LATTICE OF IDEMPOTENT SEPARATING CONGRUENCES IN A ℙ-REGULAR SEMIGROUP

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In this paper we introduce the concept of ℙ-regular, normal subset of a ℙ-regular semigroup, give an alternate characterisation of the maximum idempotent separating congruence in a ℙ-regular semigroup and finally describe the lattice of the idempotent separating congruences in a ℙ-regular semigroup.

1. INTRODUCTION

Yamada and Sen introduced the concept of ℙ-*regularity* [4] in a regular semigroup as a generalisation of both the concept of "orthodox" and the concept of "(special) involution" [3]. In [4] they characterised the maximum idempotent separating congruence τ on a ℙ-regular semigroup S. In this paper firstly we give an alternate characterisation of τ and secondly we give a description of the lattice of idempotent separating congruences on a ℙ-regular semigroup S, which generalises Feigenbaum's result [1] for orthodox semigroups. Unless otherwise defined, our notation will be that of [2].

2. PRELIMINARY RESULTS AND DEFINITIONS

Let S be a regular semigroup and E_S the set of idempotents of S. Let $P \subseteq E_S$. Then (S, P) is called a ℙ-*regular* semigroup [5] if it satisfies the following:

(1) $P^2 \subseteq E_S$

(2)   for each $q \in P$, $qPq \subseteq P$.

(3)   for any $a \in S$, there exists $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$. ($P^1 = P \cup \{1\}$ and $V(a) = \{a^+ \in S : aa^+a = a, a^+aa^+ = a^+\}$).

Hereafter (S, P) will be denoted by $S(P)$. Let $S(P)$ be a ℙ-regular semigroup. For any $a \in S$, $a^+ \in V(a)$ such that $aP^1a^+ \subseteq P$ and $a^+P^1a \subseteq P$ is called a ℙ-*inverse of* a and $V_P(a)$ denotes the set of all ℙ-inverses of a. Now $V_P(a) \neq \emptyset$ for any $a \in S$. Also for any $p \in P$, $p \in V(P)$ and $pP^1p \subseteq P$. So $p \in V_P(p)$. The following lemma follows from Lemma 2.5 of [5].

LEMMA 2.1. *In a $\mathbb{P}$-regular semigroup $S(P)$, if $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, then $b^+a^+ \in V_P(ab)$.*

The following lemma, which will be used frequently in this paper, is taken from [5].

LEMMA 2.2. *Let a, b be two elements of a $\mathbb{P}$-regular semigroup $S(P)$. Then $a\mathcal{H}b$ (where $\mathcal{H}$ is Green's $\mathcal{H}$-relation) if and only if there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$ and $a^+a = b^+b$.*

NOTE. Actually if $a\mathcal{H}b$ then for all $a^+ \in V_P(a)$ there exists $b^+ \in V_P(b)$ such that $aa^+ = bb^+$ and $a^+a = b^+b$. Because for any $a^+ \in V_P(a)$ we have $a^+a\mathcal{L}a\mathcal{R}aa^+$. Consequently $a^+a\mathcal{L}b\mathcal{R}aa^+$. So by Lemma 2.7 of [5] there exists a unique $b^+ \in V_P(b)$ such that $bb^+ = aa^+$ and $b^+b = a^+a$. Moreover we notice that if $(a, p) \in \mathcal{H}$ for some $p \in P$ then as $p \in V_P(p)$ there exists a unique $a^+ \in V_P(a)$ such that $aa^+ = pp(=p) = a^+a$. But this implies $(a^+, p) \in \mathcal{H}$ by Lemma 2.2. Hence $a^+ \in V_P(a) \cap H_a$.

A subset B of a $\mathbb{P}$-regular semigroup $S(P)$ will be called $\mathbb{P}$-*regular* if for every $x \in B$ there exists $x^+ \in V_P(x)$ such that $x^+ \in B$.

