} P(2m, 2r, 2rm+r) + \sum_{r=0}^{m-1} P(2m, 2r+1, 2rm+m+r) \\ &= 1 + \sum_{r=0}^{m-1} P(2m, 2r, 2rm+r) + \sum_{r=0}^{m-1} P(2m, 2r+1, 2rm+m+r), \end{aligned}$$

$$a_{2m-1} = \sum_{r=0}^{m-1} P(2m-1, 2r, 2rm) + \sum_{r=0}^{m-1} P(2m-1, 2r+1, 2rm+m).$$

Applying the previous lemma to the terms of the sums expressed above implies that

$$a_{2m+1} > a_{2m} > a_{2m-1}.$$ ∎

6.2 Generating Functions for the Sequence $\{a_n\}$

In this part, we express the rank of the K-groups of $\mathscr{A}_{n,\theta}$ as explicitly as possible. In fact, we present them as the constant terms in the polynomial expansions of certain functions. First of all, we need the following basic lemma.

Lemma 6.6 *Let $P(n, r, k)$ denote the number of partitions of k to r distinct positive integers not greater than n. Then $P(n, r, k)$ is the coefficient of $u^r t^k$ in the polynomial expansion of $F_n(u, t) := \prod_{i=1}^{n}(1 + ut^i)$. In other words,*

$$\sum_{r,k \geq 0} P(n, r, k) u^r t^k = \prod_{i=1}^{n} (1 + ut^i).$$

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Proof

$$\prod_{i=1}^{n}(1+ut^i) = 1+\sum_{r=1}^{n}\sum_{\substack{(i_1,\dots,i_r)\\ 1\le i_1<\cdots<i_r\le n}}(ut^{i_1})\cdots(ut^{i_r})$$

$$= 1+\sum_{r=1}^{n}\sum_{k\ge 1}P(n,r,k)u^r t^k = \sum_{r,k\ge 0}P(n,r,k)u^r t^k. \qquad \blacksquare$$

Now, we have the following result for the rank a_n of the K-groups of $\mathscr{A}_{n,\theta}$.

Theorem 6.7 *Let $a_n = \operatorname{rank} K_0(\mathscr{A}_{n,\theta}) = \operatorname{rank} K_1(\mathscr{A}_{n,\theta})$. Then for a nonnegative integer m we have*

(i) *a_{2m+1} is the constant term in the Laurent polynomial expansion of*

$$\prod_{j=-m}^{m}(1+z^j);$$

(ii) *a_{2m} is the constant term in the Laurent polynomial expansion of*

$$(1+z)\prod_{j=-m+1}^{m}(1+z^{2j-1}).$$

Proof We know that $a_n = \sum_{r=0}^{n} a_{n,r}$ and $a_{n,r} = P(n,r,[r(n+1)/2])$ by Theorem 6.3. We have $a_{2m+1} = \sum_{r=0}^{2m+1} P(2m+1,r,r(m+1))$. Now, take $y = ut^{m+1}$ and use the preceding lemma to get

$$F_{2m+1}(u,t) = F_{2m+1}(yt^{-m-1},t) = \prod_{i=1}^{2m+1}(1+yt^{i-m-1}) = \sum_{r,k\ge 0}P(2m+1,r,k)y^r t^{k-r(m+1)}.$$

In particular, we get the following identity for $y=1$:

$$\prod_{i=1}^{2m+1}(1+t^{i-m-1}) = \sum_{r,k\ge 0}P(2m+1,r,k)t^{k-r(m+1)},$$

or equivalently, by setting $z=t$ and $j=i-m-1$, we have

$$\prod_{j=-m}^{m}(1+z^j) = \sum_{r,k\ge 0}P(2m+1,r,k)z^{k-r(m+1)}.$$

In particular, the constant term in the Laurent polynomial expansion of $\prod_{j=-m}^{m}(1+z^j)$ is obtained when we take the sum of those terms for which $k = r(m+1)$ holds, namely

$$\sum_{r=0}^{2m+1}P\big(2m+1,r,r(m+1)\big),$$

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which is precisely the expression for a_{2m+1}.

For part (ii), write

$$\begin{aligned} a_{2m} &= \sum_{r=0}^{2m} P\Big(2m, r, \Big[r\Big(m+\frac{1}{2}\Big)\Big]\Big) \\ &= \sum_{r=0}^{m} P\big(2m, 2r, r(2m+1)\big) + \sum_{r=0}^{m-1} P(2m, 2r+1, 2rm+m+r) \\ &=: A_m + B_m. \end{aligned}$$

Let us determine A_m first. Note that using Lemma 6.6 we have

$$\begin{aligned} \frac{1}{2}\Big\{\prod_{i=1}^{2m}(1+ut^i) + \prod_{i=1}^{2m}(1-ut^i)\Big\} &= \sum_{r,k\geq 0} P(2m,r,k)\Big\{\frac{1+(-1)^r}{2}\Big\}u^r t^k \\ &= \sum_{r,k\geq 0} P(2m,2r,k)u^{2r}t^k. \end{aligned}$$

If we define $y := u^2 t^{2m+1}$, we have the identity

$$\frac{1}{2}\Big\{\prod_{i=1}^{2m}\big(1+y^{\frac{1}{2}}t^{i-(m+\frac{1}{2})}\big) + \prod_{i=1}^{2m}\big(1-y^{\frac{1}{2}}t^{i-(m+\frac{1}{2})}\big)\Big\} = \sum_{r,k\geq 0} P(2m,2r,k)y^r t^{k-r(2m+1)},$$

which for $y=1$ yields

$$\frac{1}{2}\Big\{\prod_{i=1}^{2m}\big(1+t^{i-(m+\frac{1}{2})}\big) + \prod_{i=1}^{2m}\big(1-t^{i-(m+\frac{1}{2})}\big)\Big\} = \sum_{r,k\geq 0} P(2m,2r,k)t^{k-r(2m+1)}.$$

Hence A_m is the constant term in the polynomial expansion of

$$\frac{1}{2}\Big\{\prod_{i=1}^{2m}\big(1+t^{i-(m+\frac{1}{2})}\big) + \prod_{i=1}^{2m}\big(1-t^{i-(m+\frac{1}{2})}\big)\Big\}.$$

Similarly, for B_m we have

$$\begin{aligned} \frac{1}{2}\Big\{\prod_{i=1}^{2m}(1+ut^i) - \prod_{i=1}^{2m}(1-ut^i)\Big\} &= \sum_{r,k\geq 0} P(2m,r,k)\Big\{\frac{1-(-1)^r}{2}\Big\}u^r t^k \\ &= \sum_{r,k\geq 0} P(2m,2r+1,k)u^{2r+1}t^k. \end{aligned}$$

If we define $y^2 := u^2 t^{2m+1}$, we have the identities

$$\begin{aligned} &\frac{1}{2}\Big\{\prod_{i=1}^{2m}\big(1+y^{\frac{1}{2}}t^{i-(m+\frac{1}{2})}\big) - \prod_{i=1}^{2m}\big(1-y^{\frac{1}{2}}t^{i-(m+\frac{1}{2})}\big)\Big\} \\ &\quad= \sum_{r,k\geq 0} P(2m,2r+1,k)y^{2r+1}t^{k-(2rm+r+m)-\frac{1}{2}} \\ &\quad= t^{-\frac{1}{2}}\sum_{r,k\geq 0} P(2m,2r+1,k)y^{2r+1}t^{k-(2rm+r+m)}, \end{aligned}$$

