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ROUND SUBSETS OF WALLMAN-TYPE COMPACTIFICATIONS*

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Abstract

Let $\mathscr{Z}$ be a normal base of a Tychonoff space X and $\omega(\mathscr{Z})$ ($\nu(\mathscr{Z})$) denote the Wallman-type (real-) compactification of X generated by $\mathscr{Z}$. This Wallman-type compactification is known to associate with a unique proximity δ. A $\mathscr{Z}$-filter $\mathscr{F}$ is round if for each $F \in \mathscr{F}$ there is an $F_0 \in \mathscr{F}$ such that $F_0\bar{\delta}(X-F)$. A subset A of $\omega(\mathscr{Z})$ is called a round subset of $\omega(\mathscr{Z})$ iff for each $Z \in \mathscr{Z}$, if $Cl_{\omega(\mathscr{Z})}Z$ contains A, then it is a neighborhood of A. Properties of round $\mathscr{Z}$-filters and round sets of $\omega(\mathscr{Z})$ are introduced. We also prove that the intersection of all the free $\mathscr{Z}$-ultrafilters is $\mathscr{F} = \{Z \in \mathscr{Z}: Cl_X(X-Z) \text{ is compact}\}$ iff $\omega(\mathscr{Z}) - X$ is a round subset of $\omega(\mathscr{Z})$; if $\mathscr{Z}$ is a separating nest generated intersection ring with property (α) then $\omega(\mathscr{Z}) - \nu(\mathscr{Z})$ is a round subset of $\omega(\mathscr{Z})$.

1. Introduction

Let $\mathscr{Z}$ be a normal base for a Tychonoff space X. Recently, the Wallman-type (real-) compactification $\omega(\mathscr{Z})$, ($\nu(\mathscr{Z})$ respectively) has been studied. (See Alo and Shapiro (1968), Gagrat and Naimpally (1973), Njåstad (1966), Steiner and Steiner (1970), Su (1975).) Mandelker (1969) studied the round z-filters and round subsets of βX. Njåstad (1966) proved that for each normal base there is a unique proximity corresponding to $\omega(\mathscr{Z})$. This enables us to study round $\mathscr{Z}$-filters and round subsets in $\omega(\mathscr{Z})$ in this note. In Section One and Two, we will give some properties of round $\mathscr{Z}$-filters and round subsets of $\omega(\mathscr{Z})$. In Section Three, we will prove that $\mathscr{F} = \{Z \in \mathscr{Z}: Cl_X(X-Z) \text{ is compact}\}$ is exactly the intersection of all the free $\mathscr{Z}$-ultrafilters iff $\omega(\mathscr{Z}) - X$ is a round subset of $\omega(\mathscr{Z})$ and some other results related to round subsets and $\omega(\mathscr{Z}) - \nu(\mathscr{Z})$.

The topological spaces are always Tychonoff spaces. A *normal* base $\mathscr{Z}$ of a space X is a base for closed subsets of X which satisfies the following conditions: (i) $\mathscr{Z}$ is a *ring* (i.e., closed under finite unions and intersections), (ii) $\mathscr{Z}$ is

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disjunctive (i.e., if x is not contained in the closed subset A of X, then there is a $Z \in \mathscr{Z}$ such that $x \in Z \subset X - A$), (iii) $\mathscr{Z}$ is *normal* (i.e., $A, B \in \mathscr{Z}$ and $A \cap B = \emptyset$, then there exist sets $C, D \in \mathscr{Z}$ such that $A \subset X - C$, $B \subset X - D$ and $C \cup D = X$. (See Alo and Shapiro (1968), Gagrat and Naimpally (1973), Njåstad (1966), Steiner and Steiner (1970) and Su (1975).) Let $\mathscr{Z}$ be a family of closed subsets of X. $\mathscr{Z}$ is called an *intersection* (or *delta*) *ring* if it is a ring which is also closed under countable intersections. $\mathscr{Z}$ is called an *intersecting normal base* iff $\mathscr{Z}$ is a normal base which is also an intersection ring. A sequence $\{Z_n\}$ of sets in $\mathscr{Z}$ is called a *nest* in $\mathscr{Z}$ if there is a sequence $\{H_n\}$ in $\mathscr{Z}$ such that $X - H_{n+1} \subset Z_{n+1} \subset X - H_n \subset Z_n$ for $n = 1, 2, \cdots$. $\mathscr{Z}$ is *nest generated* if for each member Z of $\mathscr{Z}$ there is a nest $\{Z_n\}$ in $\mathscr{Z}$ such that $Z = \cap\{Z_n : n \in N\}$. (See Alo and Shapiro (1968), Alo, Shapiro and Weir, Steiner (1966) and Steiner and Steiner (1970).) $\mathscr{Z}$ is said to be *complement* generated if for each $Z \in \mathscr{Z}$ there is a sequence $\{Z_n\}$ of $\mathscr{Z}$ such that $Z = \cap\{X - Z_n : n \in N\}$. $\mathscr{Z}$ is a *strong delta* normal base of X if it is a normal base that is a delta ring and complement generated (see Alo and Shapiro (1969), Alo, Shapiro and Weir). $\mathscr{Z}$ is said to be *separating* if for each closed set A in X and $x \notin A$ there are disjoint sets Z_1, Z_2 in $\mathscr{Z}$ with $Z_1 \supset A$ and $x \in Z_2$. It is easy to show that a family of closed subsets of a space X is a *separating nest generated* intersection ring (see Steiner (1966)) iff it is a strong delta normal base (see Alo, Shapiro and Weir). Let $\mathscr{Z}$ be a normal base and let $\omega(\mathscr{Z})$ be the set of all $\mathscr{Z}$-ultra-filters. $\omega(\mathscr{Z})$ with topology defined as usual (see Alo and Shapiro (1968) and Gagrat and Naimpally (1973)) is called a Wallman-type compactification. If in addition $\mathscr{Z}$ is an intersection ring then $\nu(\mathscr{Z})$ denotes the subspace of $\omega(\mathscr{Z})$ which consists of all $\mathscr{Z}$-ultrafilters with the countable intersection property and $\nu(\mathscr{Z})$ is called a Wallman-type real-compactification. A *separated proximity* on X is a binary relation δ satisfying the following conditions ($\bar{\delta}$ denotes the negation of δ): (P1) if $A\delta B$, then $B\delta A$; (P2) $(A \cup B)\delta C$ iff $A\delta C$ or $B\delta C$; (P3) $\{x\}\delta\{y\}$ iff $x = y$; (P4) $\phi\bar{\delta}X$; and (P5) if $A\bar{\delta}B$, then there are sets C, D such that $X = C \cup D$, $A\bar{\delta}C$ and $B\bar{\delta}D$. (See Naimpally and Warrack (1970), Njåstad (1966), and Smirnov (1964).) A set X with a proximity δ on it is a *proximity space, denoted by* (X, δ). The topology which δ induces on X is defined by the closure operation $\bar{A} = C1\ A = \{x \in X : \{x\}\delta A\}$. We will write $A \Subset B$ and read A is *strongly contained* in B, if $A\bar{\delta}(X - B)$. A family $\mathscr{B}$ of subsets of X is a *base* for the proximity δ iff (B.1) for every two disjoint sets A, B of $\mathscr{B}$, $A\bar{\delta}B$; and (B.2) for every two subsets $A, B \subset X$ with $A\bar{\delta}B$ are *separated* by sets of $\mathscr{B}$, i.e., there are sets $C, D \in \mathscr{B}$ such that $A \subset C$, $B \subset D$ and $C\bar{\delta}D$. (See Naimpally and Warrack (1970) and Njåstad (1966).)

