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CONSTRUCTING ISOSPECTRAL BUT NON-ISOMETRIC RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper we examine the examples of isospectral but non-isometric Riemannian manifolds given by Milnor, Ikeda, and Vignéras. Of these, only Milnor's example is accounted for by Sunada's method of constructing isospectral manifolds, and even then only as an "unnatural" construction.

1. **Introduction.** In this paper, "manifold" means compact manifold. Let M be a (compact) Riemannian manifold. The eigenvalues of the Laplace operator Δ on the space of $L^2(M)$ functions form a discrete sequence in $\mathbb{R}$: $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, called the spectrum of the Laplace operator. The zeta function of M is $\zeta_M(s) = \sum \lambda_i^{-s}$.

Two compact Riemannian manifolds are said to be *isospectral* if their Laplace operators have the same spectrum, or equivalently, if they have the same zeta functions. The first example of isospectral non-isometric manifolds was given by Milnor [8] in 1964. Some years later, in 1980, further examples were constructed by Ikeda [7], Vignéras [14], and others ([4], [5], [6]). In particular, Vignéras gave examples of isospectral non-isometric *Riemann surfaces* of constant curvature -1.

In 1985, Sunada [12] gave a systematic method of constructing isospectral non-isometric manifolds which involves essentially finite groups and "may be looked upon as a geometric analogue of a routine method in number theory" ([10], [12]). This method has subsequently been exploited to produce many new isospectral non-isometric manifolds ([2], [3]).

In [2] R. Brooks raised the natural question of whether Sunada's construction exhausts all posibilities of finding examples of isospectral manifolds. In this paper, we examine the examples of Milnor, Ikeda, and Vignéras. We show that Ikeda's examples do not arise from Sunada's construction and Milnor's example does, but only as an "unnatural" construction. Vignéras' method gives many different examples of pairs of isospectral but non-isometric Riemannian manifolds, and for some of them we cannot decide (see the remark at the end of the paper). But we show that her basic examples do not arise from Sunada's construction. We have been informed that Alan Reid has also shown (unpublished) that Vignéras' basic examples are not of Sunada's type.

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2. **Sunada's construction.** Let G be a finite group and H_1 and H_2 be two subgroups of G. We say that the triplet (G, H_1, H_2) of groups satisfies condition (*) if

(*) Each conjugacy class of G meets H_1 and H_2 in the same number of elements, *i.e.*, denoting the conjugacy class of g in G by g^G,

$$\#(g^G \cap H_1) = \#(g^G \cap H_2),$$

for all $g \in G$.

(A fancier way to say the same thing is that the trivial representations of H_1 and H_2 induce isomorphic representations of G.)

Let M_1 and M_2 be Riemannian manifolds and let H_1 and H_2 be subgroups of a finite group G. We say "M_1 and M_2 are sandwiched by M and M/G with the finite triplet (G, H_1, H_2)", or simply, M_1 and M_2 are sandwiched with the triplet (G, H_1, H_2), if there is a Riemannian manifold M and an isomorphism from G into the group of isometries of M such that $M \longrightarrow M/H_i$ are Riemannian coverings and that M_i are isometric to M/H_i $(i = 1, 2)$.

$$\begin{tikzcd}
 & M \arrow[dl, "H_1"'] \arrow[d, "G"] \arrow[dr, "H_2"] & \\
M_1 \cong M/H_1 \arrow[r] & M/G & M/H_2 \cong M_2 \arrow[l]
\end{tikzcd}$$

NOTE. The maps $M \longrightarrow M/H_i, i = 1, 2$, are normal Riemannian coverings. However, M/G is only an orbifold, *i.e.*, a quotient of M by a finite group of isometries, and is not necessarily a Riemannian manifold.

The following result is due to Sunada [12].

THEOREM A [12]. *Let M_1 and M_2 be two Riemannian manifolds. If they can be sandwiched with a finite triplet (G, H_1, H_2) satisfying condition (*), then M_1 and M_2 are isospectral.*

REMARK. Sunada states the assumption that M/G is also a Riemannian manifold. W. D. Neumann has kindly pointed out to us that Sunada's proof is equally valid when M/G is an orbifold. Also, as pointed out by Brooks [2], Gordon and Wilson constructed a family M_t of isospectral but non-isometric Riemannian manifolds [4]. Their work came earlier than the work of Sunsda and can be regarded as an analogue of Sunada's construction for infinite groups.