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which for $y = 1$ yield

$$\frac{t^{\frac{1}{2}}}{2}\Big\{\prod_{i=1}^{2m}\big(1+t^{i-(m+\frac{1}{2})}\big) - \prod_{i=1}^{2m}\big(1-t^{i-(m+\frac{1}{2})}\big)\Big\} = \sum_{r,k\geq 0} P(2m,2r+1,k)t^{k-(2rm+r+m)}.$$

Hence B_m is the constant term in the polynomial expansion of

$$\frac{t^{\frac{1}{2}}}{2}\Big\{\prod_{i=1}^{2m}\big(1+t^{i-(m+\frac{1}{2})}\big) - \prod_{i=1}^{2m}\big(1-t^{i-(m+\frac{1}{2})}\big)\Big\}.$$

Therefore, $a_{2m} = A_m + B_m$ is the constant term in the polynomial expansion of

$$\frac{1}{2}\Big\{\prod_{i=1}^{2m}\big(1+t^{i-(m+\frac{1}{2})}\big) + \prod_{i=1}^{2m}\big(1-t^{i-(m+\frac{1}{2})}\big) + \frac{t^{\frac{1}{2}}}{2}\prod_{i=1}^{2m}\big(1+t^{i-(m+\frac{1}{2})}\big) - \frac{t^{\frac{1}{2}}}{2}\prod_{i=1}^{2m}\big(1-t^{i-(m+\frac{1}{2})}\big)\Big\},$$

or equivalently, the constant term in the polynomial expansion of

$$\frac{1}{2}\Big\{(1+z)\prod_{i=1}^{2m}\big(1+z^{2i-(2m+1)}\big) + (1-z)\prod_{i=1}^{2m}\big(1-z^{2i-(2m+1)}\big)\Big\},$$

which equals the constant term in the Laurent polynomial expansion of

$$(1+z)\prod_{j=-m+1}^{m}(1+z^{2j-1}).$$

■

Thanks to this theorem, one can compute a_n for large values of n using a computer algebra program. Many more terms are also available online at OEIS (The Online Encyclopedia of Integer Sequences at http://www.oeis.org). Moreover, as the following corollaries suggest, such recognitions as constant terms of certain Laurent polynomials opens the door to finding even more interesting combinatorial properties of the sequence $\{a_n\}$ that have been of interest to Erdős, J. H. van Lint, and R. C. Entringer, to name a few (*cf.* [5,6,23]).

Corollary 6.8 *Let n be a nonnegative integer.*

(i) *The integer a_{2n+1} is the number of solutions of the equation*

$$\sum_{k=-n}^{k=n} k\,\epsilon_k = 0,$$

where $\epsilon_k = 0$ or 1 for $-n \leq k \leq n$. In other words, a_{2n+1} is the number of ways that a sum of integers between $-n$ and n (with no repetitions) equals 0.

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(ii) *The integer a_{2n} is the number of solutions of the equation*

$$\sum_{k=-n+1}^{k=n} (2k-1)\,\epsilon_k = 0 \text{ or } 1,$$

where $\epsilon_k = 0$ or 1 for $-n+1 \le k \le n$. In other words, a_{2n} is the number of ways that a sum of half-integers between $-n+1/2$ and $n-1/2$ (with no repetitions) equals 0 or $1/2$.

Proof Using Theorem 6.7, the number a_{2n+1} is the constant term in the Laurent polynomial expansion of $\prod_{k=-n}^{n}(1+z^k)$, which is a finite sum of the form $\sum A(n,m)z^m$. Obviously, the integer coefficient $A(n,m)$ is the number of all possible combinations from the terms $z^{-n},\dots,z^0,\dots,z^n$, whose product makes a z^m. In other words, by putting $\epsilon_k = 1$ when z^k contributes to such a product making a z^m, and $\epsilon_k = 0$ otherwise, we conclude that

$$A(n,m) = \#\Big\{ (\epsilon_{-n},\dots,\epsilon_0,\dots,\epsilon_n) \in \{0,1\}^{2n+1} : \sum_{k=-n}^{k=n} k\,\epsilon_k = m \Big\}.$$

In particular, the constant term of the Laurent polynomial expansion is $A(n,0)$, and we have $a_{2n+1} = A(n,0)$. This proves part (i). Fort part (ii), we use the same idea for the Laurent polynomial expansion of

$$(1+z)\prod_{k=-m+1}^{m}(1+z^{2k-1}) = \prod_{k=-m+1}^{m}(1+z^{2k-1}) + z\prod_{k=-m+1}^{m}(1+z^{2k-1})$$

as suggested by Theorem 6.7(ii). ∎

In [23], van Lint answered a question of Erdős by determining the asymptotic behavior of

$$A(n,0) = \#\Big\{ (\epsilon_{-n},\dots,\epsilon_0,\dots,\epsilon_n) \in \{0,1\}^{2n+1} : \sum_{k=-n}^{k=n} k\,\epsilon_k = 0 \Big\}.$$

The idea in his proof is as follows. Since $A(n,0)$ is the constant term of the Laurent polynomial expansion of $\prod_{k=-n}^{n}(1+z^k)$, we can compute it as the Cauchy integral

$$\frac{1}{2\pi i}\oint_C \frac{\prod_{k=-n}^{n}(1+z^k)}{z}\,dz,$$

where C denotes the unit circle. By parameterizing C by $z = e^{2ix}$ for $x \in [0,\pi]$, applying the elementary identity $(1+e^{2ikx})(1+e^{-2ikx}) = 4\cos^2 kx$, and a simple calculation we arrive at

$$A(n,0) = \frac{2^{2n+2}}{\pi}\int_0^{\frac{\pi}{2}} \prod_{k=1}^{n} \cos^2 kx\,dx.$$

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We can then proceed by estimating the integrand near and far from 0 using some elementary inequalities, which lead to the asymptotic formula $A(n,0) \sim (3/\pi)^{\frac{1}{2}} 2^{2n+1} n^{-\frac{3}{2}}$ [23]. This will immediately give the asymptotic behavior of the sequence $\{a_{2n+1}\}$ by the previous corollary. One can adapt the arguments used by van Lint to obtain a similar asymptotic behavior for the sequence $\{a_{2n}\}$ by estimating the corresponding integral

$$\frac{2^{2n+2}}{\pi} \int_0^{\frac{\pi}{2}} \cos^2 x \prod_{k=1}^{n} \cos^2(2k-1)x \, dx,$$

which leads to the asymptotic formula $a_{2n} \sim (3/\pi)^{\frac{1}{2}} 2^{2n} n^{-\frac{3}{2}}$. This gives rise to the following result.