A subset N of a $\mathbb{P}$-regular semigroup $S(P)$ will be called *normal* if for any a, $b \in S$, and $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, such that $aa^+ = bb^+$, $a^+a = b^+b$ then ab^+, $a^+b \in N$ imply $aNb^+ \subseteq N$ and $a^+Nb \subseteq N$.

LEMMA 2.3. *In a $\mathbb{P}$-regular semigroup $S(P)$ the set P is a $\mathbb{P}$-regular normal subset of S.*

PROOF: For each $p \in P$, $p \in V(p)$ and $pP^1p \subseteq P$. So $p \in V_P(p) \cap P$. Thus P is a $\mathbb{P}$-regular subset of S. Also we note that if for any $a, b \in S$ there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$ and $a^+a = b^+b$ then ab^+, $a^+b \in P$ imply $aPb^+ = aa^+aPb^+bb^+ = bb^+(aPa^+)ab^+ = bb^+bb^+(aPa^+)ab^+ab^+ = b(a^+(ab^+(aPa^+)ab^+)a)b^+ \subseteq P$ and $a^+Pb = a^+aa^+Pbb^+b = a^+b(b^+Pb)a^+a = a^+ba^+b(b^+Pb)a^+aa^+a = a^+(b(a^+b(b^+Pb)a^+b)b^+)a \subseteq P$. Thus P is a normal subset of S. □

3. THE LATTICE OF IDEMPOTENT SEPARATING CONGRUENCES

Yamada and Sen [4] characterised τ, the maximum idempotent separating congruence on a $\mathbb{P}$-regular semigroup $S(P)$, as $\tau = \{(a, b) \in S \times S : \text{ there exist } a^+ \in V_P(a) \text{ and } b^+ \in V_P(b) \text{ such that } axa^+ = bxb^+ \text{ and } a^+xa = b^+xb \text{ for all } x \in P\}$. Here we shall present an alternate characterisation of τ. For this we define *centraliser of P* to be $C(P) = \{x \in S : (x, p) \in \tau \text{ for some } p \in P\}$.

THEOREM 3.1. *Let $\delta = \{(a, b) \in S \times S : \text{ there exists } a^+ \in V_P(a), b^+ \in V_P(b) \text{ such that } aa^+ = bb^+, a^+a = b^+b, ab^+, a^+b \in C(P)\}$. Then $\tau = \delta$.*

PROOF: Let $(a, b) \in \tau$. Then as τ is an idempotent separating congruence on a regular semigroup, $\tau \subseteq \mathcal{H}$. Hence by Lemma 2.2, there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$. Also $(ab^+, bb^+) \in \tau$, $(a^+a, a^+b) \in \tau$ where bb^+ and $a^+a \in P$. Therefore ab^+, $a^+b \in C(P)$ and so $(a, b) \in \delta$. Conversely let $(a, b) \in \delta$. Then there are inverses $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and ab^+, $a^+b \in C(P)$. So $a\mathcal{H}b$ and $a^+\mathcal{H}b^+$ which gives $ab^+\mathcal{H}bb^+$. Since $ab^+ \in C(P)$, we have $(ab^+)\tau = (bb^+)\tau$. Therefore $a\tau = (a\tau)(a^+a)\tau = (a\tau)(b^+b)\tau = (ab^+)\tau \dot{b}\tau = (bb^+)\tau b\tau = b\tau$. Thus $\delta = \tau$.   ☐

LEMMA 3.2. $C(P)$ *is a* $\mathbb{P}$*-regular, normal subset of* S.

PROOF: Let $a \in C(P)$; then $(a, p) \in \tau$, for some $p \in P$. So $(a, p) \in \mathcal{H}$. Hence by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = p$. Also $(a^+a, a^+p) \in \tau$ implies that $(p, a^+) \in \tau$. Thus $a^+ \in C(P)$. Next let $a, b \in S$ be such that there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and ab^+, $a^+b \in C(P)$. Then by Theorem 3.1 $(a, b) \in \tau$. So $(axb^+, bxb^+) \in \tau$ for all $x \in C(P)$. Now $x \in C(P)$ implies that $(x, p) \in \tau$ for some $p \in P$. So $(bxb^+, bpb^+) \in \tau$ where $bpb^+ \in P$. Therefore $axb^+ \in C(P)$. Similarly, $a^+xb \in C(P)$ for all $x \in C(P)$. So $C(P)$ is a $\mathbb{P}$-regular, normal subset of S.   ☐

LEMMA 3.3.   *Let* $A = \{a \in S : \text{ there exists } a^+ \in V_P(a) \text{ for which } a^+pap = aa^+p$, *and* $papa^+ = pa^+a$ *for each* $p \in P\}$. *Then* $C(P) = A$.