3. **Milnor's example.** Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the usual Riemannian metric. The group of isometries of $\mathbb{R}^n$ is $O(n) \ltimes \mathbb{R}^n$ (the rigid motions) acting on $\mathbb{R}^n$ as $\big((P, c), x\big) \longrightarrow Px + c$ where Px is the multiplication on the left by the matrix P on the column vector x. It is known that every n-dimensional Riemannian manifold of constant curvature 0 is a quotient of $\mathbb{R}^n$ by a discrete subgroup Γ of $O(n) \ltimes \mathbb{R}^n$ acting on $\mathbb{R}^n$ freely. When there is no danger of confusion, we will not distinguish the elements x in $\mathbb{R}^n$ from the pairs (id, x) in $O(n) \ltimes \mathbb{R}^n$. An isometry $\Phi: \mathbb{R}^n/\Gamma \longrightarrow \mathbb{R}^n/\Gamma'$ is an isometry of $\mathbb{R}^n$ such that $\Phi\Gamma = \Gamma'\Phi$.

The following is due to Milnor [8].

THEOREM B [8]. *There are two lattices Λ_1 and Λ_2 in $\mathbb{R}^{16}$ such that $\mathbb{R}^{16}/\Lambda_1$ and $\mathbb{R}^{16}/\Lambda_2$ are isospectral but not isometric.*

More specifically, the two lattices are $\Lambda_1 = \Gamma_8 \oplus \Gamma_8$ and $\Lambda_2 = \Gamma_{16}$, which may be described as follows: For any positive integer m, let L_{4m} be the lattice consisting of $x = (x_i) \in \mathbb{Z}^{4m}$ with $\sum x_i \equiv 0 \pmod 2$. Then Γ_{4m} is generated by L_{4m} and e_{4m} where $e_n = (1/2, \ldots, 1/2) \in \mathbb{R}^n$. In [8], it is shown that the quotient manifolds are isospectral by utilizing the fact that $\Gamma_8 \oplus \Gamma_8$ and Γ_{16} have the same number of vectors of any given length. Furthermore $\mathbb{R}^{16}/\Lambda_1$ and $\mathbb{R}^{16}/\Lambda_2$ are not isometric because $\Gamma_8 \oplus \Gamma_8$ and Γ_{16} are not. ($\Gamma_8 \oplus \Gamma_8$ is generated by elements of length 2 while Γ_{16} is not. See [8, p. 51]).

As a first attempt to determine if $\mathbb{R}^{16}/\Lambda_1$ and $\mathbb{R}^{16}/\Lambda_2$ can be sandwiched with a triplet satisfying (*), it is natural to try $M = \mathbb{R}^{16}/\Lambda_1 \cap \Lambda_2$ and the triplet

$$\big((\Lambda_1 + \Lambda_2)/\Lambda_1 \cap \Lambda_2,\ \Lambda_1/\Lambda_1 \cap \Lambda_2, \Lambda_2/\Lambda_1 \cap \Lambda_2\big)$$

Indeed, $(\Lambda_1 + \Lambda_2)/\Lambda_1 \cap \Lambda_2$ is a finite group. However as $(\Lambda_1 + \Lambda_2)/\Lambda_1 \cap \Lambda_2$ is abelian, the triplet does not satisfy condition (*). In fact, we have the following simple lemma:

LEMMA 1. *For any triplet (G, H_1, H_2) of finite groups, if H_1 is a normal subgroup of G, then the triplet satisfies condition (*) if and only if $H_1 = H_2$.*

PROOF. For any $g \in G$, let $h \in g^G \cap H_1$. There is an $r \in G$ such that $rgr^{-1} = h$, hence $g = r^{-1}hr \in H_1$. So $g^G \cap H_1 \neq \emptyset \iff g \in H_1$. Thus by the assumption $\#(g^G \cap H_1) = \#(g^G \cap H_2)$, we have $g \notin H_1 \Rightarrow g \notin H_2$, *i.e.*, $H_1 \supseteq H_2$. Therefore $H_1 = H_2$ as it is clear that (*) implies $\#H_1 = \#H_2$. ∎

So, with the "obvious" choice of the triplet above, we see that condition (*) fails. It would appear that Milnor's example does not arise from Sunada's construction. But surprisingly, M_1 and M_2 do arise from Sunada's construction by a proper choice of the triplet (G, H_1, H_2). And with little more effort, we even can arrange matters so that the orbifold M/G is a Riemannian manifold. In preparation, let us refine Lemma 1.

