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TOPOLOGICAL PROPERTIES OF THE SET OF NORM-ATTAINING LINEAR FUNCTIONALS

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ABSTRACT. If X is a separable non-reflexive Banach space, then the set NA of all norm-attaining elements of X^* is not a w^*-G_δ subset of X^*. However if the norm of X is locally uniformly rotund, then the set of norm attaining elements of norm one is w^*-G_δ. There exist separable spaces such that NA is a norm-Borel set of arbitrarily high class. If X is separable and non-reflexive, there exists an equivalent Gâteaux-smooth norm on X such that the set of all Gâteaux-derivatives is not norm-Borel.

1. **Introduction and examples.** Let X be a Banach space equipped with a norm $\|\cdot\|$. Let $S_X = \{x \in X : \|x\| = 1\}$. We denote

$$\mathrm{NA}(\|\cdot\|) = \{f \in X^* : f(x) = \|f\| \text{ for some } x \in S_X\}.$$

This set will also be denoted NA if there is no ambiguity on the norm. Similarly, we denote $\mathrm{NA}_1(\|\cdot\|) = \mathrm{NA}(\|\cdot\|) \cap S_{X^*}$.

Fundamental results of Bishop-Phelps [1] and James [4] assert that NA is always norm-dense X^*, and is equal to X^* exactly when X is reflexive. Since the set

$$F = \{(x,f) \in X \times X^* : \|x\|^2 = \|f\|^2 = f(x)\}$$

is closed in $(X, \|\cdot\|) \times (X^*, w^*)$, for all separable Banach spaces the set $\mathrm{NA}(\|\cdot\|) = \pi_2(F)$ is w^*-analytic in X^* [5]. It is shown in [5] that this statement is optimal in the sense that for any non-reflexive separable space X, there is an equivalent norm $\|\cdot\|$ such that $\mathrm{NA}(\|\cdot\|)$ is not norm-Borel.

In this work we conduct a further investigation of the topological properties of the set NA. In the simplest cases this set is w^*-F_σ. However (Proposition 1) it can be a Borel set of arbitrarily high class. Theorem 3 asserts that if X is separable and non-reflexive, the set NA is not w^*-G_δ. However (Theorem 9.1) if $\|\cdot\|$ is locally uniformly rotund (l.u.r.)—it is $x_n \to x$ whenever $\|x_n\| \to \|x\|$ and $\|\frac{x+x_n}{2}\| \to \|x\|$—then $\mathrm{NA}_1(\|\cdot\|)$ is w^*-G_δ, and $\mathrm{NA}(\|\cdot\|)$ is norm-G_δ. This shows in particular that one cannot "convexify" a norm without altering the structure of the set NA. However, it is possible to "smooth up" (in the Gâteaux sense) a norm without changing the set NA. It follows that there exists on any separable non-reflexive Banach space an equivalent Gâteaux smooth norm $\|\cdot\|$ such that the set $\mathrm{NA}_1(\|\cdot\|)$ of its Gâteaux derivatives is not norm-Borel (Theorem 9.4).

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For any set S we denote by $S^{<\omega}$ the set of all finite sequences of elements of S. The Cantor set $\{0,1\}^\omega$ is denoted $\mathbf{2}^\omega$. Let

$$\mathbb{Q} = \{\varepsilon \in \mathbf{2}^\omega : \exists i_0 \text{ s.t. } \forall i \geq i_0,\ \varepsilon(i) = 0\}.$$

We will frequently use the following easy consequence of Baire's theorem: if Z is a topological space, $\Phi\colon \mathbf{2}^\omega \longrightarrow Z$ is a continuous map, and $E \subseteq Z$ is such that $\Phi^{-1}(E) = \mathbb{Q}$, then E is not a G_δ subset of Z.

Before proceeding to the main results, let us present various examples.

EXAMPLES. 1) If the norm $\|\cdot\|$ of a separable space X is strictly convex, then $\mathrm{NA}(\|\cdot\|)$ is w^*-Borel [5]. It suffices indeed to observe, in the notation of the introduction, that $\mathrm{NA}(\|\cdot\|) = \pi_2(F)$ is the injective image of a countable union of Polish spaces.

2) If $X = \big(c_0(\mathbb{N}), \|\cdot\|_\infty\big)$, then NA is the set of all elements of $\ell_1(\mathbb{N})$ with finite support, and hence NA is w^*-F_σ but not norm-G_δ. For this latter fact we consider the map $\Phi\colon \mathbf{2}^\omega \longrightarrow \ell_1(\mathbb{N})$ defined by $\Phi(\varepsilon) = \big(2^{-i}\varepsilon(i)\big)$ and we observe that $\Phi^{-1}(\mathrm{NA}) = \mathbb{Q}$.

3) If $X = \big(\ell_1(\mathbb{N}), \|\cdot\|_1\big)$, then

$$\mathrm{NA} = \{u \in \ell_\infty(\mathbb{N}) : \exists n \geq 1 \text{ such that } \|u\|_\infty = |u(n)|\}$$

hence NA is w^*-F_σ. The map $\Phi\colon \mathbf{2}^\omega \longrightarrow \ell_\infty(\mathbb{N})$ defined by

$$\Phi(\varepsilon) = \sum_{i=1}^{+\infty} 2^{-i}\varepsilon(i)\mathbf{1}_{[i,+\infty)}$$

is such that $\Phi^{-1}(\mathrm{NA}) = \mathbb{Q}$, and thus NA is not norm-G_δ.