PROOF: Let $a \in C(P)$. Then $(a, q) \in \tau$ for some $q \in P$. Since $\tau \subseteq \mathcal{H}$ by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = q$. Now $(a, q) \in \tau$ implies that $(a^+a, a^+q) \in \tau$, that is $(a^+, q) \in \tau$. For each $p \in P$, we have $(apa^+, qpq) \in \tau$ and $(a^+pa, qpq) \in \tau$ which implies that $apa^+ = qpq = a^+pa$. So we have, $a^+pap = qpqp = qp = aa^+p$ and $papa^+ = pqpq = pq = pa^+a$. Conversely, if $a \in A$, then there exists $a^+ \in V_P(a)$ such that $a^+pap = aa^+p$ and $papa^+ = pa^+a$ for all $p \in P$. Since $aa^+ \in P$, $a^+a \in P$, we have $aa^+ = aa^+aa^+ = a^+(aa^+)a(aa^+) = a^+aaa^+ = (a^+a)a(a^+a)a^+ = (a^+a)a^+a = a^+a$. So for any $p \in P$, $apa^+ = a(a^+a)pa^+ = a(aa^+p)a^+ = a(a^+pap)a^+ = (aa^+)(papa^+) = aa^+p(a^+a) = aa^+paa^+$ and $a^+pa = a^+p(aa^+)a = a^+(pa^+a)a = a^+(papa^+)a = (a^+pap)a^+a = aa^+p(a^+a) = (aa^+)p(aa^+)$. Hence $(a, aa^+) \in \tau$. Therefore $a \in C(P)$. Thus $A = C(P)$.   ☐

The following lemma will be used frequently in the proof of the next theorem.

LEMMA 3.4.   *Let* K *be a normal subset of a* $\mathbb{P}$*-regular semigroup* $S(P)$ *such that* $P \subseteq K$. *For* $a, b \in S$ *if there exist* $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ *such that* $aa^+ = bb^+$, $a^+a = b^+b$ *and* ab^+, $a^+b \in K$ *then* ba^+, $b^+a \in K$ *and* ab^*, $a^*b \in K$ *for all* $a^* \in V_P(a)$, $b^* \in V_P(b)$.

PROOF: First we note that in a normal subset $K(\supseteq P)$, $aKa^+ \subseteq K$ for all $a \in S$,

$a^+ \in V_P(a)$. Now

$$ba^+ = bb^+ba^+ = aa^+ba^+ \in aKa^+ \subseteq K,$$
$$b^+a = b^+aa^+a = b^+ab^+b \in b^+Kb \subseteq K,$$
$$ab^* = aa^+ab^* = bb^+ab^* \in bKb^* \subseteq K,$$
and $\cdot$ $$a^*b = a^*bb^+b = a^*ba^+a \in a^*Ka \subseteq K.$$

⬜

The following theorem gives a description of the lattice of idempotent separating congruences on a $\mathbb{P}$-regular semigroup $S(P)$.

Let $N_r = \{K \subseteq S : P \subseteq K \subseteq C(P)$ where K is normal and $\mathbb{P}$-regular$\}$. Clearly P and $C(P)$ belong to N_r.