Njåstad (1966) showed that for each normal base $\mathscr{Z}$ of X there is a proximity δ corresponding to the Wallman-type compactification $\omega(\mathscr{Z})$ which is defined by the statement that for subsets A and B of X, $A\delta B$ iff the closure of A in $\omega(\mathscr{Z})$ intersects the closure of B in $\omega(\mathscr{Z})$, i.e. $\bar{A} \cap \bar{B} \neq \emptyset$. X with this

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proximity δ is a proximity subspace of the space $\omega(\mathscr{Z})$ with the proximity (also denoted by δ) defined by the statement that for subsets A and B of $\omega(\mathscr{Z})$, $A\delta B$ iff $\bar{A} \cap \bar{B} \neq \emptyset$. Throughout the sequel, any proximity theoretic statement will be understood to be with respect to these special proximities on X or on $\omega(\mathscr{Z})$. In this setting, $\mathscr{Z}$ is a proximity base of δ on X and $\bar{\mathscr{Z}} = \{\bar{\mathscr{Z}}: \mathscr{Z} \in \mathscr{Z}\}$ is a proximity base of δ on $\omega(\mathscr{Z})$. Note that in $\omega(\mathscr{Z})$ an open set G contains a closed set A iff $A \Subset G$.

2. Round $\mathscr{Z}$-filters

In this section we will give some basic properties of round $\mathscr{Z}$-filters, where $\mathscr{Z}$ will always stand for a normal base of X.

DEFINITION. *A $\mathscr{Z}$-filter $\mathscr{F}$ is round iff for each $F \in \mathscr{F}$ there is an $F_0 \in \mathscr{F}$ such that $F_0 \Subset F$.*

LEMMA 2.1. *In a proximity space (X, δ_1), if $\mathscr{B}$ is a base for the proximity δ_1, then for $A, B \subset X$ such that $A \Subset B$ there is $B_0 \in \mathscr{B}$ such that $A \Subset B_0 \Subset B$.*

PROOF. It is easy, using (P5), to show that there is a $C \subset X$ such that $A \Subset C \Subset B$. Thus $A\bar{\delta}_1(X - C)$ and $C\bar{\delta}_1 X - B$. Since $\mathscr{B}$ is a base for δ_1, there are B_1, B_2, B_3 and B_4 in $\mathscr{B}$ such that $A \subset B_1$, $X - C \subset B_2$, $B_1\bar{\delta}_1 B_2$, $C \subset B_3$, $X - B \subset B_4$ and $B_3\bar{\delta}_1 B_4$. Thus, $A \subset B_1 \Subset X - B_2 \subset C \subset B_3 \Subset X - B_4 \subset B$. Let $B_0 = B_3$. Then B_0 is as desired.

In light of Lemma 2.1, we know that a $\mathscr{Z}$-filter $\mathscr{F}$ is round iff for each $F \in \mathscr{F}$ there is a $Z \in \mathscr{Z}$ and $F_0 \in \mathscr{F}$ such that $F_0 \subset X - Z \subset F$. For, if $\mathscr{F}$ is round, then for each $F \in \mathscr{F}$ there is an $F_0 \in \mathscr{F}$ such that $F_0 \Subset F$. But $F_0 \Subset F$ iff $F_0\bar{\delta}(X - F)$. Since $\mathscr{Z}$ is a proximity base for δ on X, there are $Z_1, Z_2 \in \mathscr{Z}$ such that $Z_1 \supset F_0$, $Z_2 \supset X - F$ with $Z_1\bar{\delta}Z_2$. Thus $F_0 \subset Z_1 \Subset Z - Z_2 \subset F$. Conversely, if for each $F \in \mathscr{F}$ there is a $Z \in \mathscr{Z}$ and $F_0 \in \mathscr{F}$ such that $F_0 \subset X - Z \subset F$, then since $F_0, Z \in \mathscr{Z}$, $F_0 \Subset X - Z \subset F$. (Compare with the definition in §3 of Mandelker (1969)).

