Two Geometrical Transformations.

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The transformations discussed in the present paper are, like the isogonal and isotomic transformations, particular cases of the general birational quadratic transformation, in which points correspond to points, and lines to conics passing through three fixed points. They seem to possess some interest in connection with the Geometry of the Triangle.

1. Let K be the centre of a conic which touches the sides BC, CA, AB of the triangle of reference at X, Y, Z, and S be the point of concurrence of AX, BY, CZ. Then if x, y, z be the trilinear coordinates of S, and x', y', z' those of K, we have

$$x' : y' : z' = x(by + cz) : y(cz + ax) : z(ax + by),$$

and

$$x : y : z = 1/a(by' + cz' - ax') : 1/b(cz' + ax' - by') : 1/c(ax' + by' - cz').$$

The following special cases may be noticed.

(i) If S be the Gergonne point, K is the incentre, with similar statements for the exterior Gergonne points.

(ii) If S be the orthocentre, K is the symmedian point.

(iii) If S be the symmedian point, K is the mid-point of the distance between the Brocard points.

(iv) If S be the centroid, K is also the centroid.

(v) If S be the isotomic conjugate of the circumcentre $\left(\frac{1}{a^2 \cos A}, \text{ etc.}\right)$, K is the nine-point centre.

(vi) If S be the isotomic conjugate of the orthocentre $\left(\frac{1}{a^2 \sec A}, \text{ etc.}\right)$, K is the circumcentre.

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If K be made to describe any locus, the corresponding locus of S may be called the S-transformation of the locus of K, and the locus of K may be called the K-transformation of the locus of S.

2. The S-transformation of a straight line is a conic circumscribed to the triangle.

Demonstration. Let the locus of K be the straight line $lx+my+nz=0$; then, substituting in this equation $x(by+cz)$ for x, $y(cz+ax)$ for y, and $z(ax+by)$ for z, we obtain as the locus of S,

$$(bn+cm)yz+(cl+an)zx+(am+bl)xy=0,$$

which is a conic circumscribed to the triangle.

The following special cases call for notice.

(i) When the K-locus is the line at infinity, $ax+by+cz=0$, the S-locus is the circumconic,

$$bcyz+cazx+abxy=0,$$

i.e., the minimum circumscribed ellipse, having the centroid as centre, and well-known as the Steiner ellipse.

This theorem may be stated otherwise as follows. If the lines joining the vertices of a triangle to any point on its Steiner ellipse meet the opposite sides of the triangle in X, Y, Z, the conic which touches the sides of the triangle at X, Y, Z is a parabola.

(ii) When the K-locus is the line $x\cos A+y\cos B+z\cos C=0$, the S-locus is

$$ayz+bzx+cxy=0,$$

i.e., the circumcircle of the triangle.

It may be noticed that the line $\Sigma x\cos A=0$ is perpendicular to the line joining the circumcentre and orthocentre—the Euler line. Hence using the theorem of (i), and recollecting that the fourth point of intersection of the circumcircle and the Steiner ellipse is the Steiner point of the triangle, and that the directrix of a parabola inscribed in the triangle passes through the orthocentre, we have the theorem that if the connectors of the Steiner point of a triangle with the vertices meet the opposite sides in X, Y, Z, the conic touching the sides at these points is a parabola whose directrix is the Euler line.

With regard to the Steiner point it may be remarked that it has

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been defined as the point whose isogonal and isotomic conjugates lie at infinity. It now appears, from the fact that it lies on the Steiner ellipse, that its K-transformation also lies at infinity.

(iii) When the K-line passes through the symmedian point, the S-conic passes through the orthocentre (Art. 1, ii), and is, therefore, a rectangular hyperbola.

(iv) When the K-line passes through the centroid, the S-conic also passes through the centroid and touches the K-line there, for (Art. 1, iv) S and K coincide in the centroid and nowhere else.

