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PRODUCTS OF IDEMPOTENT ENDOMORPHISMS OF AN INDEPENDENCE ALGEBRA OF FINITE RANK

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Products of idempotents are investigated in the endomorphism monoid of an algebra belonging to a class of algebras which includes finite sets and finite dimensional vector spaces as special cases. It is shown that every endomorphism which is not an automorphism is a product of idempotent endomorphisms. This provides a common generalisation of earlier results of Howie and Erdos for the cases when the algebra is a set or vector space respectively.

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Introduction

For a mathematical structure M we denote the set of endomorphisms of M by End(M) and the set of automorphisms of M by Aut(M). Under composition of mappings, End(M) is a monoid and Aut(M) is a subgroup of this monoid. We let E denote the set of non-identity idempotents of End(M). Over the last twenty-five years considerable effort has been devoted to describing the subsemigroup $\langle E\rangle$ generated by E. The first results were obtained by Howie in [7] where a set-theoretic description of $\langle E\rangle$ is given when M is simply a set and End(M) is the full transformation semigroup on M. For the case when M is a finite set, the result is:

$$\langle E\rangle = \mathrm{End}(M)\backslash \mathrm{Aut}(M).$$

When M is a finite dimensional vector space, J. A. Erdos [3] proved the same result. An alternative proof was given later by Dawlings [1].

The object of the present paper is to prove the result for a class of algebras, called independence algebras, of which sets and vector spaces are specific instances. We thus obtain a common generalisation of the theorems of Howie and Erdos.

In [7], Howie also described $\langle E\rangle$ when M is an infinite set and the analogous result for an infinite dimensional vector space M was found by Reynolds and Sullivan [11]. A common generalisation of these theorems for a special class of independence algebras is the subject of a subsequent paper.

Independence algebras were defined by Gould in [4] where she describes the basic semigroup structure of the endomorphism monoids of such algebras. In fact, independence algebras are precisely the v^*-algebras introduced by Narkiewicz [10] and described in [**5**]. However, we follow Gould's formulation of the concept as this is designed to facilitate the study of the endomorphism monoid of the algebra. We give the appropriate definitions and terminology in Section 1 and follow this with a summary of some of Gould's results on the endomorphism monoids of independence algebras. The second section of the paper is devoted to proving the main theorem.

1. Preliminaries

For standard concepts of semigroup theory see, for example, [**8**]. For universal algebra terminology and notation we follow [**9**] with the exception that we denote the subalgebra generated by a subset X of an algebra A by $\langle X \rangle$. If the algebra A contains constants, that is, values of nullary operations, then we denote the subalgebra generated by the constants by *Con* and make the convention that $\langle \varnothing \rangle = Con$. A subset X of an algebra A is said to be *independent* if $X = \varnothing$ or for every element x of X we have $x \notin \langle X \backslash \{x\} \rangle$; X is *dependent* if it is not independent. Clearly, every singleton set consisting of a non-constant element of A is independent.

A standard Zorn's lemma argument shows that, given subsets X_0, X of A with X_0 independent and contained in X, there is an independent subset Y of A with $X_0 \subseteq Y \subseteq X$ such that Y is maximal among independent sets contained in X. The following result is from [**9**, p. 50, Exercise 6].

Proposition 1.1. *For an algebra A, the following conditions are equivalent:*

(1) *For every subset X of A and all elements u, v, of A, if $u \in \langle X \cup \{v\} \rangle$ and $u \notin \langle X \rangle$, then $v \in \langle X \cup \{u\} \rangle$.*

(2) *For every subset X of A and every element u of A, if X is independent and $u \notin \langle X \rangle$, then $X \cup \{u\}$ is independent.*

(3) *For every subset X of A, if Y is a maximal independent subset of X, then $\langle X \rangle = \langle Y \rangle$.*

(4) *For subsets X, Y of A with $Y \subseteq X$, if Y is independent, then there is an independent set Z with $Y \subseteq Z \subseteq X$ and $\langle Z \rangle = \langle X \rangle$.*

An algebra A is said to have the *exchange* property or to satisfy [EP] if it satisfies the equivalent conditions of Proposition 1.1. A *basis* for A is a subset of A which generates A and is independent. It is clear from Proposition 1.1 that any algebra with the exchange property has a basis. Furthermore, for such an algebra A, bases may be characterised as minimal generating sets or maximal independent sets, and all bases for A have the same cardinality. This cardinal is called the *rank* of A and is written as *rank* A.