Corollary 6.9 *$a_n \sim \sqrt{\frac{24}{\pi}}\, 2^n n^{-\frac{3}{2}}$ when $n \to \infty$. In particular, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$.*

7 The Positive Cone of $K_0(\mathscr{F}_{\theta,f})$

In this section, we generalize a result of Kodaka on the order structure of the group K_0 of the crossed product by a Furstenberg transformation on the 2-torus [18, Theorem 5.2]. However, our approach is different, and follows the general guidelines of [30, Lemma 3.1]. We remind the reader that for a C^*-algebra A the positive cone of $K_0(A)$ is the set $K_0(A)_+ = \{[q] \in K_0(A) : q \in \mathcal{P}_\infty(A)\}$, where $\mathcal{P}_\infty(A)$ is the set of all projections in matrix algebras over A. Also, any positive trace τ on a C^*-algebra A induces a group homomorphism $\tau_* \colon K_0(A) \to \mathbb{R}$. As was indicated in the introduction, when the Furstenberg transformation $\varphi_{\theta,f}$ is minimal and uniquely ergodic, using the results of Lin and Phillips in [21] the transformation group C^*-algebra $\mathscr{F}_{\theta,f}$ is classifiable by its Elliott invariant, and the order of $K_0(\mathscr{F}_{\theta,f})$ is determined by the unique tracial state τ on $\mathscr{F}_{\theta,f}$ [22, 29]. The fact that $\tau_* K_0(\mathscr{F}_{\theta,f}) = \mathbb{Z} + \mathbb{Z}\theta$ was first proved in the unpublished thesis of Ji [16]. However, we will study the effect of the trace on the order structure of K_0 using R. Exel's machinery of rotation numbers [7].

For a C^*-algebra A, we denote by $U_p(A)$ the set of unitary elements of $M_p(A)$. The following lemma is well known, but it is convenient to state and prove it for completeness.

Lemma 7.1 *Let A and B be unital C^*-algebras and let $A \otimes B$ denote their minimal tensor product. Suppose that $u \in U_p(A)$ and $v \in U_q(B)$, and let $\phi \colon \mathcal{C}(\mathbb{T}^2) \to M_{pq}(A \otimes B)$ be the unique homomorphism mapping the coordinate unitaries $z_1, z_2 \in U(\mathcal{C}(\mathbb{T}^2))$ to the commuting unitaries $u \otimes 1_q, 1_p \otimes v \in U_{pq}(A \otimes B)$, respectively. Let $b(u,v) \in K_0(A \otimes B)$ denote the Bott element of u, v defined by $K_0(\phi)(b)$, where $\beta = [z_1] \wedge [z_2]$ is the Bott element in $K_0(\mathcal{C}(\mathbb{T}^2))$ so that $K_0(\mathcal{C}(\mathbb{T}^2)) = \mathbb{Z}[1] + \mathbb{Z}\,\beta$. Then $\tau_*(b(u,v)) = 0$ for any tracial state τ on $A \otimes B$.*

Proof Since $\tau \circ \phi$ is a trace on $\mathbb{T}^2$, there exists a Borel probability measure μ on $\mathbb{T}^2$ such that

$$(\tau \circ \phi)(f) = \int_{\mathbb{T}^2} f(x)\, d\mu(x), \quad f \in \mathcal{C}(\mathbb{T}^2).$$

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Write $\beta = [p] - [q]$, where p, q are appropriate projections in some matrix algebra over $\mathcal{C}(\mathbb{T}^2)$, so we have

$$\tau_*(b(u,v)) = \tau_*\big(K_0(\phi)(\beta)\big) = (\tau \circ \phi)_*(\beta) = \int_{\mathbb{T}^2} \mathrm{Tr}(p(x)) - \mathrm{Tr}(q(x))\, d\mu(x).$$

It is well known that for the Bott element b we have $\mathrm{Tr}(p(x)) - \mathrm{Tr}(q(x)) = 0$, namely, the projections $p(x)$ and $q(x)$ have the same rank for all $x \in \mathbb{T}^2$, and this common rank does not depend on x, since $\mathbb{T}^2$ is connected (in fact, they are rank one projections). This can be proved either by a calculation of the traces of the projections $p(x)$ and $q(x)$ explicitly (*cf.* [1, p. 7]), or by using the naturality in the Künneth formula for $\mathbb{T}^2$, which shows that the image under any point evaluation of β is zero. Briefly speaking, the map $x \mapsto \mathrm{Tr}(p(x)) - \mathrm{Tr}(q(x))$ belongs to $\mathcal{C}(\mathbb{T}^2, \mathbb{Z})$, so it has to assume a constant integer, which we call $\dim \beta$. In particular, $\dim \beta$ is invariant under the change of coordinate $(\zeta_1, \zeta_2) \mapsto (\zeta_1, \zeta_2^{-1})$, whereas the naturality of the Künneth homomorphism $\alpha_{1,1}\colon K_1(\mathcal{C}(\mathbb{T})) \otimes K_1(\mathcal{C}(\mathbb{T})) \to K_0(\mathcal{C}(\mathbb{T}^2))$, which maps $[z] \otimes [z]$ to β, implies that the Bott element β will transform into $-\beta$ under this change of coordinates since $[z] \otimes [z^{-1}] = [z] \otimes (-[z]) = -([z] \otimes [z])$. This means $\dim \beta = 0$. ∎

We denote by u the unitary in $\mathscr{F}_{\theta,f}$ implementing the action generated by the transformation $\varphi_{\theta,f}$ on $\mathbb{T}^n$ with irrational parameter θ, and by z_1 the unitary in $\mathcal{C}(\mathbb{T}^n)$ defined by $z_1(\zeta_1, \ldots, \zeta_n) = \zeta_1$ as in Section 2. Then we have $uz_1u^{-1} = z_1 \circ \varphi_{\theta,f}^{-1} = e^{2\pi i\theta} z_1$, so that $C^*(u, z_1) \cong A_\theta$, the irrational rotation algebra. Let $p_\theta \in C^*(u, z_1)$ be a Rieffel projection of trace θ as in [34]. It is obvious that $\tau_*([1]) = 1$. On the other hand, since the restriction of τ on the C^*-subalgebra $A_\theta \subseteq \mathscr{F}_{\theta,f}$ has to be the unique tracial state on A_θ, we have $\tau_*([p_\theta]) = \theta$. The main result of this section will show that all the essential information about the order structure of $K_0(\mathscr{F}_{\theta,f})$ is encoded in the embedding of A_θ in $\mathscr{F}_{\theta,f}$.

Theorem 7.2 *Let $\varphi_{\theta,f}$ be a minimal uniquely ergodic Furstenberg transformation on $\mathbb{T}^n$ with $\theta \in (0,1)$ (*e.g.*, when $\theta \in (0,1) \setminus \mathbb{Q}$ and each f_i satisfies a uniform Lipschitz condition in ζ_i for $i = 1, \ldots, n-1$). Let a_n and $\mathscr{T}_f^0$ denote, respectively, the rank and the torsion subgroup of $K_0(\mathscr{F}_{\theta,f})$ so that $K_0(\mathscr{F}_{\theta,f}) \cong \mathbb{Z}^{a_n} \oplus \mathscr{T}_f^0$. Then the isomorphism of $K_0(\mathscr{F}_{\theta,f})$ with this group can be chosen in such a way that*

(i) *the unique tracial state τ on $\mathscr{F}_{\theta,f}$ induces the map*

$$\tau_*(a[1] + b[p_\theta], c, t) = a + b\,\theta$$

on $K_0(\mathscr{F}_{\theta,f})$ for all $(a[1] + b[p_\theta], c, t) \in (\mathbb{Z}[1] + \mathbb{Z}[p_\theta]) \oplus \mathbb{Z}^{a_n - 2} \oplus \mathscr{T}_f^0 \cong \mathbb{Z}^{a_n} \oplus \mathscr{T}_f^0$;