For a triplet (G, H_1, H_2) of finite groups we say that H_1 and H_2 are *bijectively conjugate* if there exists a bijection $\psi: H_1 \longrightarrow H_2$ such that, given any $h \in H_1$, $\psi(h)^G = h^G$, *i.e.*, h and $\psi(h)$ are conjugate in G. Such a bijection ψ will be called an almost-inner bijection.

LEMMA 2. *Let (G, H_1, H_2) be a triplet of finite groups. Then this triplet satisfies condition (*) if and only if H_1 and H_2 are bijectively conjugate.*

PROOF. Suppose H_1 and H_2 are bijectively conjugate. Let $h \in g^G \cap H_1$. Then $rgr^{-1} = h$ and $\psi(h) = \psi_h h \psi_h^{-1}$ for some $r, \psi_h \in G$. So we have

$$\psi(h) = \psi_h h \psi_h^{-1} = \psi_h rgr^{-1}\psi_h^{-1} \in g^G \cap H_2.$$

That is, ψ defines an injection of $g^G \cap H_1$ into $g^G \cap H_2$, hence $\#(g^G \cap H_1) = \#(g^G \cap H_2)$, as the inverse of ψ is also an almost-inner bijection.

Conversely, write $G = \bigcup g_j^G$ as a disjoint union of conjugacy classes. We assume $\#(g^G \cap H_1) = \#(g^G \cap H_2)$. Any chosen bijection ψ_j from $g_j^G \cap H_1$ to $g_j^G \cap H_2$ defines an almost-inner bijection from H_1 to H_2. ∎

We now return to Milnor's example. Let N and P denote the 4×4 and 16×16 matrices, respectively, shown below:

$$N=(1/2)\begin{pmatrix} 1 & 1 & 1 & 1\\ -1 & 1 & -1 & 1\\ -1 & -1 & 1 & 1\\ 1 & -1 & -1 & 1\end{pmatrix} \quad\text{and } P=\begin{pmatrix} N & 0 & 0 & 0\\ 0 & 0 & I_4 & 0\\ 0 & N & 0 & 0\\ 0 & 0 & 0 & I_4\end{pmatrix},$$

where I_n is the $n\times n$ identity matrix.

Note that $N\in O(4)$ and $N^4=I_4$, so $P\in O(16)$ and $P^8=I_{16}$. Let $c\in\mathbb{R}^{16}$ be the vector whose coordinates are all 0 except that the 16-th is $1/8$. Put $\Phi=(P,c)\in O(16)\ltimes\mathbb{R}^{16}$. We have $Pc=c$ and $\Phi^i(x)=P^ix+ic$ for $0\le i\le 7$.

Define $M=\mathbb{R}^{16}/\Lambda$ where $\Lambda=(2\langle\mathbb{Z}^4,e_4\rangle)^4$. Then Φ defines an isometry of M as $N(\langle\mathbb{Z}^4,e_4\rangle)=\langle\mathbb{Z}^4,e_4\rangle$ and $P\Lambda=\Lambda$.

Put $\Lambda_0'=(\langle\mathbb{Z}^4,e_4\rangle)^4$. Then $P\Lambda_0'=\Lambda_0'$ and we have the following Riemannian covering:

$$M_0'=\mathbb{R}^{16}/\Lambda_0'\longrightarrow M_0'/\langle\Phi\rangle=\mathbb{R}^{16}/\langle\Phi,\Lambda_0'\rangle,$$

because $\langle\Phi\rangle$ acts on M_0' freely by the choice of c. (Note that, if $\lambda\in\Lambda_0'$, then the difference of the 15-th coordinate and the 16-th coordinate of λ is an integer. And for all i, $1\le i\le 7$, $\Phi^ix-x=P^ix-x+ic$. Since P fixes the last 4 coordinates of x, the 15-th and 16-th coordinates of Φ^ix-x are that of ic which are 0 and $i/8$ respectively. So, for all i, $1\le i\le 7$, $\Phi^ix-x\notin\Lambda_0'$ as its 15-th and 16-th coordinates differ by a proper fraction.)