4) If $X = \big(C(K), \|\cdot\|_\infty\big)$ where K is metrizable and compact, we denote $\{O_n : n \geq 1\}$ a basis of the topology of K, and for all $n, k \geq 1$ we let

$$L_n^k = \{x \in O_n : d(x, K \setminus O_n) \geq k^{-1}\}$$

By Tietze's lemma, for all (n,k), (n',k') such that $L_n^k \cap L_{n'}^{k'} = \emptyset$, there is a continuous function in S_X which is 1 on L_n^k and (-1) on $L_{n'}^{k'}$. We denote by $\{f_\ell : \ell \geq 1\}$ the collection of these functions. It is clear that

$$\mathrm{NA} = \{\mu \in \mathcal{M}(K) : \exists \ell \geq 1 \text{ such that } \|\mu\| = \mu(f_\ell)\}$$

hence NA is w^*-F_σ. To check that NA is not norm-G_δ if K is infinite, we pick $\{k_n : n \geq 0\}$ a convergent sequence of distinct points, and we define $\Phi\colon \mathbf{2}^\omega \longrightarrow \mathcal{M}(K)$ by

$$\Phi(\varepsilon) = \sum_{i=0}^{+\infty} 2^{-i}\varepsilon(i)(\delta_{k_{2i}} - \delta_{k_{2i+1}})$$

we have again that $\Phi^{-1}(\mathrm{NA}) = \mathbb{Q}$.

5) We denote

$$B = \Big\{(x_n) \in c_0(\mathbb{N}) : \sum_{n=0}^{+\infty} x_n^{2n+2} \leq 1\Big\}.$$

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The set B is the unit ball of an equivalent strictly convex and C^∞-smooth norm on $c_0(\mathbb{N})$ ([3]; see [2], Theorem V.1.6). By differentiation, it is easily seen that $\Lambda = (\lambda_n) \in \mathrm{NA}$ if and only if there exist $\mu \in \mathbb{R}$, $a = (a_n) \in c_0(\mathbb{N})$ such that

$$\mu\lambda_n = (2n+2)a_n^{2n+1}$$

for all $n \geq 0$, and this is equivalent to

$$\lim_{n\to\infty} |\lambda_n|^{1/2n+1} = 0.$$

This latter condition implies (see [10]) that NA is a complete $F_{\sigma\delta}$-set.

We conclude this list of examples by showing that NA_1 can be a norm-Borel set of arbitrarily high class. We use the notation Σ^0_ξ (resp. Π^0_ξ) for the additive (resp. multiplicative) class of Borel subsets of order ξ (see [6]). With this notation one has: $\Sigma^0_2 = F_\sigma$ and $\Pi^0_2 = G_\delta$. In the sequel we shall deal with these notions when the dual space X^* is equipped with the w^*-topology, or with the norm topology which in general will not be separable.

Let Γ be some fixed Borel class; we denote by $\check{\Gamma}$ the class of all complements of sets in Γ (the dual class), and by $\Gamma \setminus \check{\Gamma}$ the class of all sets in Γ which are not in $\check{\Gamma}$. Let S be a subset of some arbitrary topological space Z; we shall say that S is Γ-complete in Z if for any Γ-subset A of ω^ω there exists a continuous mapping $\phi\colon \omega^\omega \longmapsto Z$ satisfying $\phi^{-1}(S) = A$. Notice that since there are $\Gamma \setminus \check{\Gamma}$ subsets in ω^ω, if S in Γ is Γ-complete in Z then necessarily S is a $\Gamma \setminus \check{\Gamma}$ subset of Z. Conversely by a theorem of Wadge ([13]) if Z is a Polish 0-dimensional space then any $\Gamma \setminus \check{\Gamma}$ subset of Z is Γ-complete.

We now are ready to prove the following result:

PROPOSITION 1. *Let $\xi \geq 2$ be a countable ordinal.*

(a) There exists a Banach space X such that $\mathrm{NA}(X)$ *is Borel in the w^*-topology and $\Sigma^0_\xi \setminus \Pi^0_\xi$ in the norm topology.*

(b) There exists a Banach space Y such that $\mathrm{NA}(Y)$ *is Borel in the w^*-topology and $\Sigma^0_\xi \setminus \Pi^0_\xi$ in the norm topology.*

PROOF. We first observe the simple

FACT 2. $\mathrm{NA}(\|\cdot\|) \in \Sigma^0_\xi$ (resp. Π^0_ξ) if and only if $\mathrm{NA}_1(\|\cdot\|) \in \Sigma^0_\xi$ (resp. Π^0_ξ).

We denote by $\mathbb{R}^+_*$ the open half-line $(0, +\infty)$. Define the map $\psi\colon (S_X, \|\cdot\|) \times \mathbb{R}^+_* \longrightarrow (X \setminus \{0\}, \|\cdot\|)$ by $\psi(x, \lambda) = \lambda x$. Fact 2 follows easily from the fact that ψ is a homeomorphism and that $\psi(\mathrm{NA}_1 \times \mathbb{R}^+_*) = \mathrm{NA} \setminus \{0\}$.