(ii) *the positive cone $K_0(\mathscr{F}_{\theta,f})_+$ can be identified with*

$$\big\{(a[1] + b[p_\theta], c, t) \in (\mathbb{Z}[1] + \mathbb{Z}[p_\theta]) \oplus \mathbb{Z}^{a_n - 2} \oplus \mathscr{T}_f^0 : a + b\,\theta > 0\big\} \cup \{0\}.$$

Proof The idea of the proof is to show that there exists a generating set for the finitely generated abelian group $K_0(\mathscr{F}_{\theta,f})$ including $[1]$ and $[p_\theta]$ such that the induced homomorphism τ_* vanishes at all generators, except for $[1]$ and $[p_\theta]$ for which we

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have $\tau_*([1]) = 1$ and $\tau_*([p_\theta]) = \theta$. Using Theorem 2.1 and setting $\alpha = \varphi_{\theta,f}$ and $\alpha_j = K_j(\alpha)$ for $j = 1, 2$ we have

$$K_0(\mathscr{F}_{\theta,f}) \cong \operatorname{coker}(\alpha_0 - \mathrm{id}) \oplus \ker(\alpha_1 - \mathrm{id}) = \bigoplus_{r\geq 0} \left[\operatorname{coker}(\wedge^{2r}\widehat{\alpha} - \mathrm{id}) \oplus \ker(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id}) \right],$$

where $\widehat{\alpha}$ is the restriction of α_1 to the subgroup $\mathbb{Z}[z_1] + \cdots + \mathbb{Z}[z_n]$ of $K_1(\mathcal{C}(\mathbb{T}^n))$ as in Section 2, and $z_j(\zeta, \ldots, \zeta_n) = \zeta_j$ of $\mathcal{C}(\mathbb{T}^n)$ for $j = 1, \ldots, n$. Note that by Definition 1.3, $f = (f_1, \ldots, f_{n-1})$ consists of continuous functions $f_{j-1}\colon \mathbb{T}^{j-1} \to \mathbb{T}$ for $j = 2, \ldots, n$. First, we "linearize" each f_{j-1} by finding the unique "linear" function

$$(\zeta_1 \ldots, \zeta_{j-1}) \mapsto \zeta_1^{b_{1j}} \cdots \zeta_{j-1}^{b_{j-1,j}}, \qquad (b_{j-1,j} \neq 0)$$

in the homotopy class of f_{j-1}. This allows us to calculate $\widehat{\alpha}([z_j])$ by writing

$$\begin{aligned}
\widehat{\alpha}([z_1]) &= [z_1 \circ \varphi_{\theta,f}^{-1}] = [e^{2\pi i\theta} z_1] = [z_1], \\
\widehat{\alpha}([z_j]) &= [z_j \circ \varphi_{\theta,f}^{-1}] = [f_{j-1}(z_1, \ldots, z_{j-1}) z_j] \\
&= [z_1^{b_{1j}} \cdots z_{j-1}^{b_{j-1,j}} z_j] = b_{1j}[z_1] + \cdots + b_{j-1,j}[z_{j-1}] + [z_j]
\end{aligned}$$

for $j = 2, \ldots, n$. In other words, the integer matrix of $\widehat{\alpha}$ with respect to the basis $\{[z_1], \ldots, [z_n]\}$ of $\mathbb{Z}^n$ is precisely in the form (5.1) as in the proof of Theorem 5.1. Now, we can realize $\alpha_0 = \wedge^{\mathrm{even}}\widehat{\alpha}$ and $\alpha_1 = \wedge^{\mathrm{odd}}\widehat{\alpha}$ to calculate the K-groups of $\mathscr{F}_{\theta,f}$ as in Section 2.

It is important to note that, by referring to the exact sequences (2.1), (2.2) and (2.3), the isomorphic image of $\operatorname{coker}(\alpha_0 - \mathrm{id})$ in $K_0(\mathscr{F}_{\theta,f})$ is precisely the image $\operatorname{im} j_0$ of $K_0(\mathcal{C}(\mathbb{T}^n))$ and an isomorphic image of $\ker(\alpha_1 - \mathrm{id})$ in $K_0(\mathscr{F}_{\theta,f})$ is obtained by finding the image of a splitting (injective) homomorphism $s\colon \ker(\alpha_1 - \mathrm{id}) \to K_0(\mathscr{F}_{\theta,f})$ for the exact sequence (2.2) so that $\partial \circ s = \mathrm{id}$ on $\ker(\alpha_1 - \mathrm{id})$. Any such splitting homomorphism is obtained as follows: fix a basis $\{\gamma_1, \ldots, \gamma_q\}$ for the free finitely generated group $\ker(\alpha_1 - \mathrm{id})$ of rank q, and find elements $\nu_1^{(0)}, \ldots, \nu_q^{(0)} \in K_0(\mathscr{F}_{\theta,f})$ such that $\partial\nu_j^{(0)} = \gamma_j$ for $j = 1, \ldots, q$. Then define $s(\sum_j m_j\gamma_j) = \sum_j m_j\nu_j^{(0)}$ for $m_j \in \mathbb{Z}$. Clearly, $K_0(\mathscr{F}_{\theta,f}) = \operatorname{im} j_0 \oplus \operatorname{im} s$.

Now, since $\wedge^0\widehat{\alpha} = \mathrm{id}$ on $\Lambda^0\mathbb{Z}^n = \mathbb{Z}$ and $\wedge^1\widehat{\alpha} = \widehat{\alpha}$ on $\Lambda^1\mathbb{Z}^n = \mathbb{Z}^n$, we can write the isomorphism

$$K_0(\mathscr{F}_{\theta,f}) \cong \mathbb{Z} \oplus \ker(\widehat{\alpha} - \mathrm{id}) \oplus \bigoplus_{r\geq 1} \left[\ker(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id}) \oplus \operatorname{coker}(\wedge^{2r}\widehat{\alpha} - \mathrm{id}) \right].$$

In fact, a single generator for the isomorphic image of $\mathbb{Z}$ in $K_0(\mathscr{F}_{\theta,f})$ is $[1]$, and since $b_{j-1,j} \neq 0$ for $j = 2, \ldots, n$, we have $\ker(\widehat{\alpha} - \mathrm{id}) = \mathbb{Z}e_1 = \mathbb{Z}[z_1]$. It is easy to see that $\partial([p_\theta]) = [z_1]$ (see the proposition in the appendix of [31]). Therefore there exists a basis $\{\gamma_1, \ldots, \gamma_q\}$ for $\ker(\alpha_1 - \mathrm{id}) = \bigoplus_{r\geq 0} \ker(\wedge^{2r+1}\widehat{\alpha} - \mathrm{id})$ with $\gamma_1 = [z_1]$ and a splitting homomorphism $s\colon \ker(\alpha_1 - \mathrm{id}) \to K_0(\mathscr{F}_{\theta,f})$ with $s([z_1]) = [p_\theta]$. Hence a single generator for the image of $\ker(\widehat{\alpha} - \mathrm{id})$ in $K_0(\mathscr{F}_{\theta,f})$ is $[p_\theta]$.