LEMMA 2.2. *For a $\mathscr{Z}$-filter $\mathscr{F}$, define $\mathscr{F}^0 = \{F \in \mathscr{Z}: F \Supset F_0$ for some $F_0 \in \mathscr{F}\}$. Then $\mathscr{F}$ is round iff $\mathscr{F} = \mathscr{F}^0$*

PROOF. It is easy to show that $\mathscr{F}^0$ is a $\mathscr{Z}$-filter. To see that $\mathscr{F}^0$ is round, let $F \in \mathscr{F}^0$. Then there is an $F_0 \in \mathscr{F}$ such that $F_0 \Subset F$. By Lemma 2.1, since $\mathscr{Z}$ is a proximity base, there is a $Z \in \mathscr{Z}$ such that $F_0 \Subset Z \Subset F$. Thus $Z \in \mathscr{F}^0$ and $Z \Subset F$. That is, $\mathscr{F}^0$ is round. The last part is clear.

DEFINITION. *For a $\mathscr{Z}$-filter $\mathscr{F}$, we let $\theta(\mathscr{F})$ denote the set of all cluster points of $\mathscr{F}$ in $\omega(\mathscr{Z})$. That is, $\theta(\mathscr{F}) = \cap\{\bar{Z}: Z \in \mathscr{F}\}$, where $\bar{Z} = C1_{\omega(\mathscr{Z})}Z$. Now $C1_{\omega(\mathscr{Z})}Z = \{\mathscr{A} \in \omega(\mathscr{Z}): Z \in \mathscr{A}\}$. (See Alo and Shapiro* (1968) *and Gagrat and Naimpally* (1973).) *We further define, for each $p \in \omega(\mathscr{Z})$, $\mathscr{M}^p = \{Z \in \mathscr{Z}: p \in \bar{Z}\}$ and $\mathscr{O}^p =$*

$\{Z \in \mathscr{Z}: \bar{Z}$ is a neighborhood of $p\}$. It follows easily that $\theta(\mathscr{F}) = \{\mathscr{A} \in \omega(\mathscr{Z}): \mathscr{F} \subset \mathscr{A}\} = \{p \in \omega(\mathscr{Z}): \mathscr{F} \subset \mathscr{M}^p\}$. In particular, if $\mathscr{Z} = \mathscr{Z}(X)$, the family of all zero-sets on X, then $\mathscr{Z}$ is a normal base and $\omega(\mathscr{Z}) = \beta X$, the Stone-Cech compactification of X, and in this case the above notations reduce to the customary ones, as follows. Each $\mathscr{A} \in \beta X$ is a $\mathscr{Z}(X)$-ultrafilter, $\mathscr{M}^{\mathscr{A}} = \{Z \in \mathscr{Z}(X): \mathscr{A} \in Cl_{\beta X} Z\} = \{Z \in \mathscr{Z}(X): \mathscr{A} \in \bar{Z}\}$ and $\mathscr{O}^{\mathscr{A}} = \{Z \in \mathscr{Z}(X): Cl_{\beta X}\mathscr{Z}$ is a neighborhood of $\mathscr{A}\}$. (See Chapter 7 of Gillman and Jerison (1960) *and Mandelker* (1969)).

Now, it is easy to show that if $\mathscr{F}_1, \mathscr{F}$ are two $\mathscr{Z}$-filters and $\mathscr{F}_1 \subset \mathscr{F}$, then $\theta(\mathscr{F}) \subset \theta(\mathscr{F}_1)$. Moreover, since $\bar{\mathscr{Z}} = \{\bar{Z}: Z \in \mathscr{Z}\}$ is a normal base for closed subsets in $\omega(\mathscr{Z})$, (see Alo and Shapiro (1968), Gagrat and Naimpally (1973) and Njåstad (1966)), each closed subset A of $\omega(\mathscr{Z})$ is of the form $\theta(\mathscr{F})$ for some $\mathscr{Z}$-filter. Namely, $\mathscr{F} = \{Z \in \mathscr{Z}: A \subset \bar{Z}\}$ which clearly is a $\mathscr{Z}$-filter and $\theta(\mathscr{F}) = A$.

LEMMA 2.3. *If $\mathscr{F}$ is a $\mathscr{Z}$-filter and $Z_0 \in \mathscr{Z}$, then $\bar{Z}_0 \Supset \theta(\mathscr{F})$ iff there is a $W \in \mathscr{F}$ such that $\bar{W} \Supset \bar{Z}_0$.*

PROOF. "$\Rightarrow$" Consider the family $\bar{\mathscr{F}} = \{Cl_{\omega(\mathscr{Z})} Z = \bar{Z}: Z \in \mathscr{F}\}$. It is clear that $\theta(\mathscr{F}) = \cap \bar{\mathscr{F}} \neq \emptyset$ is an intersection of compact subsets of $\omega(\mathscr{Z})$. Since $\bar{Z}_0 \Supset \theta(\mathscr{F})$ there is an open set G of $\omega(\mathscr{Z})$ such that $\bar{Z}_0 \Supset G \Supset \theta(\mathscr{F}) = \cap \bar{\mathscr{F}}$. Hence there are $F_1, F_2, \cdots, F_n$ in $\mathscr{F}$ such that $\bigcap_{i=1}^n \bar{F}_i \subset G$. (See 5F of Kelley (1955)). But since $\bigcap_{i=1}^n \bar{F}_i$ is closed $\bigcap_{i=1}^n \bar{F}_i \Subset G$. Let $W = \bigcap_{i=1}^n F_i$. Then we have $\bar{W} \subset \bigcap_{i=1}^n \bar{F}_i \Subset G \Subset \bar{Z}_0$. "$\Leftarrow$" is obvious, as $\theta(\mathscr{F}) = \bigcap_{Z \in \mathscr{F}} \bar{Z} \subset \bar{W} \Subset \bar{Z}_0$.