We now construct by transfinite induction spaces X and Y such that in the w^*-topologies $\mathrm{NA}(X)$ and $\mathrm{NA}(Y)$ are Borel, and in the norm topologies $\mathrm{NA}(X)$ is Σ^0_ξ-complete and $\mathrm{NA}(Y)$ is Π^0_ξ-complete. The conclusion of Proposition 1 will then follow from the previous remarks.

We start the construction for $\xi = 2$. By example 2) above, if $X = \big(c_0(\mathbb{N}), \|\cdot\|_\infty\big)$ then $\mathrm{NA}_1(\|\cdot\|)$ is Σ^0_2 ($= F_\sigma$) but not Π^0_2 ($= G_\delta$) and $\mathrm{NA}(\|\cdot\|)$ is w^*-F_σ. If Y is any space with a separable dual Y^* then Y has an equivalent l.u.r. norm $|\cdot|$ with l.u.r. dual norm

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(see [2], Theorem II.7.1). By Theorem 9 below, $\mathrm{NA}_1(|\cdot|)$ is Π^0_2. Since $|\cdot|^*$ is l.u.r., the w^* and norm topologies agree on S_{X^*}, hence $\mathrm{NA}_1(|\cdot|)$ is w^*-G_δ, and thus $\mathrm{NA}_1(|\cdot|)$ is not w^*-F_σ by Theorem 9, hence $\mathrm{NA}_1(|\cdot|)$ is not Σ^0_2 since again, the w^* and norm topologies agree on S_{X^*}. Thus $\mathrm{NA}_1(|\cdot|)$ is not Σ^0_2. Since Y^* is separable, any norm-Borel subset of Y^* is w^*-Borel, hence $\mathrm{NA}(|\cdot|)$ is w^*-Borel, and Π^0_2 in norm since $\mathrm{NA}_1(|\cdot|)$ is. Let us also observe that $\mathrm{NA}_1(X)$ is a $\Sigma^0_2 \setminus \Pi^0_2$ subset of a Polish space, and thus is Σ^0_2-complete. Similarly we see that $\mathrm{NA}_1(Y)$ is Π^0_2-complete.

We treat simultaneously successor and limit ordinals. If (ξ_n) is a sequence of ordinals with $\xi_{n+1} \geq \xi_n$ for all n, we let $\xi = \sup\{\xi_n + 1\}$. Let $(X_n, \|\cdot\|_n)$ be such that $\mathrm{NA}(\|\cdot\|_n)$ is w^*-Borel and $\Sigma^0_{\xi_n}$, and $\mathrm{NA}_1(\|\cdot\|_n)$ is $\Sigma^0_{\xi_n}$-complete for all n. We let

$$Y = \Big(\sum \oplus (X_n, \|\cdot\|_n)\Big)_2.$$

It is easily seen that $f = (f_n) \in \mathrm{NA}(Y)$ if and only if $f_n \in \mathrm{NA}(X_n)$ for all n. It follows that $\mathrm{NA}(Y)$ is w^*-Borel and Π^0_ξ. Moreover for all $\Sigma^0_{\xi_n}$ subsets A_n of ω^ω, there exists $\varphi_n\colon \omega^\omega \longrightarrow S_{X_n^*}$ continuous such that $\varphi_n^{-1}\big(\mathrm{NA}(X_n)\big) = A_n$. If we define

$$\begin{aligned}\Phi\colon \omega^\omega &\longrightarrow (S^*_Y, \|\cdot\|)\\ x &\longmapsto \big(2^{-n}\varphi_n(x)\big)_{n\geq 1}\end{aligned}$$

then Φ is continuous and

$$\Phi^{-1}\big(\mathrm{NA}_1(Y)\big) = \bigcap_{n\geq 1} A_n.$$

Thus $\mathrm{NA}_1(Y)$ is Π^0_ξ-complete.

If now the Y_n's are such that $\mathrm{NA}(Y_n)$ is w^*-Borel and $\Pi^0_{\xi_n}$, and $\mathrm{NA}_1(Y_n)$ is $\Pi^0_{\xi_n}$-complete, we let

$$X = \Big(\sum \oplus (Y_n, \|\cdot\|_n)\Big)_1.$$

It is easily checked that $f = (f_n) \in \mathrm{NA}(X)$ if and only if there exists $n \geq 1$ such that $f_n \in \mathrm{NA}(Y_n)$ and $\|f_n\|_n = \sup\{\|f_k\|_k : k \geq 1\}$. It follows that $\mathrm{NA}(X)$ is w^*-Borel and Σ^0_ξ. Moreover if B_n is a $\Pi^0_{\xi_n}$ subset of ω^ω, there exists $\varphi_n\colon \omega^\omega \longrightarrow S_{Y_n^*}$ continuous such that $\psi^{-1}\big(\mathrm{NA}_1(Y_n)\big) = B_n$. Now

$$\Psi = (\psi_n)\colon \omega^\omega \longrightarrow S_{X^*}.$$

is such that $\Psi^{-1}\big(\mathrm{NA}_1(X)\big) = \bigcup_{n\geq 1} B_n$. Hence $\mathrm{NA}_1(X)$ is Σ^0_ξ-complete. ■

2. **Main results.** The following statement is the main result of this paper. It answers an implicit question from [5].