It remains to study the effect of τ_* on the isomorphic image of $\bigoplus_{r\geq 1} \operatorname{coker}(\wedge^{2r}\widehat{\alpha} - \mathrm{id})$, which contains the the torsion subgroup $\mathscr{T}_f^0$, and on

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the image of $\bigoplus_{r\geq 1}\ker(\wedge^{2r+1}\widehat{\alpha}-\mathrm{id})$ in $K_0(\mathscr{F}_{\theta,f})$. For more convenience, set $e_j := [z_j]$ for $j = 1,\dots,n$ as in Section 2. First, we study the isomorphic image of $\bigoplus_{r\geq 1}\mathrm{coker}(\wedge^{2r}\widehat{\alpha}-\mathrm{id})$. We show that τ_* vanishes on this whole subgroup by showing, equivalently, that τ_* vanishes on the image of the subgroup $\bigoplus_{r\geq 1}\Lambda^{2r}_{\mathbb{Z}}(e_1,\dots,e_n) \subset K_0(\mathcal{C}(\mathbb{T}^n))$ in $K_0(\mathscr{F}_{\theta,f})$ under the map j_0. Let $\eta = e_{i_1}\wedge\cdots\wedge e_{i_{2r}} \in K_0(\mathcal{C}(\mathbb{T}^n))$ for some $r\geq 1$ and $1\leq i_1<\cdots<i_{2r}\leq n$. We want to show that $\tau_*(j_0(\eta))=0$, where $j_0 := K_0(j)$ and $j\colon \mathrm{C}(\mathbb{T}^n)\to\mathscr{F}_{\theta,f}$ is the natural embedding in the structure of the crossed product $\mathscr{F}_{\theta,f}=\mathrm{C}(\mathbb{T}^n)\rtimes_\alpha\mathbb{Z}$. By Künneth formula we have $\eta = b(u,z)$, where u is a unitary in some matrix algebra over $\mathcal{C}(\mathbb{T}^{n-1})$ with $[u] = e_{i_1}\wedge\cdots\wedge e_{i_{2r-1}} \in K_1(\mathcal{C}(\mathbb{T}^{n-1}))$ and z is the canonical unitary in $\mathcal{C}(\mathbb{T})$ with $[z]=e_{i_{2r}}\in K_1(\mathcal{C}(\mathbb{T}))$. By Lemma 7.1, we have

$$\tau_*\big(j_0(\eta)\big) = \tau_*\big(K_0(j)(\eta)\big) = (\tau\circ j)_*(\eta) = (\tau\circ j)_*\big(b(u,z)\big) = 0.$$

Now, we study the isomorphic image of $\bigoplus_{r\geq 1}\ker(\wedge^{2r+1}\widehat{\alpha}-\mathrm{id})$ in $K_0(\mathscr{F}_{\theta,f})$. We will show that τ_* assumes only integer values on this whole subgroup. In other words, if $\{\gamma_1,\dots,\gamma_q\}$ is a basis for the $\ker(\alpha_0-\mathrm{id})$ as above such that $\gamma_1=[z_1]$ is a basis for $\ker(\widehat{\alpha}-\mathrm{id})$ and $\{\gamma_2,\dots,\gamma_q\}$ is a basis for $\bigoplus_{r\geq 1}\ker(\wedge^{2r+1}\widehat{\alpha}-\mathrm{id})$, then $\tau_*(\nu_1^{(0)})=\theta$, and $\tau_*(\nu_j^{(0)})=k_j$ for some $k_j\in\mathbb{Z}$ for $j=2,\dots,q$, where $\nu_j^{(0)}$'s are chosen in $K_0(\mathscr{F}_{\theta,f})$ so that $\partial(\nu_j^{(0)})=\gamma_j$ for $j=1,\dots,q$ and $\nu_1^{(0)}=[p_\theta]$ as above. To demonstrate this, we prove that the determinant of any unitary representing an element in the subgroup $\bigoplus_{r\geq 1}\Lambda^{2r+1}_{\mathbb{Z}}(e_1,\dots,e_n)$ is the constant function 1. Then the rotation number homomorphism $\rho^{\mu}_{\alpha}\colon \ker(\alpha_1-\mathrm{id})\to\mathbb{T}$ defined by Exel is the constant 1 on the subgroup $\bigoplus_{r\geq 1}\ker(\wedge^{2r+1}\widehat{\alpha}-\mathrm{id})$ [7, Theorem VI.11], hence the trace will be integer-valued on this subgroup because $\exp(2\pi i\tau_*(\eta)) = \rho^{\mu}_{\alpha}\circ\partial(\eta)$ for all $\eta\in K_0(\mathscr{F}_{\theta,f})$ [7, Theorem V.12]. To calculate the determinant on $\bigoplus_{r\geq 1}\Lambda^{2r+1}_{\mathbb{Z}}(e_1,\dots,e_n)$, let $\gamma = e_{i_1}\wedge\cdots\wedge e_{i_{2r+1}}\in K_1(\mathcal{C}(\mathbb{T}^n))$, set $\eta = e_{i_1}\wedge\cdots\wedge e_{i_{2r}}\in K_0(\mathcal{C}(\mathbb{T}^{n-1}))$, and write $\eta=[p]-[q]$ as above. Then $e_{i_{2r+1}}=[z]$ for the canonical unitary z of $\mathcal{C}(\mathbb{T})$, and using the Künneth formula we have

$$\begin{aligned}\gamma = \eta\otimes[z] &= ([p]-[q])\otimes[z] = [p]\otimes[z]+[q]\otimes[z^{-1}]\\ &= \big[((1-p)\otimes 1+p\otimes z)((1-q)\otimes 1+q\otimes z^{-1})\big].\end{aligned}$$

So, $\gamma=[\omega_1\omega_2]$, where $\omega_1 := (1-p)\otimes 1+p\otimes z$ and $\omega_2 := (1-q)\otimes 1+q\otimes z^{-1}$ are unitaries in some matrix algebra of the same size over $\mathcal{C}(\mathbb{T}^n)$. Since for all $x\in\mathbb{T}^{n-1}$ the projections $p(x)$ and $q(x)$ have the same rank ρ as in the proof of Lemma 7.1, we have the following equivalence of projections in some matrix algebra $M_l(\mathbb{C})$:

$$p(x)\sim \mathbf{1}_\rho\oplus\mathbf{0}_{l-\rho}\sim q(x),$$

where $\mathbf{1}_m,\mathbf{0}_m$ denote the identity and the zero matrix of order m, respectively, and $\oplus$ is the direct sum of matrices. This implies the following unitary equivalence of projections in $M_{2l}(\mathbb{C})$:

$$p(x)\oplus\mathbf{0}_l\sim_u \mathbf{1}_\rho\oplus\mathbf{0}_{2l-\rho}\sim_u q(x)\oplus\mathbf{0}_l.$$

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In particular, we conclude the following unitary equivalence of unitary matrices for all $(x,\zeta)\in\mathbb{T}^{n-1}\times\mathbb{T}$

$$\omega_1(x,\zeta)\oplus\mathbf{1}_l\sim_u\zeta\mathbf{1}_\rho\oplus\mathbf{1}_{2l-\rho},\qquad\omega_2(x,\zeta)\oplus\mathbf{1}_l\sim_u\zeta^{-1}\mathbf{1}_\rho\oplus\mathbf{1}_{2l-\rho}.$$

Therefore $\operatorname{Det}\omega_1(x,\zeta)=\zeta^\rho$ and $\operatorname{Det}\omega_2(x,\zeta)=\zeta^{-\rho}$, hence $\operatorname{Det}(\omega_1\omega_2)(x,\zeta)=1$, for all $(x,\zeta)\in\mathbb{T}^{n-1}\times\mathbb{T}$. This implies that $\operatorname{Det}_*(\gamma)=1\in[\mathbb{T}^n,\mathbb{T}]$, where $[\mathbb{T}^n,\mathbb{T}]$ denotes the set of homotopy classes of continuous functions from $\mathbb{T}^n$ to $\mathbb{T}$ (see [7, Definition VI.8 and Proposition VI.9]).