Define $M_0=\mathbb{R}^{16}/\Lambda_0$, where $\Lambda_0=\langle\Phi,\Lambda_0'\rangle$ is the group generated by Φ and Λ_0'. Since $\Lambda\subseteq\Lambda_1$, $\Lambda_2\subseteq\Lambda_0'$, M_1 and M_2 are sandwiched by M and $M_0=M/G$ with the triplet (G,H_1,H_2) of finite groups:

$$G=\Lambda_0/\Lambda=\langle\Phi,\Lambda_0'\rangle/\Lambda,$$
$$H_1=\Lambda_1/\Lambda=(\Gamma_8\oplus\Gamma_8)/\Lambda=\{z\mid z\in(\Lambda_1\cap\Lambda_2)/\Lambda \text{ or } z=(e_8,0_8)\},\ \text{and}$$
$$H_2=\Lambda_2/\Lambda=\Gamma_{16}/\Lambda=\{z\mid z\in(\Lambda_1\cap\Lambda_2)/\Lambda \text{ or } z=(1,0_7,1,0_7)\},$$

where 0_n is the 0 element and $e_n=(1/2,\dots,1/2)$ in $\mathbb{R}^n$.

PROPOSITION 1. *$\mathbb{R}^{16}/\Lambda_1$ and $\mathbb{R}^{16}/\Lambda_2$ are sandwiched by Riemannian manifolds $M=\mathbb{R}^{16}/\Lambda$ and M/G with the finite triplet (G,H_1,H_2) given above, and this triplet satisfies condition (*).*

PROOF. Only the last statement requires a proof. Observe $\Phi(e_8,0_8)\Phi^{-1}=P(e_8,0_8)=(1,0_7,1,0_7)$. Now define a bijection ψ from H_1 to H_2 by $\psi=\mathrm{id}$ on $\Lambda_1\cap\Lambda_2$ and $\psi=$ conjugation by Φ on $(e_8,0_8)$. Then Proposition 1 follows from Lemma 2. ∎

4. **Ikeda's example.** Let q be a positive integer and $p_1,p_2,\dots,p_n$ be integers prime to q. Let $g=g(q;p_1,p_2,\dots,p_n)$ denote the orthogonal matrix given by

$$g=g(q;p_1,p_2,\dots,p_n)=\begin{pmatrix} R(p_1/q) & & 0\\ & \ddots & \\ 0 & & R(p_n/q)\end{pmatrix}$$

where $R(\theta) = \begin{pmatrix} cos2\pi\theta & sin2\pi\theta \\ -sin2\pi\theta & cos2\pi\theta \end{pmatrix}$. Then g generates the cyclic subgroup $G = \{g^k\}_{k=1}^q$ of order q in the orthogonal group $O(2n)$ of degree $2n$. Define the *lens space* to be the Riemannian manifold:

$$L(q; p_1, p_2, \ldots, p_n) = S^{2n-1}/G.$$

THEOREM C [7]. *There are lens spaces which are isospectral but not isometric.*

For example, Ikeda proves that the lens spaces $L(11; 1, 2, 4)$ and $L(11; 1, 2, 8)$ are isospectral but non isometric. It is clear that they cannot be sandwiched by M and M/G with a triplet (G, H_1, H_2) satisfying (*). Otherwise, we must have $M = S^5$ which is the only non-trivial covering space of $L(11; 1, 2, 4)$ and $L(11; 1, 2, 8)$. Thus $H_1 = \{g^k\}_{k=1}^{11}$ and $H_2 = \{g'^k\}_{k=1}^{11}$, where

$$g = g(11; 1, 2, 4) \text{ and } g' = g(11; 1, 2, 8).$$

As H_1 and H_2 are cyclic, bijectively conjugate implies conjugate, and this contradicts the fact that these two lens spaces are not isometric. This proves

PROPOSITION 2. *The isospectral lens spaces $L(11; 1, 2, 4)$ and $L(11; 1, 2, 8)$ do not arise from Sunada's construction.*

REMARK. For the same reason, Proposition 2 holds for other examples mentioned in [7].

5. **Vignéras' example.** Let K be a number field. A quaternion algebra $\mathbb{D}$ is a 4-dimensional K-algebra generated by i, j over K, such that $i^2 = a, j^2 = b$, and $ij = -ji$ where $a, b \in K$. Such a quaternion algebra is denoted by $\mathbb{D} = \left(\frac{a,b}{K}\right)$. When $K = \mathbb{R}$ and $a = b = -1$, the quaternion algebra is the classical Hamiltonian quaternion algebra which is denoted by $\mathbb{H}$. For any $x = x_1 + x_2 i + x_3 j + x_4 ij \in \mathbb{D}$ where $x_l \in K$, $l = 1, 2, 3, 4$, the conjugate of x in $\mathbb{D}$ is $\bar{x} = x_1 - x_2 i - x_3 j - x_4 ij$. The reduced norm of x in $\mathbb{D}$ is $n(x) = x\bar{x}$. It is well known that

$$\mathbb{D} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}^r \times M(2, \mathbb{R})^s \times M(2, \mathbb{C})^{r_2}$$

where $r + s = r_1$ is the number of real imbeddings of K into $\mathbb{C}$ and $r_1 + 2r_2$ is the degree of K over $\mathbb{Q}$ and $\mathbb{H}$ is the ordinary quaternion over $\mathbb{R}$.