The fact of the contact may be confirmed as follows by analysis. Since the K-line passes through the centroid $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$, its equation may be written

$$a(b\mu - c\nu)x + b(c\nu - a\lambda)y + c(a\lambda - b\mu)z = 0.$$

The equation of the S-conic is then found to be

$$\frac{b\mu - c\nu}{a} \,.\, yz + \frac{c\nu - a\lambda}{b} \,.\, zx + \frac{a\lambda - b\mu}{c} \,.\, xy = 0.$$

The condition, then, that the line should touch the conic is that

$$\Sigma(b\mu - c\nu)^4 - \Sigma 2(c\nu - a\lambda)^2(a\lambda - b\mu)^2 = 0,$$

which is the case, since

$$(b\mu - c\nu) + (c\nu - a\lambda) + (a\lambda - b\mu) = 0.$$

The theorem may be stated in the following restricted form. If K be the centre of a conic which touches the sides BC, CA, AB of a triangle at X, Y, Z respectively, and S be the point of concurrence of AX, BY, CZ, and if G is the centroid of ABC, then the conic through A, B, C, S, G touches KG at G.

(v) Suppose that the K-line passes through the vertex A. Its equation then is $my - nz = 0$, and the equation of the S-conic is, consequently, $(-bn + cm)yz - anzx + amxy = 0.$

The coordinates of the centre of this conic are found to be 0, c, b. Hence we have the theorem that when the K-line passes through a vertex of the triangle, the centre of the S-conic is the mid-point of the opposite side.

Thus, as particular cases, we have the following theorems.

(1) If the connectors of the vertices of a triangle ABC with a variable point on the circumscribed rectangular hyperbola whose centre is the mid-point of BC, meet the opposite sides in X, Y, Z, the locus of the centre of the conic which touches the sides at X, Y, Z is the symmedian through A, for, by the present section, this locus passes through A and by (iii) it also passes through the symmedian point.

(2) If Γ, Γ_1, Γ_2, Γ_3 denote the Gergonne points of ABC, Γ_1, Γ_2, Γ_3 being respectively opposite to A, B, C, the mid-point of BC is the centre of the conics through A, B, C, Γ, Γ_1 and A, B, C, Γ_2, Γ_3 for the K-transformations of those conics are obviously the internal and external bisectors respectively of the angle A.

3. The K-transformation of a straight line is a conic medioscribed to the triangle, *i.e.*, passing through the mid-points of its sides.

Demonstration. Let the locus of S be the straight line

$$lx + my + nz = 0.$$

Then, substituting in this equation $1/a(by + cz - ax)$ for x, with similar substitutions for y and z (Art. 1), we obtain as the locus of K, the conic

$$\Sigma a^2(bcl - cam - abn)x^2 + \Sigma 2b^2c^2lyz = 0.$$

The coordinates of the mid-points of the sides, viz.,

$$(0,\ c,\ b),\quad (c,\ 0,\ a),\quad (b,\ a,\ 0)$$

satisfy this equation, and therefore the conic passes through these points.

The following particular cases require notice.

(i) When the S-locus is the line at infinity, $ax + by + cz = 0$, the K-locus is

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0.$$

As is well known, this conic is the maximum inscribed ellipse; it touches the sides of the triangle at their mid-points and has the centroid as centre.

The theorem might be stated otherwise thus. If a conic whose centre lies on the maximum inscribed ellipse of a triangle ABC touch the sides BC, CA, AB at X, Y, Z respectively, then AX, BY, CZ are parallel.

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(ii) When the S-locus is the line $a^3x + b^3y + c^3z = 0$, the K-locus is

$$\Sigma a^2(a^2 - b^2 - c^2)x^2 + \Sigma 2a^2bcyz = 0,$$

$$i.e., \quad \Sigma a\cos A \,.\, x^2 - \Sigma ayz = 0,$$

i.e., the nine-point circle of the triangle.

It may be noticed that the line $x\cos A + y\cos B + z\cos C = 0$, referred to in Art. 2, ii, and the line $a^3x + b^3y + c^3z = 0$ are parallel, and that the former is the polar of the symmedian point with respect to the maximum inscribed ellipse, while the latter is the polar of the isotomic conjugate of the orthocentre with respect to the minimum circumscribed ellipse, also that the former is related to the complementary triangle of ABC (the triangle whose vertices are the mid-points of ABC) in the same way that the latter is related to ABC.