Finally, by setting $\nu_1:=\nu_1^{(0)}=[p_\theta]$ and $\nu_k:=\nu_j^{(0)}-k_j[1]$ for $j=2,\dots,q$ so that $\tau_*(\nu_1)=\theta$ and $\tau_*(\nu_j)=0$ for $j=2,\dots,q$, we can form a generating set with the desired property for $K_0(\mathscr{F}_{\theta,f})$ by taking the union of $\{\nu_1,\dots\nu_q\}$ and a generating set including $[1]$ for the isomorphic image of $\operatorname{coker}(\alpha_0-\mathrm{id})$. This proves part (i).

For part (ii), we use part (i) together with the fact that the order on $K_0(\mathscr{F}_{\theta,f})$ is determined by the effect of the unique tracial state τ because $\mathbb{T}^n$ is a finite dimensional infinite compact metric space and $\varphi_{\theta,f}$ is a minimal homeomorphism of $\mathbb{T}^n$ (see [22, Theorem 5.1(1)] or [29, Theorem 4.5(1)]). ∎

Corollary 7.3 *Let $\varphi_{\theta,f}$ be a minimal uniquely ergodic Furstenberg transformation on $\mathbb{T}^n$ as above. Then linearizing the functions $f_i\colon\mathbb{T}^i\to\mathbb{T}$ in $f=(f_1,\dots,f_{i-1})$ does not change the isomorphism class of the transformation group C*-algebra $\mathscr{F}_{\theta,f}$.*

Proof Since $\varphi_{\theta,f}$ is minimal, θ must be irrational. So the range of the unique tracial state (by unique ergodicity) on $K_0(\mathscr{F}_{\theta,f})$ is dense in $\mathbb{R}$ as it is $\mathbb{Z}+\mathbb{Z}\theta$ by the above argument. Benefiting from the results of [21], such C^*-algebras are completely classifiable by their Elliott invariants, which remain unchanged (up to isomorphism) after the linearization process. Linearizing does not change the isomorphism classes of the K-groups, and Theorem 7.2 guarantees that the order structure of the group K_0 is precisely the regular order inherited from $\mathbb{R}$ on $\mathbb{Z}+\mathbb{Z}\theta$ before and after linearization. ∎

8 Applications to $C^*(\mathfrak{D}_n)$

This section is devoted to some applications of the results of the previous sections to the C^*-algebras studied in [33], in which the authors promised to completely classify the simple infinite dimensional quotients of the group C^*-algebra of their interest $C^*(\mathfrak{D}_n)$ by their K-theoretic invariants in a later work. In this context, the discrete group $\mathfrak{D}_n$ is a higher dimensional analogue of the discrete Heisenberg group H_3, and is defined by

$$\mathfrak{D}_n=\big\langle\,x,y_0,y_1,\dots,y_n\mid xy_0=y_0x,\ y_iy_j=y_jy_i\quad\text{for } 0\le i,j\le n,\ [x,y_j]=y_{j-1}\text{ for }1\le j\le n\,\big\rangle,$$

where $[x,y]:=xyx^{-1}y^{-1}$. The group $\mathfrak{D}_n$ can be represented as a semidirect product $\mathbb{Z}^{n+1}\rtimes_\eta\mathbb{Z}$, where the group homomorphism $\eta\colon\mathbb{Z}\to\mathrm{GL}(n+1,\mathbb{Z})$ is such that $\eta(k)$ is the matrix, whose (i,j)-entry is given by $\binom{k}{j-i}$ as defined in [33]. This realization allows us to study K-theory of $C^*(\mathfrak{D}_n)$.

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Proposition 8.1 *$K_i(C^*(\mathfrak{D}_n)) \cong K_i(\mathscr{A}_{n+1,\theta})$ for $i = 0, 1$. In particular,*

$$\operatorname{rank} K_0\big(C^*(\mathfrak{D}_n)\big) = \operatorname{rank} K_1\big(C^*(\mathfrak{D}_n)\big) = a_{n+1}.$$

Proof Since $\mathfrak{D}_n \cong \mathbb{Z}^{n+1} \rtimes_\eta \mathbb{Z}$, so $C^*(\mathfrak{D}_n) \cong C^*(\mathbb{Z}^{n+1}) \rtimes_{\widetilde{\eta}} \mathbb{Z} \cong \mathcal{C}(\mathbb{T}^{n+1}) \rtimes_{\widetilde{\eta}} \mathbb{Z}$ and the integer matrix corresponding to $\widetilde{\eta}$ is the $(n+1)\times(n+1)$ matrix M_n introduced in [33], which is precisely the matrix S_{n+1} defined in Section 5 to describe the linear structure of Anzai transformations on $\mathbb{T}^{n+1}$. The rest of the proof follows from Theorem 2.1. ∎

As some higher-dimensional analogues of the irrational rotation algebras A_θ, all simple infinite-dimensional quotients of $C^*(\mathfrak{D}_n)$ have been classified in [33, Theorem 3.2]. These consist of the C^*-algebras $\mathscr{A}_{n,\theta}$ for some irrational parameter θ, and a few more classes of C^* algebras denoted by $A_i^{(n)}$, which are of the form $\mathcal{C}(\mathbf{Y}_i \times \mathbb{T}^{n-i}) \rtimes_{\phi_i} \mathbb{Z}$ for some suitable finite sets $\mathbf{Y}_i$ and minimal homeomorphisms ϕ_i for $i = 1, 2, \ldots, n-1$. Then it is proved in [33, Theorem 4.8] that $A_i^{(n)} \cong M_{C_i}(B_i^{(n)})$, where $B_i^{(n)}$ is the transformation group C^*-algebra of some affine Furstenberg transformation on $\mathbb{T}^{n-i}$. We conclude the following results.

Corollary 8.2 *Let A be a simple infinite dimensional quotient of $C^*(\mathfrak{D}_n)$. Then* rank $K_0(A)$ = rank $K_1(A) = a_{n-i}$ *for some $i \in \{0, 1, \ldots, n-1\}$ that is uniquely determined by the isomorphism $A \cong \mathcal{C}(\mathbf{Y}_i \times \mathbb{T}^{n-i}) \rtimes_{\phi_i} \mathbb{Z}$ as in* [33, Theorem 3.2].

