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A complex Ruelle-Perron-Frobenius theorem and two counterexamples

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Abstract. In this paper a new proof of a theorem of Ruelle about real Perron-Frobenius type operators is given. This theorem is then extended to complex Perron-Frobenius type operators in analogy with Wielandt's theorem for matrices. Finally two questions raised by Ruelle and Bowen concerning analyticity properties of zeta functions for flows are answered.

0. *Introduction*

The operator $\mathscr{L}_f$ we shall be studying has its origins in statistical mechanics. In this context it is only necessary to consider its action on the space of real-valued functions (or interactions) of exponentially decreasing variation $\mathscr{F}_\theta$. Ruelle showed that the spectrum of $\mathscr{L}_f : \mathscr{F}_\theta \to \mathscr{F}_\theta$ $(f \in \mathscr{F}_\theta)$ satisfies a Perron-Frobenius type theorem (theorem 1) [**15**]. Subsequently other proofs of this, and other related results, have been developed [**20**], [**4**], ([**17**, p. 83]). In § 1, we present a new proof of the existence of a maximal eigenvalue for $\mathscr{L}_f$.

One major application of Ruelle's theorem is the construction of meromorphic extensions for certain generalized zeta functions [**16**].

It is the purpose of this paper to present a generalization of this theorem to describe the spectrum of $\mathscr{L}_f$ for complex functions of exponentially decreasing variation (theorem 2). This subsumes a complex version of the Perron-Frobenius theorem for matrices due to Wielandt (proposition 1). This new spectral theorem provides a more natural setting for the ingenious techniques developed by Ruelle ([**17**, pp. 93–95]), and enables us to produce extension results for zeta functions (theorem 3) subsuming those due to Ruelle [**16**] and Parry and the author ([**12**, theorem 1]).

In the last two sections we give examples which answer negatively questions raised by Ruelle ([**17**, p. 173]) and Bowen ([**1**, p. 31]). (During the writing of this paper the author discovered that the example in § 4 was known to Ruelle [**18**].)

This paper is an offshoot of the joint work of Parry and the author concerning asymptotic estimates for the number of closed orbits for Axiom A flows [**12**]. Parry has since derived other interesting estimates as a result of applying ideas from analytic number theory to the study of Axiom A flows [**11**].

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1. *The Ruelle operator theorem*

Let A be an aperiodic zero–one matrix of rank k and define

$$\Sigma_A^+ = \left\{ x \in \prod_0^\infty \{1, 2, \ldots, k\} \,\middle|\, A(x_n, x_{n+1}) = 1 \right\}.$$

The space Σ_A^+ is compact and zero dimensional with respect to the topology with basis consisting of sets of the form $\{x \in \Sigma_A^+ | x_i = z_i, 0 \le i \le n-1\}$.

The (*one-sided*) *shift of finite type* $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is the continuous map given by $(\sigma x)_n = x_{n+1}$. Since A is aperiodic σ is (topologically) mixing.

If $f : \Sigma_A^+ \to \mathbb{R}$ is continuous, the *pressure* is defined by

$$P(f) = \sup \left\{ h_\mu(\sigma) + \int f \, d\mu \,\middle|\, \mu \text{ is } \sigma\text{-invariant} \right\}.$$

This supremum is always attained and the measures for which $P(f) = h_\mu(\sigma) + \int f \, d\mu$ are called *equilibrium states* ([**21**, p. 224]).

Define $\text{var}_n f = \sup \{|f(x) - f(y)| \,|\, x_i = y_i, 0 \le i \le n-1\}$ then for $0 < \theta < 1$ let

$$\|f\|_\theta = \sup \left\{ \frac{\text{var}_n f}{\theta^n} \,\middle|\, n \ge 0 \right\}.$$

In this section our main interest will be in the real Banach space

$$\mathcal{F}_\theta = \{ f \in C(\Sigma_A^+) \,|\, \|f\|_\theta < \infty \}$$

with norm $|||f|||_\theta = \max \{ \|f\|_\infty, \|f\|_\theta \}$. Given $f \in \mathcal{F}_\theta$ define an operator $\mathcal{L}_f : \mathcal{F}_\theta \to \mathcal{F}_\theta$ by

$$\mathcal{L}_f h(z) = \sum_{\sigma y = z} \exp(f(y)) h(y).$$

We now present a proof of the Ruelle operator theorem which does not involve measures. The existence part was suggested by techniques employed by Krasnoselskii in [**9**] for a different problem. The rest of the proof is a combination of Ruelle ([**17**, p. 90]) and Walters [**20**].