We emphasise that (4) of Proposition 1.1 tells us that any independent subset of A can be extended to a basis for A. We also remark that it is clear that if A satisfies [EP], then so does any subalgebra of A.

We now define an *independence algebra* to be an algebra A which satisfies [EP] and also satisfies:

[F] For any basis X of A and any function $\alpha: X \to A$, there is an endomorphism $\bar{\alpha}$ of A such that $\bar{\alpha}|_X = \alpha$.

Condition [F] is equivalent to asserting that A is free in the variety it generates and that any basis is a set of free generators. We note that if A is an independence algebra and Y is an independent subset of A and $\alpha: Y \to A$ is any function, then there is a homomorphism $\bar{\alpha}: \langle Y \rangle \to A$ which extends α. This follows from [F] since, by the exchange property, Y can be extended to a basis X for A and then any extension of α to X gives rise to an endomorphism of A which restricts to give the required homomorphism.

It is easily seen that this homomorphism is uniquely determined by α. Thus if the endomorphisms θ and ψ agree on a basis for A, then $\theta = \psi$.

Familiar examples of independence algebras are sets (where all subsets are independent), vector spaces (where the independent subsets are the linearly independent subsets) and for any group G, free G-sets (where the independent sets are subsets of free generating sets).

Let A be an independence algebra. The *rank* of an endomorphism α of A is defined to be rank of the subalgebra $Im\,\alpha$. We quote the following lemma from [**4**].

Lemma 1.2. *Let A be an independence algebra. If $\alpha, \beta \in \mathrm{End}(A)$, then rank $\alpha\beta \leqq$ min $\{$rank α, rank $\beta\}$.*

As a consequence of this lemma, for each cardinal κ with $\kappa \leqq \mathrm{rank}\, A$, the set

$$T_\kappa = \{\alpha \in \mathrm{End}(A) \colon rank\ \alpha \leqq \kappa\}$$

is an ideal of $\mathrm{End}(A)$. When A has finite rank n we also use the notation $K(n,r)$ to denote T_r for $r \leqq n$.

The following description of Green's relations on the endomorphism monoid of an independence algebra is taken from [**4**].

Proposition 1.3. *Let A be an independence algebra. Then for $\alpha, \beta \in \mathrm{End}(A)$,*

(1) *$\alpha \mathscr{L} \beta$ if and only if $Im\,\alpha = Im\,\beta$,*

(2) *$\alpha \mathscr{R} \beta$ if and only if $Ker\,\alpha = Ker\,\beta$,*

(3) *$\alpha \mathscr{D} \beta$ if and only if rank $\alpha =$ rank β,*

(4) *$\mathscr{D} = \mathscr{J}$.*

It follows from this proposition that the principal ideals of $\mathrm{End}(A)$ are precisely the ideals T_κ for $\kappa \leqq$ *rank* A. Other ideals exist only when *rank* A is infinite and when this is the case the remaining ideals are the sets

$$I_\kappa = \{\alpha \in \mathrm{End}(A) \colon rank\ \alpha < \kappa\} = \bigcup \{T_\lambda \colon \lambda < \kappa\}$$

for limit cardinals κ.

If $Con \neq \varnothing$, then $T_0 \neq \varnothing$ and T_0 is a principal factor of End(A). Otherwise, $T_0 = \varnothing$ and T_1 is a principal factor. The remaining principal factors are the Rees quotients T_{κ^+}/T_κ where κ^+ is the successor of κ, and T_κ/I_κ for limit cardinals κ.

For each positive integer n we denote the principal factor T_{n+1}/T_n by P_{n+1} and the $\mathscr{D}$-class of endomorphisms of rank n by D_n. Then $P_{n+1} = D_{n+1} \cup \{0\}$ with the product of two members of D_{n+1} being zero if and only if the product in End(A) is not in D_{n+1}. If $Con \neq \varnothing$, then $P_1 = T_1/T_0$ and $P_0 = T_0$; otherwise, $P_1 = T_1$.

We require two more results from [**4**].

Proposition 1.4. *For each positive integer n, the principal factor P_n is completely 0-simple (or completely simple if $n=1$ and $P_1 = T_1$).*

In [**4**] Gould gives an explicit representation of P_n as a Rees matrix semigroup.