THEOREM 2.4. *If $\mathscr{F}$ is a $\mathscr{Z}$-filter, then the following are equivalent.*

(a) *$\mathscr{F}$ is a round $\mathscr{Z}$-filter.*
(b) *For every $Z \in \mathscr{F}$, there is $W \in \mathscr{F}$ such that $\bar{Z} \Supset \bar{W}$.*
(c) *For any $p \in \omega(\mathscr{Z})$, if $\mathscr{F} \subset \mathscr{M}^p$ then $\mathscr{F} \subset \mathscr{O}^p$.*
(d) *For every $Z \in \mathscr{F}$, $\bar{Z} \Supset \theta(\mathscr{F})$.*

PROOF. (a) $\Leftrightarrow$ (b) Since $Z \in \mathscr{F}$, there is a $W \in \mathscr{F}$ such that $W \Subset Z$, i.e., $W \bar{\delta} X - Z$. By the property of proximity $\bar{W} \bar{\delta} X - Z$ iff $W \bar{\delta} X - Z$. (See (2.8) of Naimpally and Warrack (1970)). Now since $X = (X - Z) \cup Z$, then $\omega(\mathscr{Z}) = Cl_{\omega(\mathscr{Z})}(X - Z) \cup Cl_{\omega(\mathscr{Z})} Z = \overline{X - Z} \cup \bar{Z}$ and so $X - Z \supset \omega(\mathscr{Z}) - \bar{Z}$. Thus $W \bar{\delta} X - Z$ iff $\bar{W} \bar{\delta} \omega(\mathscr{Z}) - \bar{Z}$ iff $\bar{Z} \Supset \bar{W}$.

(b) $\Rightarrow$ (c) Suppose $\mathscr{F} \subset M^p$. From (b) for each $Z \in \mathscr{F}$ there is a $W \in \mathscr{F}$ such that $\bar{W} \Subset \bar{Z}$. But $\mathscr{F} \subset \mathscr{M}^p$ which is a $\mathscr{Z}$-ultrafilter. Thus $W \in \mathscr{M}^p$, and $\mathscr{M}^p \in \bar{W} \Subset \bar{Z}$. But $\mathscr{M}^p = p$. $\{p\} \Subset \bar{Z}$, i.e., $Z \in \mathscr{O}^p$. Hence $F \subset \mathscr{O}^p$.

(c) $\Rightarrow$ (d) Suppose $p \in \theta(\mathscr{F})$. Then $p = \mathscr{A}$, a $\mathscr{Z}$-ultrafilter. $\mathscr{A} \in \bigcap_{Z \in \mathscr{F}} \bar{Z}$ implies $Z \in \mathscr{A}$ for each $Z \in \mathscr{F}$ or $\mathscr{F} \subset \mathscr{A} = \mathscr{M}^p$. And (c) says that $\mathscr{F} \subset \mathscr{O}^p$. Thus, for each $Z \in \mathscr{F}$, $\bar{Z}$ is a neighborhood of p, for each $p \in \theta(\mathscr{F})$.

This implies $\bar{Z}$ is a neighborhood of $\theta(\mathscr{F})$ which is a closed subset of $\omega(\mathscr{Z})$. Hence $\dot{\bar{Z}} \Supset \theta(\mathscr{F})$.

(d) $\Leftrightarrow$ (b) This follows immediately from Lemma 2.3.

If $A \Subset Z$, then we shall call Z a δ-neighborhood of A.

LEMMA 2.5. *For a closed subset A in $\omega(\mathscr{Z})$, there is a base of δ-neighborhoods of the form $\bar{Z}$, where $Z \in \mathscr{Z}$.*

PROOF. Let G be an open neighborhood of A in $\omega(\mathscr{Z})$. Then $G \Supset A$. By Lemma 2.1, there is a $Z \in \mathscr{Z}$ such that $A \Subset \bar{Z} \Subset G$.

THEOREM 2.6. *Let A be any closed subset of $\omega(\mathscr{Z})$. For any $\mathscr{Z}$-filter $\mathscr{F}$, we have $\theta(\mathscr{F}) = A$ iff $\bigcap_{p \in A} \mathscr{O}^p \subset \mathscr{F} \subset \bigcap_{p \in A} \mathscr{M}^p$.*

PROOF. "$\Rightarrow$" Suppose $\theta(\mathscr{F}) = A$, and $Z \in \bigcap_{p \in A} \mathscr{O}^p$. Then $\bar{Z}$ is a neighborhood of $\theta(\mathscr{F})$ which is closed and we have $\dot{\bar{Z}} \Supset \theta(\mathscr{F})$. By Lemma 2.3, there is a $W \in \mathscr{F}$ such that $\bar{Z} \Supset \bar{W}$. Thus $Z \Supset W$ and $Z \in \mathscr{F} \subset \mathscr{M}^p$ for each $p \in A$. Conversely, suppose $\mathscr{F}$ is a $\mathscr{Z}$-filter with $\bigcap_{p \in A} \mathscr{O}^p \subset \mathscr{F} \subset \bigcap_{p \in A} \mathscr{M}^p$. Then $\theta(\bigcap_{p \in A} \mathscr{M}^p) = A \subset \theta(\mathscr{F})$. However, in light of Lemma 2.5, we have $\theta(\bigcap_{p \in A} \mathscr{O}^p) = A$. Moreover $\bigcap_{p \in A} \mathscr{O}^p \subset \mathscr{F}$, $\theta(\mathscr{F}) \subset \theta(\bigcap_{p \in A} \mathscr{O}^p) = A$. Hence $\theta(\mathscr{F}) = A$.

The following is a characterization of a round $\mathscr{Z}$-filter in terms of $\mathscr{Z}$-filters of the form $\mathscr{O}^p$.

THEOREM 2.7. *For any $\mathscr{Z}$-filter $\mathscr{F}$, $\mathscr{F}^0 = \bigcap_{p \in \theta(\mathscr{F})} \mathscr{O}^p$.*

PROOF. $Z \in \mathscr{F}^0$ iff there is a $W \in \mathscr{F}$ such that $W \Subset Z$. As shown in Theorem 2.4, (a) $\Leftrightarrow$ (b), $W \Subset Z$ iff $\bar{W} \Subset \bar{Z}$. On the other hand, $Z_1 \in \bigcap_{p \in \theta(\mathscr{F})} \mathscr{O}^p$ iff Z_1 is a neighborhood of $\theta(\mathscr{F})$. Since $\theta(\mathscr{F})$ is closed, $\bar{Z}_1$ is a neighborhood of $\theta(\mathscr{F})$ iff $\bar{Z}_1 \Supset \theta(\mathscr{F})$. This, by Lemma 2.3, is equivalent to that there is a $W \in \mathscr{F}$ with $\bar{W} \Subset \bar{Z}_1$. Thus, $\mathscr{F}^0 = \bigcap_{p \in \theta(\mathscr{F})} \mathscr{O}^p$.