THEOREM 1 (Ruelle). *Let σ be a topologically mixing one-sided shift of finite type and let $f \in \mathcal{F}_\theta$. The operator $\mathcal{L}_f$ has $e^{P(f)}$ as a simple eigenvalue (with a positive eigenvector). Furthermore the rest of the spectrum is contained in a disc of radius strictly smaller than $e^{P(f)}$.*

Proof. Let Λ denote the $\| \;\|_\infty$-closed set of positive continuous functions $g : \Sigma_A^+ \to \mathbb{R}^+$ with $\|g\|_\infty \le 1$ and

$$g(x) \le g(y) \exp(\theta^n \|f\|_\theta / 1 - \theta)$$

whenever $x_i = y_i$, $0 \le i \le n-1$. Since

$$|g(x) - g(y)| \le (\theta^n / (1-\theta)) \|f\|_\theta \exp(\|f\|_\theta / (1-\theta))$$

for such x and y, Λ is equicontinuous and therefore $\| \quad \|_\infty$-compact by the Arzela-Ascoli theorem.

The continuous map $L_n:\Lambda\to\Lambda$ given by

$$L_n g = \mathscr{L}_f(g+1/n)/\|\mathscr{L}_f(g+1/n)\|_\infty \qquad n\ge 0$$

has a fixed point h_n by the Schauder-Tychonoff theorem. If λ_n denotes $\|\mathscr{L}_f(h_n+1/n)\|_\infty$ then

$$\lambda_n h_n = \mathscr{L}_f(h_n+1/n) \ge (\inf h_n + 1/n)e^{-\|f\|_\infty} > 0,$$

and so $\lambda_n > e^{-\|f\|_\infty}$. If h is a limit point of $\{h_n\}$ then $\mathscr{L}_f h = \lambda h$, where $\lambda = \|\mathscr{L}_f h\|_\infty > 0$. The eigenfunction h is strictly positive since if $h(x)=0$ for some x then

$$\mathscr{L}_f^n h(x) = \sum_{\sigma^n y = x} \exp(f^n(y))h(y) = 0,$$

(where $f^n(y) = f(y)+f(\sigma y)+\cdots+f(\sigma^{n-1}y)$). This would make h zero on the dense set $\{y|\sigma^n y = x \text{ for some } n\ge 0\}$ contradicting $\lambda>0$.

The eigenvalue λ is simple since if $\mathscr{L}_f g = \lambda g$ and $t = \inf\{g/f\}$ then $(tf-g)(x)=0$ for some x. The preceding argument applied to $tf-g\ge 0$ shows that $tf=g$.

By replacing f by $f+\log h - \log h\circ\sigma - \log\lambda$ we may assume that $\mathscr{L}_f 1 = 1$. The general effect of this change is to scale the spectrum of $\mathscr{L}_f$ by λ.

Since A is aperiodic, $A^M>0$ for some M. Given z, x choose $y\in\{\sigma^{-n-M}z\}$ with $x_i = y_i$, $0\le i\le n-1$. Then

$$f^n(x) \le f^{n+M}(y) + M\|f\|_\infty + (\|f\|_\theta/(1-\theta))$$

and

$$\sum_{\sigma^n x = x} \exp(f^n(x)) \le k\cdot \sum_{\sigma^{n+M} y = z} \exp(f^{n+M}(y)) = k\cdot 1 \qquad (k>0)$$

Thus $P(f) = \lim_{n\to\infty}(1/n)\log\sum_{\sigma^n x = x}\exp(f^n(x)) = 0$, by use of the variational principle ([**21**, p. 218]). Thus for our original f, $\log\lambda = P(f)$. It is easy to show that (for some constant $C>0$)

$$\|\mathscr{L}_f^n g\|_\theta \le C\|g\|_\infty + \theta^n\|g\|_\theta \tag{1.1}$$

for all $g\in\mathscr{F}_\theta$, $n\ge 0$. This means that $\{\mathscr{L}_f^n g\}$ is equicontinuous with respect to $\| \quad \|_\infty$ and there exists a limit point l. If we write $\alpha(g) = \sup\{g(x)\}$ and $\beta(g) = \inf\{g(x)\}$ then

$$\alpha(g) \ge \alpha(\mathscr{L}_f g) \ge \cdots \ge \alpha(l) = \alpha(\mathscr{L}_f^n l) \qquad n\ge 0$$

and

$$\beta(g) \le \beta(\mathscr{L}_f g) \le \cdots \le \beta(l) = \beta(\mathscr{L}_f^n l) \qquad n\ge 0.$$

Since σ is mixing the equalities show l is a constant. Furthermore since $\alpha(l)=\beta(l)$ the sequence $\mathscr{L}_f^n g$ converges uniformly to l.