Lemma 1.5. *Let α be an endomorphism of an independence algebra A. If $\{x_1, \ldots, x_k\}$ is a basis for $Im\,\alpha$ and if $y_1, \ldots, y_\kappa \in A$ are such that $y_i\alpha = x_i$ for $i = 1, \ldots, k$, then $\{y_1, \ldots, y_k\}$ is independent.*

2. The main theorem

Let A be an independence algebra and E be the set of idempotents in End(A)\Aut(A). We devote this section to the proof of the following theorem.

Theorem 2.1. *If rank $A = n$ is finite, then*

$$\langle E \rangle = \langle E_1 \rangle = \mathrm{End}(A) \backslash \mathrm{Aut}(A)$$

where E_1 is the set of idempotents of rank $n-1$ in End(A).

The strategy of the proof is inspired by an outline of a proof given in [**2**] for the case when A is a vector space. Let

$$S = K(n, n-1) = \mathrm{End}(A) \backslash \mathrm{Aut}(A).$$

We show first that D_{n-1} generates S; in fact, we show that D_n generates $K(n,r)$. Next we consider a group $\mathscr{H}$-class H contained in D_{n-1}. We show that any $\mathscr{H}$-class in the same $\mathscr{R}$-class as H or in the same $\mathscr{L}$-class as H contains an element which is a product of idempotents. It then follows from Green's Lemma that P_{n-1} is generated by $H \cup E_1$. Finally, this allows us to show that P_{n-1} is generated by E_1 and the theorem follows.

For the remainder of the paper, A denotes an independence algebra of rank n. If $n=1$, then either $Con = \varnothing$ and $K(1,0) = \varnothing$ or A contains constant and $K(1,0)$ consists of all endomorphisms α with $Im\,\alpha = Con$. Since all such endomorphisms are idempotent, it is certainly true that $K(1,0)$ is generated by idempotents. We may therefore assume henceforth that $n \geqq 2$.

Lemma 2.2. *Let $\alpha \in D_{r-1}$ where $r<n$. Then there are endomorphisms β, γ in D_r such that $\alpha=\beta\gamma$.*

Proof. If A contains constants, then r can be 1. In this case, let $\{x_1,\dots,x_n\}$ be a basis for A and define $\beta \in \mathrm{End}(A)$ by specifying $x_1\beta=x_1$, $x_i\beta=x_i\alpha$ for $i=2,\dots,n$. Then $Im\,\beta=\langle x_1\rangle$ so that $\beta\in D_1$. Now define $\gamma\in\mathrm{End}(A)$ by putting $x_1\gamma=x_1\alpha$, $x_i\gamma=x_1$ for $i=2,\dots,n$. Since $2\leqq n$, it is clear that γ has rank 1. Further, it is equally clear that $x_i\beta\gamma=x_i\alpha$ for $i=1,\dots,n$ so that $\beta\gamma=\alpha$ as required.

Now suppose that $1<r$. Then there is a basis $\{x_1,\dots,x_{r-1}\}$ for $Im\,\alpha$; this is contained in a basis $\{x_1,\dots,x_n\}$ for A. Choose $y_1,\dots,y_{r-1}$ in A with $y_i\alpha=x_i$ for $i=1,\dots,r-1$; then, by Lemma 1.5, $\{y_1,\dots,y_{r-1}\}$ is independent and so there is a basis $\{y_1,\dots,y_{r-1}, y_r,\dots,y_n\}$ for A. For $i=r,\dots,n$ we have $y_i\alpha\in\langle x_1,\dots,x_{r-1}\rangle$.

Define endomorphisms β and γ as follows:

$$y_i\beta=\begin{cases} x_i & \text{for } 1\leqq i\leqq r \\ y_i\alpha & \text{for } r<i\leqq r \end{cases}$$

and

$$x_i\gamma=\begin{cases} x_i & \text{for } 1\leqq i\leqq r-1 \\ y_i\alpha & \text{for } i=r \\ x_r & \text{for } r<i\leqq n. \end{cases}$$

It is readily seen that $\alpha=\beta\gamma$ and that

$$Im\,\beta=Im\,\gamma=\langle x_1,\dots,x_r\rangle$$

so that β and γ both have rank r.

As a consequence of this lemma, a set of elements of rank r generates P_r if and only if it generates $K(n,r)$.