Proof It is proved that A is isomorphic to a matrix algebra over a Furstenberg transformation group C^*-algebra $B_i^{(n)}$ on $\mathbb{T}^{n-i}$ for some suitable $i \in \{0, 1 \ldots, n\}$ [33, Theorem 4.8]. So $K_j(A) \cong K_j(B_i^{(n)})$ for $j = 0, 1$. The rest of the proof is clear from the preceding theorem. ∎

We saw in Proposition 6.5 that $\{a_n\}$ is a strictly increasing sequence. Therefore the preceding corollary is a first step towards the classification of the simple infinite dimensional quotients of $C^*(\mathfrak{D}_n)$ by means of K-theory. But as is seen, the rank of the K-groups alone can not distinguish the algebras at the same "level" (*i.e.*, those algebras that are included in the same class, but with different values of the parameters). The other powerful K-theoretic invariant that helps us do this is the trace invariant, *i.e.*, the range of the unique tracial state acting on the K_0-group.

Proposition 8.3 *Suppose $A \cong \mathcal{C}(\mathbf{Y}_i \times \mathbb{T}^{n-i}) \rtimes_{\phi_i} \mathbb{Z}$ is a simple infinite dimensional quotient of $C^*(\mathfrak{D}_n)$ as in* [33, Theorem 3.2]. *Then A has a unique tracial state $\widetilde{\tau}$ and $\widetilde{\tau}_* K_0(A) = \frac{1}{C_i}(\mathbb{Z} + \mathbb{Z}\vartheta_i)$, where $C_i = |\mathbf{Y}_i|$ and $e^{2\pi i\vartheta_i} = \zeta_i = (-1)^{C_i+1}\eta_i^{C_i}$ as in* [33, Lemma 4.6 and Theorem 4.8].

Proof Following [33, Theorem 4.8], A is isomorphic to $M_{C_i}(B_i^{(n)}) = M_{C_i}(\mathbb{C}) \otimes B_i^{(n)}$, where $B_i^{(n)}$ is a simple Furstenberg transformation group C^*-algebra with the irrational parameter $\zeta_i = (-1)^{C_i+1}\eta_i^{C_i}$. By Corollary 1.5, $B_i^{(n)}$ has a unique tracial state τ. Moreover, $\tau_* K_0(B_i^{(n)}) = \mathbb{Z} + \mathbb{Z}\vartheta_i$, where $e^{2\pi i\vartheta_i} = \zeta_i$ (see [16, Theorem 2.23] or Theorem 7.2). Thus A has the unique tracial state $\widetilde{\tau} = (1/C_i\,\mathrm{Tr}) \otimes \tau$, in which Tr is the usual trace on $M_{C_i}(\mathbb{C})$, and so $\widetilde{\tau}_* K_0(A) = 1/C_i(\mathbb{Z} + \mathbb{Z}\vartheta_i)$ [16, Lemma 3.5]. ∎

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Finally, we can characterize all simple infinite-dimensional quotients of $C^*(\mathfrak{D}_n)$.

Proposition 8.4 *$\mathscr{A}_{n,\theta} \cong \mathscr{A}_{n',\theta'}$ if and only if $n = n'$ and there exists an integer k such that $\theta = k \pm \theta'$. More generally, let $A_i^{(n)} \cong \mathcal{C}(\mathbf{Y}_i \times \mathbb{T}^{n-i}) \rtimes_{\phi_i} \mathbb{Z}$ be a simple infinite dimensional quotient of $C^*(\mathfrak{D}_n)$ with the structure constants $\lambda, \mu_1, \ldots, \mu_i$ as in* [33, pp. 165–166]*, and let $A_{i'}^{(n')} \cong \mathcal{C}(\mathbf{Y}'_{i'} \times \mathbb{T}^{n'-i'}) \rtimes_{\phi'_i} \mathbb{Z}$ be a simple infinite dimensional quotient of $C^*(\mathfrak{D}_{n'})$ with the structure constants $\lambda', \mu'_1, \ldots, \mu'_{i'}$. Suppose that $C_i = |\mathbf{Y}_i|$ and $C'_{i'} = |\mathbf{Y}'_{i'}|$. Then $A_i^{(n)} \cong A_{i'}^{(n')}$ if and only if $n - i = n' - i'$, $C_i = C'_{i'}$ and*

$$\lambda^{\binom{C_i}{i+1}} \mu_1^{\binom{C_i}{i}} \mu_2^{\binom{C_i}{i-1}} \cdots \mu_i^{C_i} = \lambda'^{\binom{C'_{i'}}{i'+1}} \mu_1'^{\binom{C'_{i'}}{i'}} \mu_2'^{\binom{C'_{i'}}{i'-1}} \cdots \mu_{i'}'^{C'_{i'}}$$

or

$$\lambda^{\binom{C_i}{i+1}} \mu_1^{\binom{C_i}{i}} \mu_2^{\binom{C_i}{i-1}} \cdots \mu_i^{C_i} = (\lambda'^{\binom{C'_{i'}}{i'+1}} \mu_1'^{\binom{C'_{i'}}{i'}} \mu_2'^{\binom{C'_{i'}}{i'-1}} \cdots \mu_{i'}'^{C'_{i'}})^{-1}.$$

Proof Use the previous proposition and the fact that $\{a_n\}$ is a strictly increasing sequence (see Proposition 6.5). Note that

$$\zeta_i = (-1)^{C_i+1} \eta_i^{C_i} = \lambda^{\binom{C_i}{i+1}} \mu_1^{\binom{C_i}{i}} \mu_2^{\binom{C_i}{i-1}} \cdots \mu_i^{C_i}$$

by the last equation on [33, p. 171]. ∎

Remark 8.5 Note that $C_i = |\mathbf{Y}_i|$ is completely determined by the structural constants $\lambda, \mu_1, \ldots, \mu_{i-1}$ (which are roots of unity). More precisely, by calculations on [33, pp. 165–166], we have

$$C_i = \min\left\{ r \in \mathbb{N} \mid \lambda^r = \lambda^{\binom{r}{2}} \mu_1^r = \cdots = \lambda^{\binom{r}{i}} \mu_1^{\binom{r}{i-1}} \mu_2^{\binom{r}{i-2}} \cdots \mu_{i-1}^r = 1 \right\}.$$

For an explicit example in lower dimensions, see [26, Lemma 5.4].