To remove the maximal eigenvalue consider $\mathscr{L}_f$ acting on the quotient space $\mathscr{F}_\theta/\mathbb{R}$. On this space (1.1) becomes

$$|\!|\!|\mathscr{L}_f^n g|\!|\!|_\theta \le C\cdot \operatorname{var}_0 g + \theta^n |\!|\!|g|\!|\!|_\theta.$$

Since $\operatorname{var}_0 \mathscr{L}_f^n g$ converges to zero we have for large n

$$\|\mathscr{L}_f^{2n} g\|_\theta \le C\cdot \operatorname{var}_0 \mathscr{L}_f^n g + \theta^n[C\cdot\operatorname{var}_0 g + \theta^n\|g\|_\theta] \tag{1.2}$$

By the uniform compactness of $\{g \mid |||g|||_\theta \leq 1\}$ we may choose n so that (1.2) holds for all g in this ball. Thus the spectral radius of $\mathscr{L}_f : \mathscr{F}_\theta/\mathbb{R} \to \mathscr{F}_\theta/\mathbb{R}$, denoted by $\rho(\mathscr{L}_f)$, satisfies

$$\rho(\mathscr{L}_f) = \inf\{\|\mathscr{L}_f^n\|_\theta^{1/n} \mid n \geq 0\} < 1.$$

This completes the proof. □

2. *A complex Ruelle operator theorem*

A continuous function $f: \Sigma_A^+ \to \mathbb{C}$ is called locally constant if there exists $n > 0$ such that $f(x)$ depends on only the first n places of x, i.e. $f(x) = f(y)$ whenever $x_i = y_i$, $0 \leq i \leq n-1$.

The operator $\mathscr{L}_f$ leaves invariant the finite dimensional subspace of $C(\Sigma_A^+, \mathbb{C})$ with base vectors

$$\delta_{x_0, \ldots, x_{n-2}}(x) = \begin{cases} 1 & x_i = z_i, 0 \leq i \leq n-2 \\ 0 & \text{otherwise.} \end{cases}$$

It is always possible to reduce locally constant functions to the case $n = 2$ by considering $x \in \Sigma_A^+$ as the sequence $(x_m, \ldots, x_{m+n-2})_{m=0}^{\infty}$ in a shift space whose symbols are words of length $n-1$.

For $n = 2$, $\mathscr{L}_f$ can be represented by a matrix

$$M = [A(x_0, x_1) \cdot \exp f(x_0, x_1)].$$

When f is real-valued the eigenvalues of M are described by the Perron–Frobenius theorem. More generally define M_+ to be the positive matrix with entries $|M(x_0, x_1)|$. The following result is due to Wielandt ([**6**, p. 57]).

PROPOSITION 1. *The eigenvalues of M have moduli less than or equal to the maximal eigenvalue β for M_+. If βe^{ia} is an eigenvalue of M (for some $0 \leq a < 2\pi$) then M takes the form $M = e^{ia} D M_+ D^{-1}$ where D is a diagonal matrix with diagonal entries of unit modulus.*

Ruelle's theorem (theorem 1) can be viewed as a generalization of the Perron-Frobenius theorem. In this section we present an analogous extension of Wielandt's result.

The space $\mathscr{F}_\theta^{\mathbb{C}} = \{f \in C(\Sigma_A^+, \mathbb{C}) \mid \|f\|_\theta < \infty\}$ is a complex Banach space with norm $|||f|||_\theta = \max\{\|f\|_\theta, \|f\|_\infty\}$. (Here $\|\cdot\|_\theta$ has the same definition as in § 1.) If $f = u + iv \in \mathscr{F}_\theta^{\mathbb{C}}$ then $u, v \in \mathscr{F}_\theta$ and we freely assume that $\mathscr{L}_u 1 = 1$, as in the proof of theorem 1.

For $g \in C(\Sigma_A^+)$ let Γ_g be the multiplicative group generated by $\langle \exp g^n(x) \mid \sigma^n x = x\rangle$ (where $g^n(x) = g(x) + g(\sigma x) + \cdots + g(\sigma^{n-1}x)$), [**13**], [**12**].

PROPOSITION 2. *For $f = u + iv \in \mathscr{F}_\theta^{\mathbb{C}}$ and $0 \leq a < 2\pi$ the following are equivalent:*

(i) *Γ_{v-a} is generated by a power of $e^{2\pi}$;*

(ii) *$\lambda_a \equiv e^{[ia + P(u)]}$ is an eigenvalue for $\mathscr{L}_f$;*

(iii) *there exists $\omega \in C(\Sigma_A^+)$ such that*

$$v - a + \omega \circ \sigma - \omega \in C(\Sigma_A^+, 2\pi\mathbb{Z}).$$

Proof. (i)⇒(ii) Choose $x \in \Sigma_A^+$ with a dense orbit and define h on $\{\sigma^n x | n \geq 0\}$ by $h(\sigma^n x) = \exp i(v-a)^n(x)$. This extends to an element of $\mathcal{F}_\theta^{\mathbb{C}}$, [**10**]. Furthermore h satisfies $h(\sigma z) = h(z) \exp i(v-a)(z)$ and consequently $\mathcal{L}_f h = e^{ia} h$.