(iii) When the S-line passes through the isotomic conjugate of the orthocentre, the K-conic passes through the circumcentre (Art. 1, vi), *i.e.*, through the orthocentre of the triangle formed by joining the mid-points of the sides. Therefore in this case the K-conic is a rectangular hyperbola.

It may be noted that the isotomic conjugate of the orthocentre is the symmedian point of the anticomplementary triangle of ABC (the triangle formed by parallels through A, B, C to the opposite sides) and is therefore related to ABC in the same way that the symmedian point is related to the complementary triangle of ABC.

(iv) When the S-line passes through the centroid, the K-conic also passes through the centroid, and touches the S-line there, for (Art. 1, iv) S and K coincide in the centroid and nowhere else.

The fact of the tangency could easily be confirmed by analysis.

Combining the statement of Art. 2, iv with the foregoing, we see that the S- and K-transformations of a straight line through the centroid touch each other and the line at the centroid. This might be stated in a variety of forms, of which the following may be given. If two conics, one circumscribed, the other medioscribed to a triangle, touch at the centroid, and if P be the centre of a conic which touches the sides of the triangle at points which connect with the opposite vertices through a point on the circumscribed conic and if Q be the point of concurrence of the lines drawn from the vertices to meet the opposite sides at points where these sides are touched by a conic whose centre lies on the medioscribed conic, then PQ touches the circumscribed and medioscribed conics at the centroid.

4. The S-transformation of a line is a parabola, hyperbola, or ellipse, according as the line touches, cuts, or does not meet the maximum inscribed ellipse; also the K-transformation of a line is a parabola, hyperbola, or ellipse, according as the line touches, cuts, or does not meet the minimum circumscribed ellipse.

This theorem depends on another theorem which is sufficiently evident, viz., that if κ be a medioscribed conic and k a line connected with it, and if the line s and the circumscribed conic σ be the S-transformations of κ and k respectively, then if k touch κ at a point, s touches σ at the S-transformation of that point, and s cuts or does not meet σ according as k cuts or does not meet κ; and *vice versa.*

If now κ be the maximum inscribed ellipse, s is the line at infinity (Art. 3, i). Therefore, when k touches κ, σ touches the line at infinity, and is, therefore, a parabola; when k cuts κ, σ meets the line at infinity in two real points and is, therefore, a hyperbola; and when k does not meet κ, σ does not meet the line at infinity in real points, and is, therefore, an ellipse.

The second half of the proposition is proved in a precisely similar manner.

5. The isogonal transformations and the K-transformations of a system of points, lines, and circumscribed conics, are projectively related; also the isogonal transformations with respect to the triangle A′B′C′, where A′, B′, C′ are the mid-points of the sides of ABC, and the S-transformations (with respect to ABC) of a system of points, lines, and medioscribed conics, are projectively related.

These statements are involved in the more general theorem that if F be a system of lines, conics, etc., F′ a system obtained by a birational quadratic transformation of F, and F″ a system obtained by another birational quadratic transformation of F, then F′ and F″ can be obtained, the one from the other, by a linear transformation, and are, consequently, projective systems.

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The substitutions in the case of the isogonal and K-transformations can be readily shown as follows.

Let $$\lambda x^{\pm 1} + \mu y^{\pm 1} + \nu z^{\pm 1} = 0$$

be a line or circumconic. Its isogonal transformation is

$$\lambda x^{\mp 1} + \mu y^{\mp 1} + \nu z^{\mp 1} = 0,$$

and its K-transformation

$$\lambda\{a(by + cz - ax)\}^{\mp 1} + \mu\{b(cz + ax - by)\}^{\mp 1} + \nu\{c(ax + by - cz)\}^{\mp 1} = 0.$$

Hence, in order to obtain the K-transformation from the isogonal transformation we require to substitute

$$a(by + cz - ax), \quad b(cz + ax - by), \quad c(ax + by - cz)$$

for x, y, z respectively.

The following tables, which might be indefinitely extended, may be given in illustration of the theorem of the present article.

TABLE I.