THEOREM 3. *Let X be a separable non-reflexive Banach space. Then the set* NA *of all elements of X^* which attain their norm is not a w^*-G_δ subset of X^*.*

PROOF. We will make use of some classical arguments from Pryce's proof [9] of James' theorem, which we recall for completeness.

FACT 4. Pick $\delta \in (0,1)$. There exist (f_n) in B_{X^*}, (x_j) in B_X, such that
(i) For every $n \geq 1$, $\lim_j f_n(x_j) > \delta$
(ii) $w^* - \lim_n(f_n) = 0$.

PROOF. Since X is not reflexive we may pick $h \in X^\perp \subset X^{***}$ with $\|h\| = 1$, and then $z \in X^{**}$ with $\|z\| \leq 1$ and $h(z) > \delta$. If $D = \{f \in B_{X^*} : f(z) > \delta\}$, h belongs to the w^*-closure of D in X^{***}. Moreover if $f_\alpha \xrightarrow{w^*} h$ in X^{***}, then $f_\alpha \xrightarrow{w^*} 0$ in X^* since $h \in X^\perp$. Finally, z can be approximated pointwise on X^* by elements of X. An easy inductive constructive now leads to the conclusion.

FACT 5. Let $C = \operatorname{conv}\{f_n : n \geq 1\}$. For every $f \in C$, $\|f\| > \delta$.
Indeed pick a w^*-cluster point t of the x_j's. We have $\|t\| \leq 1$ and $t(f) > \delta$ for all $f \in C$.

FACT 6. Let V be a vector space, $u, v \in V$, $\alpha, \beta > 0$, and $\varphi = V \longrightarrow \mathbb{R}$ a convex function. Let $w = (\alpha+\beta)^{-1}(\alpha u + \beta v)$. Then

$$\beta^{-1}[\varphi(\alpha u + \beta v) - \varphi(\alpha u)] \geq \alpha^{-1}[\varphi(\alpha w) - \varphi(0)] + \beta^{-1}[\varphi(\alpha w) - \varphi(\alpha u)].$$

PROOF. Since $(\alpha+\beta)w = \alpha u + \beta v$, we have

$$\alpha w = \frac{\alpha}{\alpha+\beta}(\alpha u + \beta v).$$

Using the convexity of φ between $(\alpha u + \beta v)$ and 0 we get

$$\varphi(\alpha w) \leq \frac{\alpha}{\alpha+\beta}\varphi(\alpha u + \beta v) + \frac{\beta}{\alpha+\beta}\varphi(0)$$

hence after multiplication by $\frac{\alpha+\beta}{\alpha\beta}$

$$\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\varphi(\alpha w) \leq \frac{1}{\beta}\varphi(\alpha u + \beta v) + \frac{1}{\alpha}\varphi(0).$$

The conclusion follows after subtraction of $\beta^{-1}\varphi(\alpha u)$ and reorganization.

FACT 7. With the above notation, if A is a convex subset of V and

$$\inf_{z \in A} \alpha^{-1}[\varphi(\alpha z) - \varphi(0)] > \delta$$

then there is $u \in A$ such that

$$\inf_{v \in A} \beta^{-1}[\varphi(\alpha u + \beta v) - \varphi(\alpha u)] > \delta.$$

Moreover if V is a topological vector space and φ is continuous, we may pick u from any prescribed dense subset of A.

PROOF. Indeed pick $\varepsilon > 0$ such that

$$\inf_{z \in A} \alpha^{-1}[\varphi(\alpha z) - \varphi(0)] > \delta + \varepsilon$$

by the definition of the infimum, there is $u \in A$ such that

$$\inf_{z\in A} \beta^{-1}[\varphi(\alpha z) - \varphi(\alpha u)] > -\varepsilon$$

and if φ is continuous this u may be found within a prescribed dense subset. Fact 6 concludes the proof, since $w = \frac{\alpha u+\beta v}{\alpha+\beta}$ belongs to the convex set A whenever u and v do.

FACT 8. Let A be a norm-open convex subset of X^*, let $\alpha_0, \alpha_1, \dots, \alpha_{n+1} > 0$ and let $g_0, g_1, \dots, g_{n-1}$ in X^* be such that

$$\inf_{g\in A}\left\{\left\|\sum_{k=0}^{n-1} \alpha_k g_k + \alpha_n g\right\| - \left\|\sum_{k=0}^{n-1} \alpha_k g_k\right\|\right\} > \alpha_n \delta.$$

Then there exists $g_n \in A$ such that

(i) $\inf_{g\in A}\{\|\sum_{k=0}^{n} \alpha_k g_k + \alpha_{n+1} g\| - \|\sum_{k=0}^{n} \alpha_k g_k\|\} > \alpha_{n+1}\delta$,

(ii) $\sum_{k=0}^{n} \alpha_k g_k \in \mathrm{NA}$.