(ii)⇒(iii) Since $\mathcal{L}_f h = e^{ia} h$ may be expressed as

$$\sum_{\sigma y = x} \exp u(y) \exp i(v-a)(y) h(y) = h(x) \tag{2.1}$$

if $|h(x)|$ is maximal then so is $|h(y)|$ when $\sigma y = x$. Because σ is mixing h is of constant modulus. Thus (2.1) represents a convex combination of points on a circle which also lies on the circle. From this we deduce

$$\exp i(v-a)(y) h(y) = h(\sigma y),$$

or equivalently $v - a + \arg h - \arg h \circ \sigma \in C(\Sigma_A^+, 2\pi\mathbb{Z})$.

(iii)⇒(i) This is immediate. □

If $f \in \mathcal{F}_\theta^{\mathbb{C}}$ satisfies one, and hence all, of the above conditions we call it an *a-function.* For example, the functions in $\mathcal{F}_\theta$ are all a-functions with $a = 0$. If f is not an a-function (for any a) then we call it *regular.*

If $f = u + iv$ where $v - a + \omega \circ \sigma - \omega \in C(\Sigma_A^+, 2\pi\mathbb{Z})$ then

$$\mathcal{L}_f = e^{ia} \Delta(e^{-i\omega}) \mathcal{L}_u \Delta(e^{i\omega})$$

where $\Delta(h)$ denotes the operator that multiplies functions by h. Therefore the spectrum of $\mathcal{L}_f$ is precisely the spectrum of $\mathcal{L}_u$ rotated through an angle a. By theorem 1:

PROPOSITION 3. *If $f = u + iv$ is an a-function then λ_a is a simple eigenvalue for $\mathcal{L}_f$ and the rest of the spectrum is contained in a disc of radius strictly smaller than $|\lambda_a| = e^{P(u)}$.*

An immediate corollary is that $f \in \mathcal{F}_\theta^{\mathbb{C}}$ can be an a-function for at most one a $(0 \leq a < 2\pi)$.

For any $f = u + iv$ (with $\mathcal{L}_u 1 = 1$) the following extension of (1.1) is true:

$$\|\mathcal{L}_f^n h\|_\theta \leq C \cdot \|h\|_\infty + \theta^n \|h\|_\theta \qquad n \geq 0, h \in \mathcal{F}_\theta^{\mathbb{C}}. \tag{2.2}$$

The operator norm satisfies $|||\mathcal{L}_f^n|||_\theta \leq C + 1$ and we have an upper bound on the spectral radius

$$\rho(\mathcal{L}_f) = \inf \{|||\mathcal{L}_f^n|||_\theta^{1/n} | n \geq 1\} \leq 1.$$

We now show that when f is regular the spectrum of $\mathcal{L}_f$ (denoted sp $(\mathcal{L}_f)$) is disjoint from the unit circle.

Choose a point e^{it} on the circle, then for $|||h|||_\theta \leq 1$ write

$$h_N = \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}_{f-it}^n h \qquad (N \geq 1).$$

By (2.2) h_N is contained in the uniformly compact set $\{g \,|\, |||g|||_\theta \leq C + 1\}$. When f is regular $\|h_N\|_\infty$ must tend to zero since any non-zero limit point of $\{h_N\}$ would be an eigenvector for $\mathcal{L}_f$ with eigenvalue e^{it}. For $k \geq 0$,

$$\|h_N\|_\theta \leq \|h_N - \mathcal{L}_{f-it}^k h_N\|_\theta + \|\mathcal{L}_{f-it}^k h_N\|_\theta,$$

where we have estimates

$$\|h_N - \mathcal{L}_{f-it}^k h_N\|_\theta = \left\| \frac{1}{N} \sum_{n=0}^{k-1} \mathcal{L}_{f-it}^n h - \mathcal{L}_{f-it}^{n+N} h \right\|_\theta$$
$$\leq (2k/N)(C+1),$$

and

$$\|\mathcal{L}_{f-it}^k h_N\|_\theta \leq C\|h_N\|_\infty + \theta^k \|h_N\|_\theta$$
$$\leq C \cdot \|h_N\|_\infty + \theta^k (C+1).$$

If we take $k = [N^{\frac{1}{2}}]$ then these bounds show that $\overline{\lim}\, \|h_N\|_\theta = 0$. By the uniform compactness of $\{h \,|\, |||h|||_\theta \leq 1\}$ we can choose N such that $|||h_N|||_\theta < 1$ for all h in this ball. Since the spectral radius of an operator is smaller than its norm

$$\rho\left(\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}_{f-it}^n\right) \leq \left|\left|\left| \frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}_{f-it}^n \right|\right|\right|_\theta < 1.$$

Thus 1 is not an element of $\operatorname{sp}((1/N)\sum_{n=0}^{N-1} \mathcal{L}_{f-it}^n)$. However we know from the spectral mapping theorem ([**19**, p. 263]) that

$$\operatorname{sp}\left(\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{L}_{f-it}^n\right) = \left\{ \frac{1}{N} \sum_{n=0}^{N-1} \lambda^n \,\middle|\, \lambda \in \operatorname{sp}(\mathcal{L}_{f-it}) \right\}.$$

Thus 1 cannot be an element of $\operatorname{sp}(\mathcal{L}_{f-it}) = e^{-it} \operatorname{sp}(\mathcal{L}_f)$, or equivalently e^{it} is not in the spectrum of $\mathcal{L}_f$. Since e^{it} was chosen arbitrarily and $\operatorname{sp}(\mathcal{L}_f)$ is closed we have the following.