Primary elements consisting of points, lines, and circumconics.	*Derived elements, projectively related.*	
	Isogonal transformations.	*K-transformations.*
line at infinity	circumcircle	maximum inscribed ellipse
circumcircle	line at infinity	line $\Sigma x \cos A = 0$
line $\Sigma a^3 x = 0$	conic $\Sigma a^3/x = 0$	nine-point circle
minimum circumscribed ellipse	line $\Sigma x/a = 0$	line at infinity
orthocentre	circumcentre	symmedian point
centroid	symmedian point	centroid
symmedian point	centroid	mid-point of distance between Brocard points Ω and Ω'
incentre	incentre	Gergonne point
isotomic conjugate of circumcentre	isogonal conjugate of isotomic conjugate of circumcentre	nine-point centre
isotomic conjugate of orthocentre	isogonal conjugate of isotomic conjugate of orthocentre	circumcentre
etc.	etc.	etc.

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TABLE II.

Primary elements consisting of points, lines, and medioscribed conics.	*Derived elements, projectively related.*	
	Isogonal transformations with respect to A′B′C′.	*S-transformations.*
line at infinity	nine-point circle	minimum circumscribed ellipse
nine-point circle	line at infinity	line $\Sigma a^3 x = 0$
line $\Sigma x \cos A$	conic $\Sigma \frac{a^2(b^2+c^2-3a^2)}{by+cz-ax} = 0$	circumcircle
maximum inscribed ellipse	line $\Sigma xa(b^2c^2 - c^2a^2 - a^2b^2) = 0$	line at infinity
circumcentre	nine-point centre	isotomic conjugate of orthocentre
nine-point centre	circumcentre	isotomic conjugate of circumcentre
mid-point of distance between Brocard points	point $a(b^4+c^4)$, etc., (viz., the isogonal conjugate of the isotomic conjugate of the symmedian point of A′B′C′)	symmedian point
centroid	symmedian point of A′B′C′	centroid
etc.	etc.	etc.

By means of these tables a great number of elementary theorems can be readily obtained. Thus from the second and third columns of Table I. we obtain the following.

(1) The circumcentre, the centroid, and the isogonal conjugate of the isotomic conjugate of the orthocentre are collinear (*i.e.*, the last-named point lies on the Euler line); for their correspondents in the third column, viz., the symmedian point, the mid-point of $\Omega\Omega'$, and the circumcentre lie on a straight line (the line of centres of the Tucker circles).

(2) The symmedian point, the isogonal conjugate of the isotomic conjugate of the orthocentre, and the isogonal conjugate of the isotomic conjugate of the circumcentre are collinear; for their

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correspondents in the third column, viz., the centroid, circumcentre, and nine-point centre are collinear.

(3) Since the line of centres of the Tucker circles in the third column corresponds to the Euler line in the second, the S-transformation of the former line is the isogonal transformation of the latter, viz., the Jĕrabek hyperbola. *

(4) Since the join of the centroid and symmedian point in the third column corresponds to the line of centres of the Tucker circles in the second, the S-transformation of the former line is the isogonal transformation of the latter, viz., the Kiepert hyperbola.

From Table II. it appears that the K-transformation of the join of the isotomic conjugates of the orthocentre and circumcentre (which corresponds in the third column to the Euler line of A′B′C′ in the second) is the Jĕrabek hyperbola of A′B′C′.

As a specimen of the theorems derivable from Table I., but not depending on the projective relation of the elements in the second and third columns, we have the following.

The isotomic conjugates of the circumcentre and orthocentre lie on a circumscribed hyperbola which passes through the centroid, and touches the Euler line at that point. This follows from the fact that the K-transformations of the three points mentioned lie on the Euler line, and from Arts. 2, iv and 4.

6. The K-transformations of a system of similar circumconics envelop a conic which has double contact with the maximum inscribed ellipse on the line $\Sigma x \cos A = 0$; and the S-transformations of a system of similar medioscribed conics envelop a conic which has double contact with the minimum circumscribed ellipse on the line $\Sigma a^3 x = 0$.