THEOREM 2.8. *If $\mathscr{F}$ is a round $\mathscr{Z}$-filter, then $\mathscr{F} = \bigcap_{p \in \theta(\mathscr{F})} \mathscr{O}^p$. Conversely if A is a nonempty closed subset of $\omega(\mathscr{Z})$, then $\bigcap_{p \in A} \mathscr{O}^p$ is a round $\mathscr{Z}$-filter and for distinct closed subsets A, $\bigcap_{p \in A} \mathscr{O}^p$ are distinct.*

PROOF. The first part follows immediately from Lemma 2.2 and Theorem 2.7. Let A be a nonempty closed subset of $\omega(\mathscr{Z})$, and $\mathscr{F} = \bigcap_{p \in A} \mathscr{O}^p$. By Theorem 2.6, $\theta(\mathscr{F}) = A$ and hence for each $Z \in \mathscr{F}$ we have $\bar{Z}$ is a neighborhood of A which is closed. Thus, $\dot{\bar{Z}} \Supset A = \theta(\mathscr{F})$ and from (a) $\Leftrightarrow$ (d) of Theorem 2.4, $\mathscr{F}$ is a round $\mathscr{Z}$-filter. Finally, let A_1 and A_2 be closed subsets of $\omega(\mathscr{Z})$, and $A_1 \neq A_2$. Then, there is an $a \in A_1 - A_2$ (or $A_2 - A_1$). Suppose that $a \in A_1 - A_2$. Then $a \bar{\delta} A_2$ or $A_2 \Subset X - \{a\}$. By Lemma 2.1, there is a $Z \in \mathscr{Z}$ such that $A_2 \Subset Z \Subset X - \{a\}$. Thus $Z \in \bigcap_{p \in A_2} \mathscr{O}^p - \bigcap_{p \in A_1} \mathscr{O}^p$. Similarly, for $a \in A_2 - A_1$.

COROLLARY 2.9. *The correspondence* $A \to \bigcap_{p\in A}\mathcal{O}^p$ *is a one-to-one order-reversing map between the nonempty closed subsets of* $\omega(\mathcal{Z})$ *and the round* $\mathcal{Z}$*-filters.*

If $\mathcal{P}$ is a prime $\mathcal{Z}$-filter, (i.e., $Z_1 \cup Z_2 \in \mathcal{P}$ implies $Z_1 \in \mathcal{P}$ or $Z_2 \in \mathcal{P}$), then $\theta(\mathcal{P}) = \bigcap_{Z\in\mathcal{P}} \bar{Z} = \{p(=\mathcal{A}) \in \omega(\mathcal{Z}): \mathcal{P} \subset \mathcal{A}\}$ is just one point. For if $\mathcal{A}_1, \mathcal{A}_2 \in \theta(\mathcal{P})$ and $\mathcal{A}_1 \neq \mathcal{A}_2$, then there are $Z_1, Z_2 \in \mathcal{Z}$ such that $Z_i \in \mathcal{A}_i$ and $Z_1 \cap Z_2 = \emptyset$. It follows that $Z_1 \bar{\delta} Z_2$ and there are subsets A, B of X such that $A \cup B = X$ with $Z_1 \bar{\delta} A$ and $Z_2 \bar{\delta} B$. Since $\mathcal{Z}$ is a proximity base, there are A_1, A_2, B_1 and $B_2 \in \mathcal{Z}$ such that $Z_1 \subset A_1, A \subset A_2, A_1 \bar{\delta} A_2$; $Z_2 \subset B_1, B \subset B_2$ and $B_1 \bar{\delta} B_2$. $Z_1 \subset A_1 \Subset X - A_2 \subset X - A \subset B \subset B_2 \Subset X - B_1 \subset X - Z_2$. Now, $A_2 \cup B_2 \supset A \cup B = X \in \mathcal{P}$. This implies $A_2 \in \mathcal{P} \subset \mathcal{A}_1$ or $B_2 \in \mathcal{P} \subset \mathcal{A}_2$. Then, we have $Z_1 \in \mathcal{A}_1$ and $Z_1 \subset X - A_2$ so $A_2 \notin \mathcal{P}$. But also $Z_2 \in \mathcal{A}_2$ and $Z_2 \subset X - B_2$ so $B_2 \notin \mathcal{P}$. This is a contradiction.

3. Round subsets of $\omega(\mathcal{Z})$

A *remote point* in $\beta\mathbf{R}$ is a point not in the closure of any discrete subset of $\mathbf{R}$. In this section we will generalize the characterization of remote points, and obtain a class of subsets of $\omega(\mathcal{Z})$ which is related to a class of round $\mathcal{Z}$-filters.

DEFINITION. *A subset A of* $\omega(\mathcal{Z})$ *is called a round subset of* $\omega(\mathcal{Z})$ *if for any* $Z \in \mathcal{Z}$, *if* $\bar{Z}$ *contains A, then* $\bar{Z}$ *is a neighborhood of A.*

From the definition, we have the following properties of round subsets in $\omega(\mathcal{Z})$.