THEOREM 2. *Let σ be a topologically mixing one-sided shift of finite type and let $f = u + iv \in \mathcal{F}_\theta^{\mathbb{C}}$.*

(i) *If f is an a-function then $\lambda_a = e^{[ia+P(u)]}$ is a simple eigenvalue for $\mathcal{L}_f$ and the rest of the spectrum is contained in a disc of radius strictly smaller than $|\lambda_a| = e^{P(u)}$.*

(ii) *If f is regular then the spectrum of $\mathcal{L}_f$ is contained in a disc of radius strictly smaller than $e^{P(u)}$.*

It is possible to formulate a proof of part (ii) closer to the proof of theorem 1 by proceeding along the lines of propositions 13 and 14 in [**12**].

An important feature of the above theorem is that the type of spectrum for $\mathcal{L}_f$ is determined by $\mathcal{I}(f)$ and the size of the spectrum is given by $\mathcal{R}(f)$.

We can obtain a *lower bound* for the spectral radius $\rho(\mathcal{L}_f)$ in the regular case. If $f = u + iv$ then $\mathcal{L}_u^n = \mathcal{L}_f^n \Delta(e^{-iv^n})$ and

$$e^{P(u)} = \rho(\mathcal{L}_u) \leq \rho(\mathcal{L}_f) \cdot \overline{\lim}\, |||\Delta(e^{-iv^n})|||_\theta^{1/n}.$$

It is simple to show $\overline{\lim}\, |||\Delta(e^{-iv^n})|||_\theta^{1/n} \leq 1/\theta$ and so $\rho(\mathcal{L}_f) \geq \theta \cdot e^{P(u)}$.

Proposition 2(i) shows that a necessary condition for $u + iv$ to be an a-function (for some a) is that Γ_v should be of rank at most two. But functions satisfying this condition can easily be approximated by functions which do not. This makes the family of regular functions dense in $\mathcal{F}_\theta^{\mathbb{C}}$. Furthermore, by theorem 2 and upper semicontinuity of $f \mapsto \rho(\mathcal{L}_f)$ this family is also open.

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3. *Extending the zeta function*
Given $f \in \mathcal{F}_\theta^{\mathbb{C}}$ define a zeta function by

$$\zeta(f) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma^m x = x} \exp f^m(x)$$

where $f^m(x) = f(x) + f(\sigma x) + \cdots + f(\sigma^{m-1}x)$. If $f = u + iv$ with $P(u) < 0$ then

$$\overline{\lim} \left| \sum_{\sigma^m x = x} \exp f^m(x) \right|^{1/m} \leq \overline{\lim} \left| \sum_{\sigma^m x = x} \exp u^m(x) \right|^{1/m} = e^{P(u)} < 1.$$

Since convergence is uniform in a neighbourhood of f it follows that ζ is non-zero and analytic on $\{f | \mathcal{R}(f) < 0\}$ ([**17**, p. 100]). We now consider the cases where $P(u) = 0$.

PROPOSITION 4. *If $f = u + iv \in \mathcal{F}_\theta^{\mathbb{C}}$ is regular with $P(u) = 0$ then ζ is analytic and non-zero in a neighbourhood of f.*

Proof. Choose $\theta' > \theta$, then $f \in \mathcal{F}_\theta^{\mathbb{C}} \subseteq \mathcal{F}_{\theta'}^{\mathbb{C}}$. For $n > 0$ define locally constant functions

$$u_n(x) = \sup\{u(z) | x_i = z_i, \quad 0 \leq i \leq n-1\}$$
$$v_n(x) = \sup\{v(z) | x_i = z_i, \quad 0 \leq i \leq n-1\}$$

and let $f_n(x) = u_n(x) + iv_n(x)$. This enables us to write

$$\zeta(f) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma^m x = x} \exp f^m(x) - \exp f_n^m(x)$$
$$\times \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma^m x = x} \exp f_n^m(x). \tag{3.1}$$

By applying theorem 2 to $\mathcal{L}_f : \mathcal{F}_{\theta'}^{\mathbb{C}} \to \mathcal{F}_{\theta'}^{\mathbb{C}}$, there exists $0 < \beta < 1$ such that spectral radius $\rho(\mathcal{L}_f) < \beta$. Since $g \mapsto \rho(\mathcal{L}_g)$ is upper semicontinuous on $\mathcal{F}_{\theta'}^{\mathbb{C}}$ and

$$|||f - f_n|||_{\theta'} \leq \|f\|_\theta (\theta/\theta')^n,$$

it follows that $\rho(\mathcal{L}_{f_n}) < \beta$ for large enough n. If ε is chosen sufficiently small then $\rho(\mathcal{L}_{g_n}) < \beta$ holds uniformly on $D = \{g \,|\, |||g - f|||_\theta < \varepsilon\}$.