PROOF. We define $\varphi: A \longrightarrow \mathbb{R}$ by $\varphi(g) = \|g + \sum_{k=0}^{n-1} \alpha_k g_k\|$. The function φ is convex and continuous on A and by assumption

$$\inf_{g\in A} \alpha_n^{-1}[\varphi(\alpha_n g) - \varphi(0)] > \delta.$$

By Bishop-Phelps' theorem (see [2], Theorem 1.3.1), the set

$$D = A \cap \left\{-\alpha_n^{-1}\left(\sum_{k=0}^{n-1} \alpha_k g_k\right) + \mathrm{NA}(\|\cdot\|)\right\}$$

is norm-dense in A, and thus by Fact 7 we can find $g_n \in D$ such that

$$\inf_{g\in A} \alpha_{n+1}^{-1}[\varphi(\alpha_n g_n + \alpha_{n+1} g) - \varphi(\alpha_n g_n)] > \delta$$

and clearly g_n satisfies (i) and (ii).

We now proceed to the proof of Theorem 3. Let (f_n) be the sequence in B_{X^*} provided by Fact 4. We fix a sequence (α_n) of positive numbers such that

$$\lim_{n+\infty} \alpha_n^{-1}\left(\sum_{k=n+1}^{+\infty} \alpha_k\right) = 0 \tag{1}$$

and for all $p \geq 1$ we let

$$A_p = \mathrm{conv}\{f_{p+k} : k \geq 0\} + 2^{-p} B_{X^*}.$$

For showing that NA is not a w^*-G_δ set, it suffices to construct a continuous map Φ: $\mathbf{2}^\omega \longrightarrow (X^*, w^*)$ such that $\Phi^{-1}(\mathrm{NA}) = \mathbb{Q}$.

For any $s \in 2^{<\omega}$,

$$\|s\| = \sum_{i\in \mathrm{Dom}(s)} s(i)$$

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and $s^* \in \omega^{<\omega}$ be the increasing enumeration of $\{i \in \mathrm{Dom}(s) : s(i) = 1\}$. Clearly, s^* has length $\|s\|$. We now define a map

$$G\colon 2^{<\omega} \longrightarrow (X^*)^{<\omega}$$

such that for any $s \in 2^{<\omega}$ the sequence $G(s) = (g_k^{(s)})_{k<\|s\|}$ is of length $\|s\|$, and such that the following conditions are satisfied

(i) $s \prec t \Rightarrow G(s) \prec G(t)$,

(ii) $g_k^{(s)} \in A_{s^*(k)}$ for all $k, 0 \le k < \|s\|$,

(iii) $h_s = \sum_{k=0}^{\|s\|-1} \alpha_k g_k^{(s)} \in \mathrm{NA}$,

(iv) $\inf_{g\in A_{\ell(s)}}\{\|h_s + \alpha_{\|s\|} g\| - \|h_s\|\} > \alpha_{\|s\|}\delta$ with $\ell(s) = s^*(\|s\|-1)+1$.

It follows from Facts 5 and 8 that such a construction can be completed. We finally define $\Phi\colon \mathbf{2}^\omega \longrightarrow X^*$ by

$$\Phi(\varepsilon) = w^* - \lim_n h_{\varepsilon\restriction n}.$$

It is easily seen that Φ is w^*-continuous (and even norm-continuous at every $\varepsilon \notin \mathbb{Q}$). If $\varepsilon \in \mathbb{Q}$ there is $s \in 2^{<\omega}$ such that $\Phi(\varepsilon) = h_s$ and thus by condition (iii), $\Phi(\varepsilon) \in \mathrm{NA}$. We claim that if $\varepsilon \notin \mathbb{Q}$ then $\Phi(\varepsilon) \notin \mathrm{NA}$. Indeed by (i) and (ii) we may write

$$\Phi(\varepsilon) = \sum_{n=0}^{+\infty} \alpha_n g_n$$

where $g_n \in A_{p_n}$ for all $n \ge 0$, with $\lim p_n = +\infty$. By condition (iv) we have for all $n > 0$,

$$\Big\|\sum_{k=0}^{n-1} \alpha_k g_k + \alpha_n g_n\Big\| > \delta\alpha_n + \Big\|\sum_{k=0}^{n-1} \alpha_k g_k\Big\|.$$

By (1) we have

$$\Big\|\sum_{k=n+1}^{+\infty} \alpha_k g_k\Big\| = o(\alpha_n).$$

If there exists $x \in X$ with $\|x\| = 1$ and $\Phi(\varepsilon)(x) = \|\Phi(\varepsilon)\|$, we may write

$$\begin{aligned}\Phi(\varepsilon)(x) &= \Big\|\sum_{k=0}^{n} \alpha_k g_k\Big\| + o(\alpha_n)\\ &> \delta\alpha_n + o(\alpha_n) + \Big\|\sum_{k=0}^{n-1} \alpha_k g_k\Big\|\\ &\ge \delta\alpha_n + o(\alpha_n) + \sum_{k=0}^{n-1} \alpha_k g_k(x).\end{aligned}$$

It follows that

$$\liminf g_n(x) \ge \delta$$

but since $g_n \in A_{p_n}$ with $\lim p_n = +\infty$, we have $\lim g_n(x) = 0$, and this contradiction concludes the proof. ■

We noticed in Example 1 that $\mathrm{NA}(\|\cdot\|)$ is w^*-Borel when $\|\cdot\|$ is strictly convex. We will see now that various convexity assumptions provide sharper conclusions. However it is not so for smoothness assumptions.