THEOREM 3.1. *Let* $A \subset \omega(\mathcal{Z})$. *Then*
(a) *A is a round subset of* $\omega(\mathcal{Z})$ *iff* $\bigcap_{p\in A}\mathcal{M}^p = \bigcap_{p\in A}\mathcal{O}^p$.
(b) *If* $\mathrm{Cl}_{\omega(\mathcal{Z})}A$ *is a round subset of* $\omega(\mathcal{Z})$, *then so is A.*
(c) *Every open subset G in* $\omega(\mathcal{Z})$ *is round.*
(d) *Any union of round subsets of* $\omega(\mathcal{Z})$ *is also round.*

PROOF. (a) Note that $\mathcal{M}^p = \{Z \in \mathcal{Z}: p \in \bar{Z}\} = \{Z \in \mathcal{Z}: Z \in \mathcal{A} = p\}$, and $\mathcal{O}^p = \{Z \in \mathcal{Z}: \{p\} \Subset \bar{Z}\}$. Now, A is a round set iff each $Z \in \mathcal{Z}$ with $A \subset \bar{Z}$ implies $\bar{Z}$ is a neighborhood of A. Hence $\bigcap_{p\in A}\mathcal{M}^p = \bigcap_{p\in A}\{Z \in \mathcal{Z}: p \in \bar{Z}\} = \{Z \in \mathcal{Z}: A \subset \bar{Z}\} = \{Z \in \mathcal{Z}: \bar{Z}$ is a neighborhood of $A\} = \bigcap_{p\in A}\{Z: \{p\} \Subset \bar{Z}\} = \bigcap_{p\in A}\mathcal{O}^p$ iff A is round.

(b) Let $A_1 = \mathrm{Cl}_{\omega(\mathcal{Z})}A$. Then since A_1 is round and closed each $Z \in \mathcal{Z}$ with $\bar{Z} \supset A_1$ implies $\bar{Z}$ is a neighborhood of A_1. Thus $\bar{Z} \supset A_1$ implies first $\bar{Z} \Supset A_1$ and then $\bar{Z} \Supset A_1 \Supset A$.

(c) and (d) are straightforward from the definitions.

THEOREM 3.2. *For any nonempty closed subset A of* $\omega(\mathcal{Z})$ *the following are equivalent.*

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(a) *A is a round subset of $\omega(\mathscr{Z})$.*
(b) $\bigcap_{p\in A}\mathscr{M}^p$ *is a round $\mathscr{Z}$-filter.*
(c) $\bigcap_{p\in A}\mathscr{M}^p = \bigcap_{p\in A}\mathscr{O}^p$.
(d) *There is a unique $\mathscr{Z}$-filter $\mathscr{F}$ such that $\theta(\mathscr{F}) = A$.*

PROOF. (a) $\Rightarrow$ (b) Since A is a round closed subset of $\omega(\mathscr{Z})$, each $Z \in \mathscr{Z}$ with $\bar{Z} \supset A$ implies $\bar{Z} \Supset A$. Consider $\mathscr{F} = \bigcap_{p\in A}\mathscr{M}^p = \{Z \in \mathscr{Z}: A \subset \bar{Z}\}$. $\mathscr{F}$ is a $\mathscr{Z}$-filter and $\theta(\mathscr{F}) = \{q \in \omega(\mathscr{Z}): \mathscr{F} \subset \mathscr{M}^q\}$. But $p \in A$ iff $\mathscr{M}^p \supset \mathscr{F}$ iff $p \in \theta(\mathscr{F})$ (by definition of $\theta(\mathscr{F})$)). Thus $\theta(\mathscr{F}) = A \Subset \bar{Z}$. It follows from Theorem 2.4, (d) $\Rightarrow$ (a), that $\mathscr{F}$ is a round $\mathscr{Z}$-filter.

(b) $\Rightarrow$ (c) is trivial from Theorem 2.8.

(c) $\Leftrightarrow$ (d) is Theorem 2.6; and (c) $\Rightarrow$ (a) is Theorem 3.1, (a).

4. The free $\mathscr{Z}$-ultrafilter and round subsets

In this section we will see a more general result (Theorem 4.1) of a known theorem: The intersection of all the free maximal ideals in $C(X)$, the ring of all continuous real-valued functions, is the family $C_K(X)$ of all functions with compact support iff $\beta X - X$ is a round subset of βX. (See 7E of Gillman and Jerison (1960)). We will also generalize the results of Mandelker (1969).

THEOREM 4.1. *For any normal base $\mathscr{Z}$, the intersection of all the free $\mathscr{Z}$-ultrafilters is $\mathscr{F} = \{Z \in \mathscr{Z}: C1_X(X - Z)$ is compact$\}$ iff $\omega(\mathscr{Z}) - X$ is a round subset of $\omega(\mathscr{Z})$.*

PROOF. It is easy to show that $\mathscr{F}$ thus defined is a $\mathscr{Z}$-filter. Let $A = \omega(\mathscr{Z}) - X$. Then by Theorem 3.1, (a), A is a round subset of $\omega(\mathscr{Z})$ iff $\bigcap_{p\in A}\mathscr{M}^p = \bigcap_{p\in A}\mathscr{O}^p$. However, if we can show that $\mathscr{F} = \bigcap_{p\in A}\mathscr{O}^p$, then A is a round subset iff $\bigcap_{p\in A}\mathscr{M}^p = \mathscr{F}$. For each $Z \in \mathscr{F}$, $C1_X(X - Z)$ is compact in X so it is compact in $\omega(\mathscr{Z})$ which is Hausdorff. Thus $C1_X(X - Z)$ is closed in $\omega(\mathscr{Z})$ and $C1_X(X - Z) = C1_{\omega(\mathscr{Z})}(X - Z)$. Since $X = Z \cup (X - Z)$, then $\omega(\mathscr{Z}) = C1_{\omega(\mathscr{Z})}X = C1_{\omega(\mathscr{Z})}Z \cup C1_{\omega(\mathscr{Z})}(X - Z) = \bar{Z} \cup C1_X(X - Z)$ and so $\omega(\mathscr{Z}) - \bar{Z} \subset C1_X(X - Z) \subset X$. We then have $A = \omega(\mathscr{Z}) - X \subset \omega(\mathscr{Z}) - C1_X(X - Z) \subset \bar{Z}$ and since $\omega(\mathscr{Z}) - C1_X(X - Z)$ is open in $\omega(\mathscr{Z})$ then $A \Subset \bar{Z}$. Thus $Z \in \bigcap_{p\in A}\mathscr{O}^p$ and so $\mathscr{F} \subset \bigcap_{p\in A}\mathscr{O}^p$. Conversely, if $Z \in \bigcap_{p\in A}\mathscr{O}^p$, then $\bar{Z}$ is a neighborhood of A. That is, there is an open set G such that $A = \omega(\mathscr{Z}) - X \subset G \subset \bar{Z}$. so $\omega(\mathscr{Z}) - \bar{Z} \subset \omega(\mathscr{Z}) - G \subset X$. It follows that $C1_{\omega(\mathscr{Z})}$ $\omega(\mathscr{Z}) - \bar{Z}) \subset \omega(\mathscr{Z}) - G \subset X$. Therefore $X - Z = X - \bar{Z} \subset \omega(\mathscr{Z}) - \bar{Z} \subset C1_{\omega(\mathscr{Z})}$ $\omega(\mathscr{Z}) - \bar{Z}) \subset X$ and so $C1_X(X - Z) \subset C1_{\omega(\mathscr{Z})}(X - Z) \subset C1_{\omega(\mathscr{Z})}(\omega(\mathscr{Z}) - \bar{Z}) \subset X$. Since $C1_{\omega(\mathscr{Z})}(\omega(\mathscr{Z}) - \bar{Z})$ is compact in X then so is $C1_X(X - Z)$ and thus $Z \in \mathscr{Z}$. Therefore $\mathscr{F} \supset \bigcap_{p\in A}\mathscr{O}^p$. Consequently, $\mathscr{F} = \bigcap_{p\in A}\mathscr{O}^p$.