THEOREM 3.5. *The map $K \to (K) = \{(a, b) \in S \times S :$ there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ for which $aa^+ = bb^+$, $a^+a = b^+b$ and ab^+, $a^+b \in K\}$ is a one to one order preserving map of N_r onto the set of all idempotent separating congruences on S.*

PROOF: First we shall show that if $K \in N_r$ then (K) is an idempotent separating congruence on S. Since $P \subseteq K$, (K) is a reflexive relation. Furthermore, $K \subseteq C(P)$ implies $(K) \subseteq \tau$ (by Theorem 3.1), so that (K) is an idempotent separating relation. Also if $(a, b) \in (K)$ then by Lemma 3.4 $(b, a) \in (K)$. Hence (K) is symmetric. To prove that (K) is transitive, let $(a, b) \in (K)$, $(b, c) \in (K)$. Then there exist $a^+ \in V_P(a)$, b^+, $b^* \in V_P(b)$, $c^* \in V_P(c)$ such that $aa^+ = bb^+$, $a^+a = b^+b$, $bb^* = cc^*$, $b^*b = c^*c$. But $(K) \subseteq \tau \subseteq \mathcal{H}$ implies that $a\mathcal{H}b\mathcal{H}c$ so that by the Note after Lemma 2.2 there exists $c^+ \in V_P(c)$ such that $aa^+ = bb^+ = cc^+$ and $a^+a = b^+b = c^+c$. Also ab^+, a^+b, bc^+, $b^+c \in K$ by Lemma 3.4. Therefore $ac^+ = aa^+ac^+cc^+ = ab^+bc^+bb^+ \in K$ and $a^+c = a^+aa^+cc^+c = a^+bb^+cb^+b \in K$, as K is normal. So $(a, c) \in (K)$. To see that (K) is compatible, let $(a, b) \in (K)$, $(c, d) \in (K)$. Since $K \subseteq C(P)$ we have $(K) \subseteq \tau$. So (a, b), $(c, d) \in \tau$. Hence there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$, $c^+ \in V_P(c)$, $d^+ \in V_P(d)$ such that $apa^+ = bpb^+$, $a^+pa = b^+pb$, $cpc^+ = dpd^+$, $c^+pc = d^+pd$ for all $p \in P$. Then it follows that $aa^+ = bb^+$, $a^+a = b^+b$, $cc^+ = dd^+$, $c^+c = d^+d$ and by Lemma 2.1 $c^+a^+ \in V_P(ac)$, $d^+b^+ \in V_P(bd)$. Now $(ac)(c^+a^+) = a(cc^+)a^+ = a(dd^+)a^+ = b(dd^+)b^+ = (bd)(d^+b^+)$, $(c^+a^+)(ac) = c^+(a^+a)c = c^+(b^+b)c = d^+(b^+b)d = (d^+b^+)(bd)$, $(ac)(d^+b^+) = a(cd^+)b^+ \in aKb^+ \subseteq K$ and $(c^+a^+)(bd) = c^+(a^+b)d \in c^+Kd \subseteq K$ as cd^+ and $a^+b \in K$ by Lemma 3.4. Thus $(ac, bd) \in (K)$. Therefore (K) is an idempotent separating congruence. Now if μ is an idempotent separating congruence on S then we define the P-kernel of μ, written as P-ker μ, by P-ker $\mu = \{a \in S : a\mu p$ for some $p \in P\}$. We shall show that P-ker $\mu \in N_r$ and $(P\text{-ker}\mu) = \mu$. To prove that P-ker μ is a $\mathbb{P}$-regular subset of S, let $a \in P$-ker