9 Concluding Remarks

(I) The method used in Section 1 for computing K-groups of the transformation group C^*-algebras of homeomorphisms of the tori may be extended to more general settings. Let G be a compact connected Lie group with torsion-free fundamental group $\pi_1(G)$. (It is well known that the fundamental group of such spaces are finitely generated and abelian, so being torsion-free means $\pi_1(G) \cong \mathbb{Z}^l$, for some l.) Some important examples are any finite Cartesian products of the groups S^3, $SO(2)$, $Sp(n)$, $U(n)$ and $SU(n)$. Then $K^*(G)$ is torsion-free and can be given the structure of a $\mathbb{Z}_2$-graded Hopf algebra over the integers [14]. Moreover, regarded as a Hopf algebra, $K^*(G)$ is the exterior algebra on the module of the primitive elements, which are of degree 1. The module of the primitive elements of $K^*(G)$ may also be described as follows. Let $U(n)$ denote the group of unitary matrices of order n and let $U := \cup_{n=1}^{\infty} U(n)$ be the stable unitary group. Any unitary representation $\rho\colon G \to U(n)$, by composition with the inclusion $U(n) \subset U$, defines a homotopy class $\beta(\rho)$ in

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$[G,U] = K^1(G)$. The module of the primitive elements in $K^1(G)$ is exactly the module generated by all classes $\beta(\rho)$ of this type. If in addition, G is semisimple and simply connected of rank l, there are l basic irreducible representations $\rho_1, \ldots, \rho_l$, whose maximum weights $\lambda_1, \ldots, \lambda_l$ form a basis for the character group $\widehat{\mathbf{T}}$ of the maximal torus $\mathbf{T}$ of G, and the classes $\beta(\rho_1), \ldots, \beta(\rho_l)$ form a basis for the module of the primitive elements in $K^1(G)$ and $K^*(G) = \Lambda^*(\beta(\rho_1), \ldots, \beta(\rho_l))$. In any case, to compute $K_*(\mathcal{C}(G) \rtimes_\alpha \mathbb{Z})$ it is sufficient to determine the homotopy classes of $\alpha \circ \rho$ for the irreducible representations ρ of G in terms of $\beta(\rho)$'s.

(II) There is a relation between K-theory of the transformation group C^*-algebras of the homeomorphisms of the tori and the topological K-theory of compact nilmanifolds. In fact, let $\alpha = (t, \mathrm{A})$ be an affine transformation on $\mathbb{T}^n$ satisfying conditions (a) through (d) of Remark 5.2. Then it has been shown in [12] that α is conjugate (in the group of affine transformations on $\mathbb{T}^n$) to the transformation $\alpha' = (t', \mathrm{A}')$, where A' is an upper triangular matrix whose bottom right $k \times k$ corner is the identity matrix I_k, and $t' = (0, \ldots, 0, t'_1, \ldots, t'_k)$. The transformation α' is called a *standard form* for α [28]. Assume that α is given in standard form. Then Packer associates an *induced flow* $(\mathbb{R}, N/\Gamma)$ to the flow $(\mathbb{Z}, \mathbb{T}^n)$ generated by α, where N is a simply connected nilpotent Lie group of dimension $n+1$, the discrete group Γ is a cocompact subgroup of N, and the action of $\mathbb{R}$ is given by translation on the left by $\exp sX$ for $s \in \mathbb{R}$ and some $X \in \mathfrak{n}$, the Lie algebra of N. One of the most important facts is that the C^*-algebra $\mathcal{C}(N/\Gamma) \rtimes_\beta \mathbb{R}$ corresponding to the induced flow is strongly Morita equivalent to $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ [28, Proposition 3.1]. Consequently, one has

$$K_i(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong K_i(\mathcal{C}(N/\Gamma) \rtimes_\beta \mathbb{R}) \cong K^{1-i}(N/\Gamma), \quad i = 0, 1. \tag{9.1}$$

The second isomorphism here is Connes' Thom isomorphism. So the K-theory of $\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}$ is converted to the topological K-theory of the compact nilmanifold N/Γ. Following the proof of [28, Proposition 3.1], one can conclude that for the special case of Anzai transformations we can take $N = \mathfrak{F}_{n-1}$ (the generic filiform Lie group of dimension $n+1$) and $\Gamma = \mathfrak{D}_{n-1}$, which were defined in [33]. On the other hand, following [35, Theorem 3.6], one has the isomorphism

$$K_i(C^*(\Gamma)) \cong K^{i+n+1}(N/\Gamma), \quad i = 0, 1. \tag{9.2}$$

Combining (9.1) and (9.2) one gets

$$K_i(\mathcal{C}(\mathbb{T}^n) \rtimes_\alpha \mathbb{Z}) \cong K^{i+1}(N/\Gamma) \cong K_{i+n}(C^*(\Gamma)), \quad i = 0, 1.$$

Using the above isomorphisms, one can relate the algebraic invariants of the involved C^*-algebras and topological information of the corresponding nilmanifold. For example, since N/Γ is a classifying space for Γ, one has the isomorphisms

$$H^*_{\mathrm{dR}}(N/\Gamma) \cong \check{H}^*(N/\Gamma, \mathbb{R}) \cong H^*(\Gamma, \mathbb{R}) \cong H^*(N, \mathbb{R}) \cong H^*(\mathfrak{n}, \mathbb{R}), \tag{9.3}$$

where $H^*_{\mathrm{dR}}(N/\Gamma)$ denotes the de Rham cohomology of the manifold N/Γ, $H^*(N/\Gamma, \mathbb{R})$ denotes the Čech cohomology of N/Γ with coefficients in $\mathbb{R}$, $H^*(\Gamma, \mathbb{R})$

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denotes the group cohomology of Γ with coefficients in the trivial Γ-module $\mathbb{R}$, $H^*(N,\mathbb{R})$ denotes the Moore cohomology group of N (as a locally compact group) with coefficients in the trivial Polish N-module $\mathbb{R}$, and $H^*(\mathfrak{n},\mathbb{R})$ denotes the cohomology of the Lie algebra $\mathfrak{n}$ with coefficients in the trivial $\mathfrak{n}$-module $\mathbb{R}$. Now, using the Chern isomorphisms $\mathrm{ch}_0\colon K^0(N/\Gamma)\otimes\mathbb{Q}\to \check{H}^{\mathrm{even}}(N/\Gamma,\mathbb{Q})$ and $\mathrm{ch}_1\colon K^1(N/\Gamma)\otimes\mathbb{Q}\to \check{H}^{\mathrm{odd}}(N/\Gamma,\mathbb{Q})$, one concludes that the even and odd cohomology groups stated in (9.3) are all isomorphic to $\mathbb{R}^k$, where k is the (common) rank of the K-groups of $\mathcal{C}(\mathbb{T}^n)\rtimes_\alpha\mathbb{Z}$ as in Corollary 2.2. As an example, if $N=\mathfrak{F}_{n-1}$, $\Gamma=\mathfrak{D}_{n-1}$, and $\mathfrak{n}=\mathfrak{f}_{n-1}$, then the even and odd cohomology groups stated in (9.3) are all isomorphic to $\mathbb{R}^{a_n}$, where a_n is the rank of the K-groups of $\mathscr{A}_{n,\theta}$ that was studied in detail in Sections 5 and 6. Conversely, one may use the topological tools for N/Γ to get some information about $\mathcal{C}(\mathbb{T}^n)\rtimes_\alpha\mathbb{Z}$ and $C^*(\Gamma)$. For example, we know that N/Γ as a compact nilmanifold can be constructed as a principal $\mathbb{T}$-bundle over a lower dimensional compact nilmanifold [10]. Then we can compute the topological K-groups of N/Γ using the six term Gysin exact sequence [17, IV.1.13, p. 187]. As an example, one can see that $\mathfrak{F}_{n-1}/\mathfrak{D}_{n-1}$ is a principal $\mathbb{T}$-bundle over $\mathfrak{F}_{n-2}/\mathfrak{D}_{n-2}$, and the corresponding Gysin exact sequence is in fact the topological version of the Pimsner–Voiculescu exact sequence for the crossed product $\mathscr{A}_{n,\theta}\cong\mathscr{A}_{n-1,\theta}\rtimes_\alpha\mathbb{Z}$ as in [33, Theorem 2.1(d)].

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