Let $\lambda_1, \ldots, \lambda_{N(n)}$ be the eigenvalues of $\mathcal{L}_{f_n}$ acting on the finite dimensional invariant subspace of § 2 then $|\lambda_j| < \beta$, $1 \leq j \leq N(n)$. Furthermore

$$\sum_{\sigma^m x = x} \exp f_n^m(x) = \text{trace } \mathcal{L}_{f_n}^m = \lambda_1^m + \cdots + \lambda_{N(n)}^m$$

where $N(n) \leq k^n$, (k is the dimension of A). Choose α satisfying $\beta k^\alpha < 1$ and take $n = [m\alpha]$. Then

$$\left| \sum_{\sigma^m x = x} \exp f_n^m(x) \right|^{1/m} \leq (\beta^m k^n)^{1/m} \leq \beta k^\alpha < 1$$

for sufficiently large m. Since this holds uniformly on D

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma^m x = x} \exp g_n^m(x)$$

is analytic for $g \in D$. From the definitions of u_n and v_n

$$\|u_n^m - u^m\|_\infty \leq m \|u_n - u\|_\infty \leq m\theta^n \|u\|_\theta$$
$$\|v_n^m - v^m\|_\infty \leq m\theta^n \|v\|_\theta. \tag{3.2}$$

Since

$$\sum_{\sigma^m x = x} \exp f^m(x) - \exp f_n^m(x) = \sum_{\sigma^m x = x} [\exp u^m(x) - \exp u_n^m(x)] \exp iv^m(x) + \sum_{\sigma^m x = x} \exp u_n^m(x)[\exp iv^m(x) - \exp iv_n^m(x)]$$

it follows from (3.2) that

$$\overline{\lim} \left| \sum_{\sigma^m x = x} \exp f^m(x) - \exp f_n^m(x) \right|^{1/m} \leq \theta^\alpha e^{P(u)}.$$

If we choose $\theta^\alpha < \Phi < 1$ and ε sufficiently small then for large m

$$\left| \sum_{\sigma^m x = x} \exp g^m(x) - \exp g_n^m(x) \right|^{1/m} < \Phi$$

uniformly on D and so

$$\sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma^m x = x} \exp g^m(x) - \exp g_n^m(x)$$

is analytic on this disc. This completes the proof. □

When f is an a-function then $\rho(\mathscr{L}_f) = e^{P(\mathscr{R}(f))}$. However the isolated eigenvalue λ_a can be dealt with using perturbation theory. In a neighbourhood of f the operator $\mathscr{L}_g$ still has an isolated eigenvalue β ([**3**, p. 587]). This leads to a natural definition of the complex pressure (in a neighbourhood of an a-function) as $P(g) = \log \beta$.

By developing an approach due to Ruelle ([**17**, pp. 93–95]), Parry has proved the following result ([**11**, proposition 3]).

PROPOSITION 5. *If $f \in \mathscr{F}_\theta$ and $P(f) = 0$ then there exists $\varepsilon > 0$ such that P extends to an analytic function in*

$$D = \{g \mid |||f - g|||_\theta < \varepsilon\}$$

and

$$\sum_{m=1}^{\infty} (e^{iam}/m) \left(\sum_{\sigma^m x = x} \exp g^m(x) - e^{mP(g)} \right)$$

converges uniformly in D.

Propositions 4 and 5 together give the following result; (the version for two-sided shifts is theorem 1 in [**11**]).

THEOREM 3. *Let $f = u + iv \in \mathscr{F}_\theta^{\mathbb{C}}$.*

(i) *If $P(u) < 0$ or f is regular with $P(u) = 0$ then ζ is non-zero and analytic in a neighbourhood of f.*

(ii) *If f is an a-function with $P(u) = 0$ then ζ has a non-zero analytic extension to a set $\{g \mid |||g - f|||_\theta < \varepsilon,\ P(g) \neq 0\}$ given by*

$$\zeta(g) = \frac{1}{1 - e^{P(g)}} \exp \sum_{m=1}^{\infty} \frac{1}{m} \left(\sum_{\sigma^m x = x} \exp g^m(x) - e^{mP(g)} \right).$$

The above theorem extends a result of Ruelle [**16**], ([**17**, pp. 100–101]).