THEOREM 9. *Let* $(X, \|\cdot\|)$ *be a Banach space. Then*

1) *If* $\|\cdot\|$ *is locally uniformly rotund (*l. u. r.*), then* $\mathrm{NA}_1(\|\cdot\|)$ *is a* w^**-*G_δ *subset of* S_{X^*}*, and* $\mathrm{NA}(\|\cdot\|)$ *is a norm-*G_δ *subset of* X^*.
2) *If X is separable and non-reflexive, then* NA_1 *is not both a* w^**-*G_δ *and* w^**-*F_σ *subset of* S_{X^*}.
3) *If X is separable and the dual norm* $\|\cdot\|^*$ *is Gâteaux-differentiable, then* $\mathrm{NA}(\|\cdot\|)$ *is norm-*$F_{\sigma\delta}$.
4) *If X is separable and non-reflexive, there exists a Gâteaux-differentiable equivalent norm* $\|\cdot\|$ *on X, such that* $\mathrm{NA}_1(\|\cdot\|)$ *is not norm-Borel.*

PROOF. 1) We start with a statement of independent interest.

LEMMA 10. *Let* $(X, \|\cdot\|)$ *be a Banach space. The following are equivalent:*

a) $\|\cdot\|$ *is* l. u. r.
b) *There exists* $\sigma\colon \mathrm{NA}_1(\|\cdot\|) \longrightarrow S_X$ *which is* w^**-to-norm continuous, and such that* $\langle f, \sigma(f)\rangle = 1$ *for all* $f \in \mathrm{NA}_1(\|\cdot\|)$.

PROOF OF LEMMA 10. a) $\Rightarrow$ b): Since $\|\cdot\|$ is in particular strictly convex, every $f \in \mathrm{NA}_1(\|\cdot\|)$ attains its norm in a unique $x \in S_X$ and this determines $\sigma(f)$. For a given $\varepsilon < 0$, there is a $\delta > 0$ such that

$$\|y\| \leq 1, \quad \|\sigma(f)+y\| > 2-\delta \Rightarrow \|\sigma(f)-y\| < \varepsilon.$$

If $g \in \mathrm{NA}_1(\|\cdot\|)$ satisfies $g\big(\sigma(f)\big) > 1-\delta$, we have

$$g\big(\sigma(f)+\sigma(g)\big) > 2-\delta$$

and thus $\|\sigma(f)-\sigma(g)\| < \varepsilon$. Hence σ is $(w^* - \|\cdot\|)$-continuous.

b) $\Rightarrow$ a): Note first that if there exists such a map σ which is only norm-to-norm continuous then the norm $\|\cdot\|$ is strictly convex. Indeed, since $\mathrm{NA}_1(\|\cdot\|)$ is norm-dense in S_{X^*}, we may extend σ to $\tilde{\sigma}\colon S_{X^*} \longrightarrow S_{X^{**}}$ by taking $\tilde{\sigma}(f)$ $\big(f \in S_{X^*} \setminus \mathrm{NA}_1(\|\cdot\|)\big)$ a w^*-cluster point in X^{**} of $\sigma(g)$ $\big(g \in \mathrm{NA}_1(\|\cdot\|), \|g-f\| \longrightarrow 0\big)$. Then

$$\begin{aligned}|\langle\tilde{\sigma}(f),f\rangle - 1| &\leq |\langle\tilde{\sigma}(f)-\sigma(g),f\rangle| + |\langle\sigma(g),f-g\rangle| \\ &\leq |\langle\tilde{\sigma}(f)-\sigma(g),f\rangle| + \|f-g\|\end{aligned}$$

which is less than ε if g is chosen in $\mathrm{NA}_1(\|\cdot\|)$ such that $\|f-g\| < \frac{\varepsilon}{2}$ and $|\langle\tilde{\sigma}(f)-\sigma(g),f\rangle| < \frac{\varepsilon}{2}$. Thus $\langle\tilde{\sigma}(f),f\rangle = 1$ for any f any S_{X^*}.

Since the bidual norm is w^*-l.s.c., $\tilde{\sigma}$ is still norm-to-norm continuous at all points of $\mathrm{NA}_1(\|\cdot\|)$. Indeed, if $f \in \mathrm{NA}_1(\|\cdot\|)$, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$g \in \mathrm{NA}_1(\|\cdot\|) \quad \text{and} \quad \|g-f\| > \delta \Rightarrow \|\sigma(g)-\sigma(f)\| \leq \varepsilon.$$

Then, for any $f_0 \in S_{X^*}$ such that $\|f-f_0\| < \delta$, $\tilde{\sigma}(f_0)$ is w^*-cluster point of points $\sigma(g)$ lying in $\sigma(f)+\varepsilon\cdot B_{X^{**}}$. Thus $\tilde{\sigma}(f_0) \in \sigma(f)+\varepsilon\cdot B_{X^{**}}$.