Let $\mathbb{D} = \left(\frac{a,b}{K}\right)$ be a quaternion algebra over K. If v is a place of K, then $\mathbb{D}_v = \left(\frac{a,b}{K_v}\right)$ is a quaternion algebra over the completion K_v of K. We say v is unramified in $\mathbb{D}$ if $\mathbb{D}_v$ is isomorphic to the matrix algebra over K_v. It is well known that the ramified places in $\mathbb{D}$ are non-complex and the number of such places is finite and even (*cf.* [10, Chapter III]). Conversely, given any finite even number of non-complex places, there is, up to isomorphism, exactly one quaternion algebra over K which ramifies exactly at the given places (A consequence of theorem of Brauer-Hasse-Albert-Noether). If $\mathbb{D}$ ramifies at a non-empty set of places of K, then $\mathbb{D}$ is a division algebra over K. Hereafter we will assume that all quaternion algebras under discussion are division algebras and are unramified at at least one archimedean place, *i.e.*, $\mathbb{D} \ncong M(2, K)$ and $s + r_2 > 0$.

Let $\mathrm{UR}_\infty(\mathbb{D})$ be the set of the archimedean places of K unramified in $\mathbb{D}$. Set $\mathbb{D}^* = \{x \in \mathbb{D} \mid n(x) \neq 0\}$. and $\mathbb{D}^1 = \{x \in \mathbb{D} \mid n(x) = 1\}$. Then there exists a (non canonical) isomorphism ρ of $\mathbb{D}^*$ into $G^* = \mathrm{GL}(2,\mathbb{R})^s \times \mathrm{GL}(2,\mathbb{C})^{r_2}$:

$$\mathbb{D}^* \hookrightarrow \prod_{v \in \mathrm{UR}_\infty(\mathbb{D})} \mathbb{D}_v^* \longrightarrow G^*,$$

so that on each factor of G^*, the determinant respects the reduced norm in $\mathbb{D}$. The restriction of ρ on $\mathbb{D}^1$ gives rise to an isomorphism ρ of $\mathbb{D}^1$ into $G^1 = \mathrm{SL}(2,\mathbb{R})^s \times \mathrm{SL}(2,\mathbb{C})^{r_2}$:

$$\rho\colon \mathbb{D}^1 \hookrightarrow G^1.$$

Let N be a maximal compact subgroup of G^1. Then $X = G^1/N = \mathbf{H}_2^s \mathbf{H}_3^{r_2}$ is a product of 2 or 3 dimensional hyperbolic spaces.

We recall that an *order* O in $\mathbb{D}$ is a subring of $\mathbb{D}$ that is also a finitely generated module over the ring R of integers of K and contains a K basis of $\mathbb{D}$. We say that O is a maximal order in $\mathbb{D}$ if O is not properly contained in any orders in $\mathbb{D}$.

For any order O in $\mathbb{D}$, the group $O^1 = \{x \in O \mid n(x) = 1\}$ of units of reduced norm 1 is isomorphic under ρ to a discrete subgroup Γ of G^1. The following is due to Vignéras [14].

THEOREM D [14]. *There exist quaternion division algebras $\mathbb{D}$ having maximal orders O_1 and O_2 such that the quotients $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are isospectral but not isometric compact Riemannian manifolds.*

As explicit examples we have [14]:

1). $K = \mathbb{Q}(\sqrt{10})$ and $\mathbb{D}$ is the quaternion algebra over K which ramifies exactly at the following places: $(7), (11), (11 + 3\sqrt{10})$, and one real infinite place. Note that, in this case, the resulting isospectral manifolds are *Riemann surfaces* of constant curvature -1.

2). $K = \mathbb{Q}(\sqrt{-5})$ and $\mathbb{D}$ is the quaternion algebra over K which ramifies exactly at the places: $(11), (3 + 2\sqrt{-5})$.

Now we proceed to show that the isospectrtal non-isometric Riemannian manifolds constructed in this manner are not of Sunada's type.