4. *A counter-example to Ruelle's question*

Let $\Sigma_A = \{x \in \prod_{-\infty}^{\infty}\{1,\ldots,k\} \mid A(x_n, x_{n+1}) = 1\}$ then $\sigma: \Sigma_A \to \Sigma_A$ given by $(\sigma x)_n = x_{n+1}$ is a (*two-sided*) *shift of finite type*. Let $f: \Sigma_A \to \mathbb{R}^+$ be a strictly positive continuous function for which there exists $0 < \theta < 1$, $C > 0$ satisfying $|f(x) - f(y)| \le C\theta^n$ whenever $x_i = y_i$, $|i| \le n-1$. Define

$$\Sigma_A^f = \{(x, t) \mid 0 \le t \le f(x)\},$$

where $(x, f(x))$ and $(\sigma x, 0)$ are identified. The *f suspension* $\sigma_t^f: \Sigma_A^f \to \Sigma_A^f$ is the flow defined by $\sigma_t^f(x, s) = (x, t+s)$ with appropriate identifications. Thus σ^f can be interpreted as flowing vertically under the graph of f. The flow σ^f is (*topologically*) *weak mixing* if the rank of $\Gamma_f = \langle \exp f^n(x) \mid \sigma^n x = x\rangle$ is greater than one [**12**]. The *topological entropy* of σ^f is the unique $h \in \mathbb{R}^+$ satisfying $P(-hf) = 0$ [**12**]. The zeta function associated with σ^f is

$$Z(s) = \zeta(-sf) = \exp \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\sigma^m x = x} \exp - sf^m(x).$$

In [**12**] Parry and the author partially answered a question of Ruelle ([**17**, p. 173]) by showing that $Z(s)$ has an analytic extension to a neighbourhood of $\{s \mid \mathcal{R}(s) \ge h\}$, except for a simple pole at $s = h$. We shall now complete this analysis by presenting a flow for which $Z(s)$ is not analytic on any strip $h - \delta < \mathcal{R}(s) < h$.

Let $\sigma: \Sigma_A \to \Sigma_A$ be a full shift on two symbols $\{1, 2\}$. Choose $0 < p < \frac{1}{2}$, $p + q = 1$, and define a locally constant function f by

$$f(x) = \begin{cases} -\log p & \text{if } x_0 = 1 \\ -\log q & \text{if } x_0 = 2. \end{cases}$$

If

$$P^s = \begin{pmatrix} p^s & q^s \\ p^s & q^s \end{pmatrix}$$

where $s \in \mathbb{C}$, then

$$\sum_{\sigma^m x = x} \exp - sf^m(x) = \operatorname{trace}(P^s)^m = (p^s + q^s)^m.$$

Thus

$$\begin{aligned} Z(s) &= \exp \sum_{m=1}^{\infty} \frac{1}{m}(p^s + q^s)^m \\ &= \exp(-\log(1 - p^s - q^s)) = 1/(1 - p^s - q^s), \end{aligned}$$

and the poles for $Z(s)$ are the solutions to $p^s + q^s = 1$. In particular the first part of the question shows $h = 1$ and for σ^f to be weak mixing we require $\log p/\log q$ to be irrational.

Let $\varepsilon > 0$ satisfy $p^{-\varepsilon} - q^{-\varepsilon} = 1$, then the poles are contained in the strip $-\varepsilon \le \mathcal{R}(s) \le 1$. If $-\varepsilon < \sigma < 1$, then zero is a limit point of $\{p^s + q^s - 1 \mid \mathcal{R}(s) = \sigma\}$. Since $p^s + q^s - 1$ is an analytic almost periodic function it has a zero in every vertical strip containing σ([**2**, p. 75]). We conclude that the poles $\{\sigma_n + it_n\}$ for Z are distributed with $\{\sigma_n\}$ dense in the interval $[-\varepsilon, 1]$. (In fact sharper estimates about the distribution of poles are possible (cf. [**8**])).

5. *A counter-example to Bowen's question*

In [**5**] Gallovotti gave an example of a suspension for which the corresponding zeta function has an essential singularity at $s_1 < 0$. Bowen asked whether the zeta function for flows could always be extended to $s = 0$ ([**1**, p. 31]). In this section we give an example where this is not the case. In fact it is possible to construct a suspension with an essential singularity at $s_0 > 0$.

Let $\sigma_n : \Sigma_n \to \Sigma_n$ be a full shift on n-symbols and let $\{\beta_k\}$ be a convergent sequence with limit β. For $n = 3$ define $g \in C(\Sigma_3)$ by

$$g(z) = \begin{cases} \beta_k & \text{if } z_k = 2,\ z_i \in \{1, 3\}, 0 \le i \le k-1 \\ \beta & \text{if } z_i \in \{1, 3\},\ i \ge 0. \end{cases}$$

Let $\sigma^m z = z$ and assume the cycle $(z_0, \ldots, z_{m-1})$ contains disjoint blocks of 1's and 3's of lengths $k_1, \ldots, k_r$ with $k_1 + \cdots + k_r = N$. Then

$$g^m(z) = (m - N)\beta_0 + \sum_{p=1}^{r} (\beta_1 + \cdots + \beta_{k_p}).$$

Thus $g^m(z)$ is independent of the 2^N possible combinations of 1's and 3's.