If, in particular, $\mathscr{Z} = Z(X)$, then we have the result stated above.

Before we state the next result, let us recall *Q-closedness*. A subset A of X

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is Q-closed in X if for each $p \in X - A$ there is a G_δ-set containing p and disjoint from A. (See Mrówka (1957)).

THEOREM 4.2. *Let $\mathscr{Z}$ be a separating nest generated intersection ring. Then the following are equivalent.*

(a) *X is $\mathscr{Z}$-realcompact, i.e., every $\mathscr{Z}$-ultrafilter with the countable intersection property is fixed.*

(b) *$\omega(\mathscr{Z}) - X$ is a union of zero-sets in $\omega(\mathscr{Z})$.*

(c) *$\omega(\mathscr{Z}) - X$ is a union of G_δ-sets in $\omega(\mathscr{Z})$.*

PROOF. (a) $\Rightarrow$ (b) Since X is $\mathscr{Z}$-realcompact, $X = \nu(\mathscr{Z})$. In Steiner and Steiner (1970), $\nu(\mathscr{Z})$ is proved to be realcompact by showing for each $p \in \omega(\mathscr{Z}) - \nu(\mathscr{Z})$ there is a zero-set $Z \in Z[\omega(\mathscr{Z})]$ containing p and disjoint from $\nu(\mathscr{Z})$. (See Steiner and Steiner (1970), Theorem 3.2.) Thus $\omega(\mathscr{Z}) - \nu(\mathscr{Z}) = \omega(\mathscr{Z}) - X$ is a union of zero-sets of $\omega(\mathscr{Z})$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Since $\omega(\mathscr{Z}) - X$ is a union of G_δ-sets in $\omega(\mathscr{Z})$, X is Q-closed in $\omega(\mathscr{Z})$. Since $\mathscr{Z}$ is an intersecting normal base of X, Theorem 4 of Alo and Shapiro (1969) states that $\nu(\mathscr{Z})$ is a subset of X^Q, the Q-closure of X in $\omega(\mathscr{Z})$. Thus $X \subset \nu(\mathscr{Z}) \subset X^Q = X$. This implies $X = \nu(\mathscr{Z})$. Hence X is $\mathscr{Z}$-realcompact.

Let $\mathscr{Z}$ be a normal base of X. Then $\mathscr{Z}$ is said to have property (α) if for every C-embedded closed subset S of X (i.e., every continuous real-valued function on S has a continuous extension on X) which is disjoint from a member Z of $\mathscr{Z}$ there are $Z_1, Z_2 \in \mathscr{Z}$ such that $Z_1 \supset S$, $Z_2 \supset Z$ and $Z_1 \cap Z_2 = \emptyset$.

THEOREM 4.3. *Let $\mathscr{Z}$ be a normal base of X which has property (α). Then any zero-set Z_0 of $\omega(\mathscr{Z})$ contained in $\omega(\mathscr{Z}) - X$ is a round subset of $\omega(\mathscr{Z})$.*

PROOF. Since Z_0 is a zero-set in $\omega(\mathscr{Z})$, let $f \in C(\omega(\mathscr{Z}))$ such that $Z(f) = Z_0$. To show that Z_0 is a round set, let $Z \in \mathscr{Z}$ be arbitrary such that $\bar{Z} \supset Z_0$. We need to show that $\bar{Z}$ is a neighborhood of Z_0. Let $T = \omega(\mathscr{Z}) - Z_0$. Then $T \supset X$. Define $h(t) = 1/f(t)$ for each $t \in T$. Then h is a continuous function on T. Suppose $Z_0 \cap Cl_{\omega(\mathscr{Z})}(X - Z) \neq \emptyset$. Then h would be unbounded on $X - Z$. Thus $X - Z$ contains a noncompact closed subset S which is C-embedded in T. (See Gillman and Jerison (1960; 1.20). Thus S is closed in X and disjoint from Z, and by hypothesis there are disjoint sets $Z_1, Z_2 \in \mathscr{Z}$ such that $Z_1 \supset S$ and $Z_2 \supset Z$. Hence $Cl_{\omega(\mathscr{Z})}S \cap Cl_{\omega(\mathscr{Z})}Z \subset Cl_{\omega(\mathscr{Z})}Z_1 \cap Cl_{\omega(\mathscr{Z})}Z_2 = \emptyset$. But S is a noncompact closed subset in T. We must have $q \in Cl_{\omega(\mathscr{Z})}S - T$. Hence $q \in Z_0$ but $q \notin \bar{Z} = Cl_{\omega(\mathscr{Z})}Z$. This is a contradiction. It follows $Z_0 \cap Cl_{\omega(\mathscr{Z})}(X - Z) = \emptyset$, i.e., $Z_0 \subset \omega(\mathscr{Z}) - Cl_{\omega(\mathscr{Z})}(X - Z) \subset \bar{Z}$. This shows that $\bar{Z}$ is a neighborhood of Z_0.