Demonstration. For convenience the elements belonging to the third column of Table I. will be called the K-projections of the corresponding elements in the second column, and the elements in the third column of Table II., the S-projections of those in the second column.

* See Mathesis, VIII., p. 81, and Casey's Conics, p. 448.

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Now the isogonal transformations of a system of similar circumconics are, as is well known, tangents to a circle, U, concentric with the circumcircle, and consequently having (imaginary) double contact with the circumcircle on the line at infinity. Therefore, by the preceding article, the K-transformations of the same system are tangents to the conic which is the K-projection of U, *i.e.*, to a conic having double contact with the K-projection of the circumcircle (the maximum inscribed ellipse) on the K-projection of the line at infinity (the line $\Sigma x \cos A = 0$). Thus the first part of the proposition is proved.

Similarly the isogonal transformations, with respect to the triangle A′B′C′, of a system of similar medioscribed conics, are tangents to a circle, V, concentric with the nine-point circle, and consequently having (imaginary) double contact with the nine-point circle on the line at infinity. Therefore the S-transformations of the system are tangents to the conic which is the S-projection of V, *i.e.*, to a conic having double contact with the S-projection of the nine-point circle (the minimum circumscribed ellipse) on the S-projection of the line at infinity (the line $\Sigma a^3 x = 0$). Thus the proposition is completely proved.

The converse may be stated. The S-transformations of tangents to a conic having double contact with the maximum inscribed ellipse on the line $\Sigma x \cos A = 0$, are circumconics having the same eccentricity; and the K-transformations of tangents to a conic having double contact with the minimum circumscribed ellipse on the line $\Sigma a^3 x = 0$, are medioscribed conics having the same eccentricity.

The special cases when the transformations are circles, equilateral hyperbolas and parabolas have already been dealt with in Arts. 2 ii, 3 ii, 2 iii, 3 iii and 4.

7. As another special case of the general theorem given in Art. 5, we have the following theorems which indicate the remarkable connection existing between the isotomic transformation on the one hand (where $1/a^2x$, $1/b^2y$, $1/c^2z$ are substituted for x, y, z in the equations of the original elements) and the S- and K-transformations on the other.

(i) If S and K be two points (or loci) connected with a triangle

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ABC in such a manner that the first is the S-transformation of the second, and consequently the second the K-transformation of the first, the isotomic transformation, S′, of S bears to ABC the same relation that K bears to the complementary triangle A′B′C′.

Demonstration. Let the coordinates of K be x, y, z. Then those of S are

$$1/a(by+cz-ax), \quad 1/b(cz+ax-by), \quad 1/c(ax+by-cz)$$

and consequently those of S′

$$(by+cz-ax)/a, \quad (cz+ax-by)/b, \quad (ax+by-cz)/c.$$

The coordinates of K referred to the triangle A′B′C′ are

$$p-x, \quad q-y, \quad r-z,$$

where p, q, r are the altitudes of A′B′C′,

$$i.e., \quad \frac{\triangle}{a}-x, \quad \frac{\triangle}{b}-y, \quad \frac{\triangle}{c}-z,$$

where $\triangle$ is the area of ABC,

$$i.e., \quad (by+cz-ax)/2a, \quad (cz+ax-by)/2b, \quad (ax+by-cz)/2c.$$

Therefore K is related to the triangle A′B′C′ in the same way that S′ is related to the triangle ABC.

Another statement of the same fact is that K is related to ABC in the same way that S′ is related to the anticomplementary triangle of ABC.

The following are obvious corollaries.

(*a*) The join of K to any of the vertices of ABC (or A′B′C′) is parallel to the join of S′ to the corresponding vertex of the anticomplementary triangle of ABC (or ABC itself). Also if K_1, K_2 be two positions of K, and S'_1, S'_2 the corresponding positions of S′, then K_1K_2 and $S'_1S'_2$ are parallel and have opposite directions, and $K_1K_2 = \frac{1}{2}S'_1S'_2$.

(*b*) The centroid is a point of trisection of KS.

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(*c*) The loci of K and S′ are homothetic, the centroid being the homothetic centre.