For $n = 2$ define $f \in C(\Sigma_2)$ by

$$f(x) = \begin{cases} \beta_k + \log 2 & \text{if } x_k = 2,\ x_i = 1 \\ & \quad 0 \le i \le k-1, \quad (k \ne 0) \\ \beta_0 & \text{if } x_0 = 2 \\ \beta + \log 2 & \text{if } x_i = 1,\ i \ge 0. \end{cases}$$

The functions f and g are related by

$$\sum_{\sigma^m x = x} \exp f^m(x) = \sum_{\sigma^m z = z} \exp g^m(z)$$

and so $\zeta(f) = \zeta(g)$.

The function f is similar to the Fisher potential used by Gallovotti [**5**].

Define a locally constant function f_N by replacing β_k by β for $k \ge N$. The zeta function $\zeta(f_N)$ can be calculated simply. Define

$$P_N = \begin{pmatrix} e^{\beta_0} & 2e^{\beta_1} & \cdots & 2e^{\beta_{N-1}} & 2e^{\beta} \\ e^{\beta_0} & & & & 0 \\ & 2e^{\beta_1} & & 0 & \vdots \\ & & & & 0 \\ 0 & & \ddots & 2e^{\beta_{N-1}} & 2e^{\beta} \end{pmatrix}$$

then by ([**14**, p. 82])

$$1/\zeta(f_N) = \det(I - P_N)$$

$$= (1 - 2e^{\beta})\left(1 - \sum_{n=0}^{N-1} 2^n e^{\beta_0 + \cdots + \beta_n}\right) - 2^N \cdot e^{\beta + \beta_0 + \cdots + \beta_{N-1}}.$$

Assume that $g \in \mathcal{F}_\theta$ and $g > 0$. By replacing β_k by $-s\beta_k$ (and β by $-s\beta$) we have from (5.1) and § 3

$$1/Z(s) = 1/\zeta(-sg) = (1-2e^{-s\beta})\left(1 - \sum_{n=0}^{\infty} 2^n e^{-s(\beta_0+\cdots+\beta_n)}\right) \quad (5.2)$$

(for $\mathcal{R}(s)$ large). In particular, for $2e^{-\mathcal{R}(s)\beta} < 1$ we have $\lim_{N\to\infty} 2^N e^{-s(\beta_0+\cdots+\beta_{N-1})} = 0$. Following Gallovotti we set

$$\beta_m = \begin{cases} -\log\left(\dfrac{1+\theta^m/m}{1+\theta^{m-1}/m-1}\right) + C & m \geq 2 \\ -\log(1+\theta) + C & m = 1 \\ C & m = 0 \end{cases}$$

and $\beta = C$ (where $C > 0$ is chosen to make $\beta_m > 0$). From (5.2)

$$1/Z(s) = (1-2e^{-sC})\left(1 - \tfrac{1}{2}\sum_{m=1}^{\infty} (1+\theta^m/m)^s (2e^{-sC})^{m+1} - e^{-sC}\right)$$

Thus the entropy of σ^g is the solution $h > 0$ to

$$1 = \tfrac{1}{2}\sum_{m=1}^{\infty} (1+\theta^m/m)^h (2e^{-hC})^{m+1} + e^{-hC}.$$

$Z(s)$ has a meromorphic extension to $s = h$ given by

$$1/Z(s) = (1-2e^{-sC})(1-e^{-sC}[F(s)+1]) - 2e^{-2sC},$$

$$F(s) = \sum_{m=1}^{\infty} (2e^{-sC})^m[(1+\theta^m/m)^s - 1].$$

For $0 < s \leq h$ there exist $B, D > 0$ such that

$$B \cdot s \cdot \theta^m/m \leq (1+\theta^m/m)^s - 1 \leq D \cdot s \cdot \theta^m/m.$$

Thus

$$B \cdot \log(1-2e^{-sC}\theta) \leq F(s)/s \leq D \cdot \log(1-2e^{-sC}\theta).$$

Consider $s_0 = 1/C \log 2\theta$. If $s_0 > 0$ (or equivalently $\theta > \frac{1}{2}$) then as s approaches s_0 from above $|F(s)|$ is unbounded but $(s-s_0)F(s)$ tends to zero. If $s_0 = 0$ (or equivalently $\theta = \frac{1}{2}$) then as s approaches zero from above $|sF(s)|$ is unbounded but $s^2F(s)$ tends to zero. We conclude that in either case s_0 is an essential singularity.

Remark. Hofbauer used the Fisher potential to produce examples of functions with two equilibrium states (one a single atom) [7]. The type of functions studied in this section give examples with two *non-atomic* equilibrium states (one with support a Cantor set).

Remark. Our example extends in a natural way to suspensions over Σ_n, $n > 3$. This enables us to give an example with an essential singularity s_0 arbitrarily close to $h(\sigma^g) = 1$.

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