And thus (see [2], Lemma I.4.13) the dual norm is Fréchet-smooth at these points. Now Smulyan's lemma (see [2], Theorem I.1.4) shows that all $x \in S_X$ are strongly exposed and *a fortiori* $\|\cdot\|$ is strictly convex. If $\|\cdot\|$ is not l.u.r. there exist $x \in S_X$, $(x_n) \subset S_X$ and $\varepsilon > 0$ such that $\lim \|x+x_n\| = 2$ and $\|x-x_n\| \geq \varepsilon$ for all n. Let $f_n \in S_{X^*}$ be such that

$$f_n(x+x_n) = \|x+x_n\|.$$

Since $\lim f_n(x+x_n) = 2$, we have $\lim f_n(x) = 1$, hence any w^*-cluster point f of $\{f_n\}$ satisfies $f(x) = 1$. If we let

$$y_n = \frac{x+x_n}{\|x+x_n\|}$$

we have $f_n(y_n) = \|y_n\| = 1$. Since $\|\cdot\|$ is strictly convex, $\sigma(f) = x$ and $\sigma(f_n) = y_n$. But for all n, $\|x - y_n\| \geq \varepsilon/2$, and this contradicts the w^*-to-norm continuity of σ. ∎

We now come back to the proof of 1). We use the notation of Lemma 10. Since $\mathrm{NA}_1(\|\cdot\|)$ is w^*-dense in B_{X^*}, σ has a continuous extension $\bar{\sigma}$ to a w^*-G_δ subset Ω of B_{X^*}. Indeed if E is a topological space, (M,d) a complete metric space, and $\sigma = D \longrightarrow M$ is a continuous map from a dense subset D of E to M, then σ can be extended to $\Omega = \bigcap_{n\geq 1} O_n$, where O_n is the union of all open subsets V of E such that

$$\sup\bigl\{d\bigl(\sigma(x),\sigma(y)\bigr) : x,y \in V \cap D\bigr\} < n^{-1}.$$

Indeed if $x \in \Omega$, it suffices to let

$$\bar{\sigma}(x) = \lim_{\substack{y \to x \\ y \in D}} \sigma(y)$$

since this limit exists by definition of Ω.

We observe now that $\langle f, \bar{\sigma}(f)\rangle = 1 = \|\bar{\sigma}(f)\|$ for all $f \in \Omega$. It follows that $\Omega \cap S_{X^*} = \mathrm{NA}_1(\|\cdot\|)$ and thus $\mathrm{NA}_1(\|\cdot\|)$ is w^*-G_δ in S_{X^*}.

Since $\mathrm{NA}_1(\|\cdot\|)$ is w^*-G_δ in S_{X^*}, it is *a fortiori* norm-G_δ, hence by Fact 2 $\mathrm{NA}(\|\cdot\|)$ is norm-G_δ as well. This shows 1). ∎

2) For any Banach space X, S_{X^*} is a G_δ-subset of the compact set (B_{X^*}, w^*) and thus (S_{X^*}, w^*) is a Baire space. Hence 2) follows from Baire's theorem and the following.

LEMMA 11. *Let X be a separable non-reflexive space. The set* NA_1 *has an empty interior in* (S_{X^*}, w^*).

PROOF OF LEMMA 11. Let $V \neq \emptyset$ be a w^*-open subset of S_{X^*}. It is easy to construct a convex w^*-open subset U of B_{X^*} such that for all $g \in \bar{U}^{w^*}$, $(\|g\|^{-1})g \in V$. We will localize to U the construction of the proof of Theorem 3.

There is $f \in U$ with $\|f\| = 1 - \eta < 1$. Pick $t \in (f + X^\perp) \cap S_{X^{***}}$. It is easily seen that t belongs to the w^*-closure of U in X^{***}. It follows that there exists a sequence (f_n) in U such that

$$\begin{cases} f = w^* - \lim_{n\to\infty} f_n \text{ in } (X^*, w^*) \\ \|g\| > 1 - \eta/2 \text{ for all } g \in \mathrm{conv}\{f_n : n \geq 1\}. \end{cases}$$

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These conditions, and Simons' inequality ([11]; see [2], Lemma I.3.7) show that there exists $\lambda_n \geq 0$, with $\sum_{n=1}^{+\infty} \lambda_n = 1$, such that

$$g = \sum_{n=1}^{+\infty} \lambda_n f_n \notin \mathrm{NA}$$

and we have $(\|g\|^{-1})g \in V \setminus \mathrm{NA}_1$. This proves Lemma 11, and 2). ∎

3) Since $\|\cdot\|^*$ is Gâteaux-differentiable, the map $J: X^* \longrightarrow X^{**}$ defined for all $f \in X^*$ by

$$\|J(f)\|^2 = \|f\|^2 = \langle J(f), f\rangle$$

is norm-to-w^* continuous, and $\mathrm{NA}(\|\cdot\|) = J^{-1}(X)$. If (x_n) is a dense sequence in X, we can write

$$X = \bigcap_{k=1}^{+\infty} \bigcup_{n=1}^{+\infty} B_{X^{**}}(x_n, k^{-1})$$

thus X is $w^* - K_{\sigma\delta}$. Hence $\mathrm{NA}(\|\cdot\|)$ is norm-$F_{\sigma\delta}$ in X^*. ∎