μ. Then $(a, p) \in \mu \subseteq \mathcal{H}$ for some $p \in P$. So by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = p$. Now $(a^+a, a^+p) \in \mu$ or, $(p, a^+) \in \mu$. Therefore $a^+ \in P$-ker μ. To prove that P-ker μ is normal, let $a, b \in S$, $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $a^+b, ab^+ \in P$-ker μ. We shall show that axb^+ and a^+xb belong to P-ker μ for all $x \in P$-ker μ. From the given condition $a\mathcal{H}b$, $a^+\mathcal{H}b^+$. Therefore $ab^+\mathcal{H}bb^+$. As $ab^+ \in P$-ker μ, we have $(ab^+)\mu = (bb^+)\mu$. Therefore $a\mu = a\mu(a^+a)\mu = a\mu(b^+b)\mu = (ab^+)\mu b\mu = (bb^+)\mu b\mu = b\mu$. Hence $(a, b) \in \mu$. So $(axb^+, bxb^+) \in \mu$ for all $x \in P$-ker μ. Now $x \in P$-ker μ implies that $(x, p) \in \mu$ for some $p \in P$. Hence $(bxb^+, bpb^+) \in \mu$. Thus $(axb^+, bpb^+) \in \mu$. But $bpb^+ \in P$. So $axb^+ \in P$-ker μ. Similarly $a^+xb \in P$-ker μ for all $x \in P$-ker μ. Also $P \subseteq P$-ker $\mu \subseteq C(P)$. Therefore P-ker $\mu \in N_r$. Now we shall show that $(P$-ker $\mu) = \mu$. For this let $(a, b) \in (P$-ker $\mu)$; then there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$ and $ab^+, a^+b \in P$-ker μ. Therefore $a\mathcal{H}b$, $a^+\mathcal{H}b^+$ which implies that $ab^+\mathcal{H}bb^+$. As $ab^+ \in P$-ker μ we have $(ab^+)\mu = (bb^+)\mu$. Therefore $a\mu = a\mu(a^+a)\mu = a\mu(b^+b)\mu = (ab^+)\mu b\mu = (bb^+)\mu b\mu = b\mu$. Next let $(a, b) \in \mu$. Then $(a, b) \in \mathcal{H}$. So there exist $a^+ \in V_P(a)$, $b^+ \in V_P(b)$ such that $aa^+ = bb^+$, $a^+a = b^+b$. Also $(ab^+, bb^+) \in \mu$ and $(a^+a, a^+b) \in \mu$. Therefore $ab^+, a^+b \in P$-ker μ. Hence $(a, b) \in (P$-ker $\mu)$. Thus the given map is onto. The given map is clearly order preserving. We shall now show that the given map is one to one. For this let $K, L \in N_r$ with $(K) = (L)$. Let $a \in K$, since $K \subseteq C(P)$, $a \in C(P)$. Therefore $(a, p) \in \tau$ for some $p \in P$. Since $\tau \subseteq \mathcal{H}$ by the Note after Lemma 2.2 there exists $a^+ \in V_P(a) \cap H_a$ such that $aa^+ = a^+a = p$. Now K is ℙ-regular so there exists $a^* \in V_P(a) \cap K$. Then $a^+ = a^+aa^+ = (a^+a)a^*(aa^+) = pa^*p \in K$. Thus $a\mathcal{H}a^+a$, $a(a^+a) = a \in K$ and $a^+(a^+a) = a^+(aa^+) = a^+ \in K$. Therefore $(a, a^+a) \in (K) = (L)$. So $a(a^+a)^* \in L$ for all $(a^+a)^* \in V_P(a^+a)$ by Lemma 3.4. So in particular $aa^+a \in L$ so that $a \in L$. Thus $K \subseteq L$. Similarly we can prove $L \subseteq K$. Thus $K = L$.   □

NOTE. If we take $P = E$ then S becomes an orthodox semigroup and N_r becomes the collection of all self conjugate, regular subsemigroups K such that $E \subseteq K \subseteq C(E)$ so we get the Theorem 3.3 of [1] due to Feigenbaum, because in that case $a, b \in K$ implies $a, b \in C(E)$. So $a\tau e$, $b\tau f$ for some $e, f \in E$ and there exist $a^+ \in V(a)$, $b^+ \in V(b)$ such that $aa^+ = a^+a = e$, $bb^+ = b^+b = f$ and $ae = a \in K$, $a^+e = a^+ \in K$ and $abef = ab = efab$. Since K is normal we have $aKe \subseteq K$. Therefore $efabeef \in efKef$ or, $ab \in efKef \subseteq K$. If we take $P = \{e\}$ then S becomes a group and N_r becomes the collection of all normal subgroups of the group S.

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