Lemma 2.3. *If ϕ, γ are idempotents in D_{n-1}, then there is an idempotent ε in D_{n-1} such that $\phi\varepsilon\gamma\in D_{n-1}$.*

Proof. Let $\{x_1,\dots,x_{n-1}\}$ be a basis for $Im\,\phi$; then $x_i\phi=x_i$ for $i=1,\dots,n-1$ since ϕ is idempotent. Let $x_n\in A$ be such that $\{x_1,\dots,x_n\}$ is a basis for A. Then

$$Im\,\gamma=\langle x_1\gamma,\dots,x_n\gamma\rangle$$

and since γ has rank $n-1$, there is an independent subset of $\{x_1\gamma,\dots,x_n\gamma\}$ of cardinality $n-1$.

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If $\{x_1\gamma,\ldots,x_{n-1}\gamma\}$ is independent, then since $x_i\phi\gamma=x_i\gamma$ for $i=1,\ldots,n-1$, it follows that $\phi\gamma$ has rank $n-1$ and so taking $\varepsilon=\phi$ we have $\phi\varepsilon\gamma$, $\varepsilon\in D_{n-1}$.

Now suppose that $\{x_1\gamma,\ldots,x_{n-1}\gamma\}$ is dependent; then without loss of generality we may suppose that $\{x_2\gamma,\ldots,x_n\gamma\}$ is independent.

If $x_n\gamma\in\langle x_1,\ldots,x_{n-1}\rangle$, then $x_n\gamma=(x_n\gamma)\gamma$ is in $\langle x_1\gamma,\ldots,x_{n-1}\gamma\rangle$ so that $Im\,\gamma=\langle x_1\gamma,\ldots,x_{n-1}\gamma\rangle$. But γ has rank $n-1$ and so we have $\{x_1\gamma,\ldots,x_{n-1}\gamma\}$ is independent, a contradiction. Hence $x_n\gamma\notin\langle x_1,\ldots,x_{n-1}\rangle$ and it follows that $\{x_1,\ldots,x_{n-1},x_n\gamma\}$ is independent. Now we define $\varepsilon\in\mathrm{End}(A)$ by putting $x_1\varepsilon=x_n\gamma$, $(x_n\gamma)\varepsilon=x_n\gamma$ and $x_i\varepsilon=x_i$ for $2\leqq i\leqq n-1$. Then $\varepsilon^2=\varepsilon$ and ε has rank $n-1$. We also have $\phi\varepsilon\gamma\in D_{n-1}$ as required.

Corollary 2.4. *Every $\mathscr{H}$-class contained in D_{n-1} contains an element which is a product of idempotents.*

Proof. Let H be an $\mathscr{H}$-class contained in D_{n-1} and α be a member of H. Since $\mathrm{End}(A)$ is regular, there are idempotents γ,ϕ in R_α and L_α respectively. By Lemma 2.3, there is an idempotent ε such that $\gamma\varepsilon\phi\in D_{n-1}$. From the fact that P_{n-1} is completely 0-simple, it follows that $\gamma\mathscr{R}\gamma\varepsilon\phi\mathscr{L}\phi$ so that $\gamma\varepsilon\phi\in H$.

An immediate consequence of this corollary and Green's Lemmas (see [**8**, Lemmas II.2.1 and II.2.2]) is the following result.

Corollary 2.5. *Let H be a group of $\mathscr{H}$-class in D_{n-1}. Then every element in D_{n-1} can be written as a product of elements from $H\cup E_1$.*

Lemma 2.6. *Every element of D_{n-1} is a product of elements of E_1.*

Proof. We use induction on n. When $n=1$, D_0 is the set of endomorphisms of rank 0. Either $D_0=\varnothing$ and there is nothing to prove or A contains some constants and

$$D_0=\{\alpha\in\mathrm{End}(A)\colon Im\,\alpha=Con\}.$$

In this case, D_0 consists of idempotents and the result is true.