An interesting particular case of (i) arises when K is at infinity, *i.e.*, when the inconic of which K is the centre is a parabola. We then obtain, with the help of (*a*) above, the following theorem which I cannot recollect having seen stated, viz., that if ABC is the triangle formed by three tangents to a parabola, and if X be the point of contact on BC, and X′ be the point where the diameter through A meets BC, then BX = CX′. From this can be derived another theorem relating to the parabola, also new to me, viz., that if ABC be the triangle formed by three tangents to a parabola and if X be the point of contact on BC, then if AB = AC, A, X and the focus are collinear, and conversely.

When K is an incentre or excentre, we have, as a particular case, the well-known theorem that the isotomic conjugate of the corresponding Gergonne point (Nagel point) is the incentre or excentre as the case may be of the anticomplementary triangle.

It may be noted also that since any line passing through the centroid is related in the same way to both ABC and A′B′C′, the isotomic transformation of such a line is the same as its S-transformation.

(ii) It follows from (i) above that S bears to ABC the same relation that the isotomic conjugate of K, with respect to A′B′C′, bears to A′B′C′.

Combining (i) and (ii) we have the following remarkable theorem, viz., that if P be an element (point or locus) connected with the triangle ABC, and P′ be the corresponding element connected with the complementary triangle A′B′C′, then the K-transformation of P is related to A′B′C′ in the same way that the S-transformation of P′ is related to ABC.

The following table illustrates the correspondence between the isotomic and the K-transformations. A similar table could easily be constructed showing the correspondence between the isotomic transformation, with respect to A′B′C′, and the S-transformation.

TABLE III.

Primary elements consisting of points, lines, and circumconics.	*Derived elements.*	
	Isotomic transformations.	*K-transformations.*
line at infinity	minimum circumscribed ellipse	maximum inscribed ellipse
minimum circumscribed ellipse	line at infinity	line at infinity
line $\Sigma a^3x=0$	circumcircle	nine-point circle
circumcircle	line $\Sigma a^3x=0$	line $\Sigma x\cos A=0$
centroid	centroid	centroid
orthocentre	isotomic conjugate of orthocentre (symmedian point of anticomplementary triangle)	symmedian point
isotomic conjugate of orthocentre	orthocentre	circumcentre (orthocentre of A′B′C′)
symmedian point	isotomic conjugate of symmedian point (mid-point of the distance between the Brocard points of the anticomplementary triangle)	mid-point of the distance between the Brocard points
isotomic conjugate of circumcentre	circumcentre	nine-point centre
Gergonne point	isotomic conjugate of Gergonne point (Nagel point)	corresponding in- or ex-centre
etc.	etc.	etc.

It may be noticed that points in the third column are derived from points in the second by the substitution $(by+cz)/a$ for x, etc., and points in the second from points in the third by the substitution $(by+cz-ax)/a$ for x, etc. These substitutions have already been discussed by Mons. E. Lemoine,* who, however, makes no mention of the isotomic or K-transformation in connection with them.

* E. Lemoine : Notes de Géométrie, Deux modes de génération de quelques points remarquables, Association française, Congrès de Besançon, 1893.

8. The following generalisations seem to be worth stating.

Let a variable conic touch the sides BC, CA, AB of a triangle at X, Y, Z, and let S be the point of concurrence of AX, BY, CZ. Let a fixed transversal meet BC, CA, AB in L, M, N, and let L′, M′, N′ be the harmonic conjugates of these points with respect to the ends of the sides on which they lie. Let P be the pole of LMN with respect to the conic.

Then, by projection, we obtain the following from theorems already established.

(i) If the locus of P is a straight line, the locus of S is a circumconic (general theorem of Art. 2).

(ii) If the locus of P is LMN, *i.e.*, if the conic touches LMN as well as the sides of the triangle, the locus of S is the circumconic which touches AL, BM, CN (Art. 2, i).

(iii) If the locus of S is a straight line, the locus of P is a conic through L′, M′, N′ (general theorem of Art. 3).

(iv) If the locus of S is LMN, the locus of P is the conic which touches the sides of ABC at L′, M′, N′ (Art. 3, i).

From each of the foregoing a correlative theorem could be obtained by reciprocation.

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