COROLLARY 4.4. *Let $\mathscr{Z}$ be a separating nest generated intersection ring which has property (α). Then $\omega(\mathscr{Z}) - \nu(\mathscr{Z})$ is a round subset of $\omega(\mathscr{Z})$.*

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PROOF. As shown in Theorem 3.2 of Steiner and Steiner (1970), for each $p \in \omega(\mathscr{Z}) - \nu(\mathscr{Z})$ there is a zero-set zero–set of $\omega(\mathscr{Z})$ containing p and missing $\nu(\mathscr{Z})$. Thus $\omega(\mathscr{Z}) - \nu(\mathscr{Z})$ is a union of zero-sets of $\omega(\mathscr{Z})$. Use an argument similar to the one in the proof of Theorem 4.3 to show that each zero-set of $\omega(\mathscr{Z})$ disjoint from $\nu(\mathscr{Z})$ is a round subset in $\omega(\mathscr{Z})$. Thus by Theorem 3.1, (d), $\omega(\mathscr{Z}) - \nu(\mathscr{Z})$ is a round subset.

COROLLARY 4.5. *Let $\mathscr{Z}$ be an intersecting normal base which has property (α). Let $\eta^*(\mathscr{Z}) = \{\mathscr{A} \in \omega(\mathscr{Z})$: $\mathscr{A}^0$ has countable intersection property$\}$, where $\mathscr{A}^0$ is defined in Lemma 2.2. Then $\omega(\mathscr{Z}) - \eta^*(\mathscr{Z})$ is a round subset.*

PROOF. As shown in Theorem 1 of Su (1975), for each $p \in \omega(\mathscr{Z}) - \eta^*(\mathscr{Z})$ there is a zero-set of $\omega(\mathscr{Z})$ which contains p and is disjoint from $\eta^*(\mathscr{Z})$. Thus $\omega(\mathscr{Z}) - \eta^*(\mathscr{Z})$ is a union of zero–sets of $\omega(\mathscr{Z})$. By Theorem 4.3 and Theorem 3.1, (d), $\omega(\mathscr{Z}) - \eta^*(\mathscr{Z})$ is a round subset.

COROLLARY 4.6. *If X is $\mathscr{Z}$-realcompact for a separating nest generated intersection ring $\mathscr{Z}$ which has property (α), then $\omega(\mathscr{Z}) - X$ is a round subset and hence the intersection of all the free $\mathscr{Z}$-ultrafilters is $\mathscr{F} = \{Z \in \mathscr{Z}$: $\mathrm{Cl}_X(X - Z)$ is compact$\}$.*

PROOF. It follows from Corollary 4.4 and Theorem 4.1.

REMARK. (1) If $\mathscr{Z} = Z(X)$, then $\mathscr{Z}$ is a separating nest generated intersection ring which has property (α).

(2) There is a separating nest generated intersection ring other than $Z(X)$ which has property (α). Let X be a non-Lindelof normal space. Since X is not Lindelof, there is a filter $\mathscr{F}$ of zero-sets which is closed under countable intersection but $\cap \mathscr{F} = \varnothing$. Let $\mathscr{Z} = \{Z \in Z(X)$: $Z \in \mathscr{F}$ or $Z \cap A = \varnothing$ for some $A \in \mathscr{F}\}$. It is easy to show that $\mathscr{Z}$ is a separating nest generated intersection ring (see Lemma 3.5 in Steiner and Steiner (1970)). We need to show that $\mathscr{Z}$ has property (α). Let S be any closed subset disjoint from a $Z \in \mathscr{Z}$. Since X is a normal space, there are Z_1 and $Z_2 \in Z(X)$ such that $Z_1 \supset S$, $Z_2 \supset Z$ and $Z_1 \cap Z_2 = \varnothing$. Now, since $Z \in \mathscr{Z}$, we have either (i) $Z \in \mathscr{F}$ or (ii) there is an $A \in \mathscr{F}$ such that $A \cap Z = \varnothing$. If it is case (i), then it is clear by definition of $\mathscr{Z}$, $Z_2 \in \mathscr{F} \subset \mathscr{Z}$ and thus $Z_1 \in \mathscr{Z}$. If it is case (ii), let $Z_1 \cup A = Z_0$. Then since $Z_1 \cap Z = \varnothing$, $Z_0 \cap Z = (Z_1 \cup A) \cap Z = \varnothing$. Moreover, since $Z_0 \in Z(X)$ and $Z_0 \supset A$, $Z_0 \in \mathscr{Z}$. Thus Z_0 and $Z \in \mathscr{Z}$ such that $Z_0 \supset S$, $Z = Z$ and $Z_0 \cap Z = \varnothing$.

(3) Let X be a zero-dimensional T_1 space, i.e., it has a base consisting of clopen (both closed and open) subsets of X. Let $\mathscr{Z}$ be a family of clopen subsets of X such that (i) $\mathscr{Z}$ is a base for closed subsets of X, (ii) $\mathscr{F}$ is an intersection ring, (iii) $X - F \in \mathscr{Z}$ for each $F \in \mathscr{Z}$. Then it is clear $\mathscr{Z}$ is a separating nest generated intersection ring. Moreover, if S is any closed subset disjoint from a $Z \in \mathscr{Z}$, then $S \subset X - Z$ which is in $\mathscr{Z}$ (by (iii)). Thus we have Z and $X - Z$ in $\mathscr{Z}$ such that

$X - Z \supset S$, $Z \supset Z$ and $Z \cap (X - Z) = \emptyset$. Hence $\mathscr{Z}$ has property (α). It turns out that $\omega(\mathscr{Z})$ is a zero-dimensional Wallman-type compactification of X and $\nu(\mathscr{Z})$ is N-compact (see Su (1974), Theorem D).

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