When $n=2$, let $\{x,y\}$ be a basis for A and consider the $\mathscr{H}$-class

$$H=\{\alpha\in\mathrm{End}(A)\colon Im\,\alpha=\langle y\rangle,\ Ker\,\alpha=Cg^A(x,y)\}.$$

Certainly H is a group $\mathscr{H}$-class because it contains the idempotent η given by $x\eta=y$, $y\eta=y$. If $\alpha\in H$, then $x\alpha=y\alpha=a$ for some element a of $\langle y\rangle$. Define ε_1 and ε_2 in $\mathrm{End}(A)$ by putting $x\varepsilon_1=y\varepsilon_1=x$ and $x\varepsilon_2=a, y\varepsilon_2=y$. Then $\alpha=\varepsilon_1\varepsilon_2$ and $\varepsilon_1,\varepsilon_2$ are clearly idempotents of rank 1. Thus every member of H is a product of idempotents (of rank 1) and it follows from Corollary 2.5 that the same is therefore true of D_1.

Now assume that the result holds for $n-1$ where $3\leqq n$. Let $\{x_1,\ldots,x_n\}$ be a basis for A and consider the $\mathscr{H}$-class

$$H=\{\alpha\in\text{End}(A)\colon Im\,\alpha=\langle x_2,\dots,x_n\rangle,\ Ker\,\alpha=Cg^A(x_1,x_2)\}.$$

The idempotent θ is in H where $x_1\theta=x_2\theta=x_2$ and $x_i\theta=x_i$ for $i=3,\dots,n$. Thus H is a group $\mathscr{H}$-class. For $\alpha\in H$ we have $x_1\alpha=x_2\alpha$ and hence

$$Im\,\alpha=\langle x_2\alpha,\dots,x_n\alpha\rangle$$

and consequently, $\{x_2\alpha,\dots,x_n\alpha\}$ is independent. Since $x_1\notin Im\,\alpha$ it follows that $\{x_1,x_2\alpha,\dots,x_n\alpha\}$ is independent and hence this set is a basis for A. We use this basis to define $\psi\in\text{End}(A)$ by putting $x_1\psi=x_2\alpha$ and $(x_i\alpha)\psi=x_i\alpha$ for $i=2,\dots,n$. Then ψ is an idempotent of rank $n-1$.

We define ϕ to be the idempotent endomorphism of rank $n-1$ given by $x_1\phi=x_2\phi=x_1$ and $x_i\phi=x_i$ for $i=3,\dots,n$.

Now consider the algebra $B=\langle x_2,\dots,x_n\rangle$ and define the endomorphism β' of B by specifying

$$x_2\beta'=x_3\beta'=x_3\alpha \text{ and } x_i\beta'=x_i\alpha \text{ for } i=4,\dots,n.$$

Then $Im\,\beta'=\langle x_3\alpha,\dots,x_n\alpha\rangle$ so that β' has rank $n-2$. By the induction assumption, $\beta'=\varepsilon_1'\dots\varepsilon_k'$ for some idempotents of rank $n-2$ in End(B). Now define $\varepsilon_i\in\text{End}(A)$ for $i=1,\dots,k$ by putting $x_1\varepsilon_i=x_1$ and $x_j\varepsilon_i=x_j\varepsilon_i'$ for $j=2,\dots,n$. Clearly, each ε_i is an idempotent of rank $n-1$. If we put $\beta=\varepsilon_1\dots\varepsilon_k$, then it is readily verified that $\alpha=\phi\beta\psi$ so that the members of H are products of idempotents of rank $n-1$. It now follows from Corollary 2.5 that every member of D_{n-1} is a product of idempotent of rank $n-1$ and this completes the proof by induction.

Theorem 2.1 now follows immediately from Lemmas 2.6 and 2.2. We can deduce a stronger result from Theorem 2.1 and Lemma 2.2. Let E_{n-r} be the set of idempotents of End(A) having rank r.

Corollary 2.7. *If A is an independence algebra of finite rank n, then $K(n,r)=\langle E_{n-r}\rangle$ for $r=1,\dots,n-1$.*

Proof. The case $r=n-1$ is simply a restatement of the theorem and so we may assume that $r<n-1$. In view of Theorem 2.1 every element of the $\mathscr{D}$-class D_r is certainly a product of idempotents. Hence by Lemma 1 of [**6**], any element α of D_r is a product of idempotents all of which are $\mathscr{D}$-related to α, that is, in D_r. The result now follows from Lemma 2.2.

Finally, we remark that both Theorem 2.1 and Corollary 2.7 specialise immediately to give the corresponding results for the full transformation semigroup on a finite set, the monoid of endomorphisms of a finite dimensional vector space and the endomorphism monoid of a free G-set of finite rank.

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