LEMMA 3. *Let O be a maximal order in $\mathbb{D}$ and $L \supseteq O^1$ be a subgroup of $\mathbb{D}^1$. If $[L : O^1] < \infty$, then $L = O^1$.*

PROOF. Let $\mathcal{L} = R[L]$ be the ring generated by L over the ring R of integers of K. Since $[L : O^1] < \infty$, we can write $L = \bigcup_{j=1}^{s}\{g_j O^1\}$ and $\mathcal{L} = \sum R\{g_j O^1\}$. Thus $\mathcal{L}$ is a finitely generated R-module containing $R[O^1]$, hence $\mathcal{L}$ is an order in $\mathbb{D}$. As $\mathcal{L}^1 \supseteq L \supseteq O^1$, the maximality of O yields that $O^1 = \mathcal{L}^1 = L$ (*e.g.*, by comparing their covolumes. *cf.* [14]). ∎

LEMMA 4. *Let O_1 and O_2 be any two orders in $\mathbb{D}$ and let Γ_1 and Γ_2 be the images of O_1^1 and O_2^1 in G^*. For $\gamma \in G^*$, if Γ_1 and $\gamma\Gamma_2\gamma^{-1}$ are commensurable, i.e., if the*

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intersection $\Gamma_1 \cap \gamma\Gamma_2\gamma^{-1}$ *has finite index in both* Γ_1 *and* $\gamma\Gamma_2\gamma^{-1}$, *then* $\gamma\Gamma_2\gamma^{-1} = g\Gamma_2 g^{-1}$ *for some* $g \in \rho(\mathbb{D}^*)$.

PROOF. It is well known that Γ_1 and Γ_2 are commensurable (see [11, Chapter IV]) and that commensurability is transitive. Hence, Γ_2 and $\gamma\Gamma_2\gamma^{-1}$ are commensurable. Thus, Lemma 4 follows from Corollory 1.5 in [11, Chapter IV, p. 106]. ∎

PROPOSITION 3. *Let* $\Gamma_1\backslash X$ *and* $\Gamma_2\backslash X$ *be the isospectral but non-isometric Riemannian manifolds constructed as above from a quaternion division algebra* $\mathbb{D}$ *which is unramified at only one archimedean place, (i.e.,* $s + r_2 = 1$*). Then* $\Gamma_1\backslash X$ *and* $\Gamma_2\backslash X$ *cannot be sandwiched with any finite triplet* (H, H_1, H_2).

PROOF. Suppose $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are sandwiched by M and M/H with some finite triplet (H, H_1, H_2). Since X is the universal covering of M, we have a diagram of coverings:

$$\begin{tikzcd}
& M \arrow[d, equal] & \\
& \Gamma\backslash X \arrow[dl] \arrow[dr] \arrow[dd] & \\
\Gamma_1\backslash X \cong \Gamma_1'\backslash X \arrow[dr] & & \Gamma_2'\backslash X \cong \Gamma_2\backslash X \arrow[dl] \\
& \Gamma_0\backslash X \arrow[d, equal] & \\
& M/H &
\end{tikzcd}$$

where $\Gamma_0 \supseteq \Gamma_1', \Gamma_2' \supseteq \Gamma$ are discrete subgroups of G^*. So we have $\Gamma_1' = \gamma_1\Gamma_1\gamma_1^{-1}$ and $\Gamma_2' = \gamma_2\Gamma_2\gamma_2^{-1}$ for some γ_1, γ_2 in G^*. Up to a conjugation, we may assume that $\gamma_1 = \mathrm{id}$. Since Γ is of finite index in both Γ_1 and $\gamma_2\Gamma_2\gamma_2^{-1}$, by Lemma 4 $\gamma_2\Gamma_2\gamma_2^{-1} = g\Gamma_2 g^{-1}$ for some $g \in \rho(\mathbb{D}^*)$. As $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are not isometric, $\Gamma_1 \neq g\Gamma_2 g^{-1}$. Since Γ_0 contains both Γ_1 and $g\Gamma_2 g^{-1}$, by Lemma 3, it cannot be of finite index over Γ_1, which contradicts to the assumption that $M \longrightarrow M/H$ is a finite quotient. ∎

REMARK. When $\mathbb{D}$ is unramified at more than one archimedean place, *i.e.*, when $s + r_2 > 1$, the group of isometries of X properly contains G^*, preventing us for concluding that $\Gamma_1' = \gamma_1\Gamma_1\gamma_1^{-1}$ and $\Gamma_2' = \gamma_2\Gamma_2\gamma_2^{-1}$ for some $\gamma_1, \gamma_2 \in G^*$ as in the proof of Proposition 3. So it remains an open question whether these examples of Vignéras arise from Sunada's construction.

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