4) By [5], there is an equivalent norm $|\cdot|$ on X such that $\mathrm{NA}(|\cdot|)$ is not norm-Borel. Let $\{x_n\}$ be a dense subset of B_X. We define T: $\ell_2(\mathbb{N}) \longrightarrow X$ by

$$T(\alpha) = \sum_{n=1}^{+\infty} 2^{-n} \alpha_n x_n$$

and we let $K = T(B_{\ell_2})$. The set K is convex symmetric and norm-compact. Let $\|\cdot\|$ be the norm whose unit ball satisfies

$$B_X(\|\cdot\|) = B_X(|\cdot|) + K.$$

Since K is compact, we clearly have

$$\mathrm{NA}(\|\cdot\|) = \mathrm{NA}(|\cdot|)$$

and thus $\mathrm{NA}(\|\cdot\|)$ is not norm-Borel. Since $X \setminus \{0\}$ is homeomorphic to $(S_X \times \mathbb{R}_*^+)$ through $x \longmapsto (\|x\|^{-1}x, \|x\|)$ it follows that $\mathrm{NA}_1(\|\cdot\|)$ is not norm-Borel. We now compute the dual norm $\|f\|^*$ of $f \in X^*$. By definition

$$\begin{aligned}
\|f\|^* &= \sup\{|f(x+x')| : |x| \leq 1, x' \in K\} \\
&= \sup\{|f(x)| : |x| \leq 1\} + \sup\{|f(x')| : x' \in K\} \\
&= |f|^* + \sup\{\left|f\big(T(y)\big)\right| : y \in B_{\ell_2}\} \\
&= |f|^* + \|T^*(f)\|_2.
\end{aligned}$$

Since T^* is one-to-one and $\|\cdot\|_2$ is strictly convex, it follows that $\|\cdot\|^*$ is strictly convex, and thus $\|\cdot\|$ is Gâteaux-smooth. ∎

REMARKS. 1) It follows classically from Smulyan's lemma (see [2], Theorem I.1.4) that if $\|\cdot\|$ is l.u.r. then $\mathrm{NA}(\|\cdot\|)$ is exactly the set of points where $\|\cdot\|^*$ is Fréchet-smooth. This gives an alternative proof of the fact that $\mathrm{NA}(\|\cdot\|)$ is norm-G_δ and in fact (by [8]) a special kind of norm-G_δ, since its complement is "porous".

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2) The proof of Lemma 10 shows that there exists $\sigma\colon \mathrm{NA}_1(\|\cdot\|) \longrightarrow S_X$ norm-to-norm continuous such that $\langle f, \sigma(f)\rangle = 1$ for all $f \in \mathrm{NA}_1$ if and only if every $x \in S_X$ is strongly exposed in B_X. Note that it follows from [12] (see [2], Theorem IV.3.5) that such a norm has an equivalent l.u.r. norm.

3) If Γ is uncountable, then $\ell_\infty(\Gamma)$ equipped with any equivalent norm contains an isometric copy of $\ell_\infty(\mathbb{N})$ ([7]), and this copy is 1-complemented since $\ell_\infty(\mathbb{N})$ is injective. The set $\mathrm{NA}(\|\cdot\|_\infty)$ is not norm-Borel. To show this we pick a non-trivial ultrafilter $\mathcal{U}$ and we consider the norm-continuous map $\Phi\colon \mathbf{2}^\omega \longrightarrow \ell_\infty(\mathbb{N})^*$ such that

$$\Phi(\varepsilon) = \sum_{i\geq 0} 2^{-i}\varepsilon(i)e_i - \delta_{\mathcal{U}}$$

where (e_i) is the canonical basis of $\ell_1(\mathbb{N})$. It is easily seen that $\Phi(\varepsilon) \in \mathrm{NA}$ if and only if there is an x in the unit sphere of $\ell_\infty(\mathbb{N})$ such that $x(i) = 1$ for all $i \in A(\varepsilon) := \{j : \varepsilon(j) = 1\}$ but $\lim_{i,\mathcal{U}} x(i) = -1$, that is if and only if $A(\varepsilon) \notin \mathcal{U}$. Since $\mathcal{U}$ is not Borel in $\mathbf{2}^\omega$, it follows that $\ell_\infty(\Gamma)$ has no equivalent norm such that NA is norm-Borel if Γ is uncountable.

4) It is easily seen that the weak and norm topologies agree on the unit sphere of $\big(\ell_1(\mathbb{N}), \|\cdot\|_1\big)$. Example 3) shows that this condition does not suffice for ensuring that NA_1 is norm-G_δ.

We now conclude with

QUESTION A. Do there exist strictly convex norms $\|\cdot\|$ such that $\mathrm{NA}(\|\cdot\|)$ is w^*-Borel of arbitrarily high class?

QUESTION B. Does there exist a Fréchet-differentiable norm $\|\cdot\|$ such that $\mathrm{NA}_1(\|\cdot\|)$ is not Borel? Can such a norm be constructed on any non-reflexive space with separable dual?

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