POLARIZED RADIATION FROM WHITE DWARFS AND ATOMS IN STRONG MAGNETIC FIELDS

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Abstract. We present the recent results of our continuing program of investigation of the behavior of matter in strong to super-strong magnetic fields ($B \sim 10^6$–10^{12} G). This work was motivated by the discovery of strong magnetic fields ($B \sim 10^7$ G) in some white dwarfs and the likely existence of super-strong fields ($B \sim 10^{12}$ G) in pulsars. Magnetic white dwarfs were discovered from observations of the continuous spectrum and one of the most intriguing challenges for the theorist is to provide an explanation for the observed wavelength dependence of the fractional circularly and linearly polarized radiation. Our initial response to this question was the determination of an exact solution of Kemp's harmonic oscillator model. These results are used as input to the ATLAS model atmosphere program and then comparison is made with observations. The disparities still existing between theory and observation convince us of the necessity for developing a new model of the continuum radiation, two likely possibilities being photoionization and free-free absorption. This leads us to present a general formulation of radiation absorption and emission processes in a magnetic field. Next we calculate the cross section for the photoionization, correct to first order in B. For the purpose of obtaining exact results for this cross section, the effect of a magnetic field on the energy spectrum and wave functions of hydrogen, helium, etc. must be obtained. The results for hydrogen are presented here. They will be useful also in determining accurate values for the displacements due to the quadratic Zeeman effect in the line spectra of DA stars, particularly for the higher excited states.

1. Introduction

The discovery of magnetic fields, of the order of 10^7 G, in some, but not all, white dwarfs must surely rank high among the many exciting astronomical discoveries of today. Such fields are orders of magnitude larger than laboratory produced fields. The magnitude is consistent with the idea that magnetic flux is conserved during the evolution preceding the formation of white dwarfs and hence lends support to the generally accepted conclusion that even higher fields, of the order of the critical magnetic field $B_C = (m^2c^3/e\hbar) = 4.4 \times 10^{13}$ G, exist in pulsars (Gold, 1969).

Initial attempts to positively detect magnetic fields were based on observations of *line* spectra. Preston (1970) and Trimble (1971) failed to find evidence for displacement due to the quadratic Zeeman effect in the lines of the DA stars whose radial velocities had been measured by Greenstein and Trimble (1967). In addition, Angel and Landstreet (1970a) searched unsuccessfully for evidence of the Zeeman effect by looking for circular polarization in the wings of Balmer lines.

A dramatic breakthrough was achieved when Kemp (1970) predicted that the *continuous* spectrum of light from white dwarfs should exhibit a fractional circular polarization q, given by

$$q \equiv [\mathrm{P}_+(\omega) - \mathrm{P}_-(\omega)]/[\mathrm{P}_+(\omega) + \mathrm{P}_-(\omega)] \simeq -(\Omega/\omega). \tag{1}$$

Here $\mathrm{P}_\pm(\omega)$ are the intensities of right and left circularly polarized light (RCP and

C. J. Hansen (ed.), Physics of Dense Matter, 287–300.

LCP, respectively) of angular frequency ω, and $\Omega = (eB/2\mu c)$ is the Larmor frequency, where B is the magnetic field. The basic feature of this idea was confirmed in the laboratory by Kemp *et al.* (1970a) who placed incandescent sources between the pole pieces of a 25 kG magnet, and observed q values of 10^{-5}–10^{-4}, at a mean wavelength in the near infrared.

Soon afterwards, the first discovery of circularly polarized light from a white dwarf was reported (Kemp *et al.*, 1970b), amounting to 1–3% in visible light from the semi-DC white dwarf Grw + 70°8247, from which one calculates a B field of about 10^7 G. Further work on this star (Angel and Landstreet, 1970b) led to no evidence for variation by more than 0.1% over a period of 4 days but it also gave results in conflict with Kemp's (1970) theory.

The second discovery, by Angel and Landstreet (1971a) and Kemp *et al.* (1971), of circular polarization was made in the DC white dwarf G195–19 + GR250, with a q value of about 0.42%, implying a B value of the order of 3×10^6 G. It was later shown (Angel and Landstreet, 1971b, 1972) that the polarization is periodically variable with a period of 1.33 days, the variation in the red being different from that in the blue-green. No detectable linear polarization was found.

The third discovery (Angel and Landstreet, 1971c) of circular polarization was made in the DG p white dwarf G99-37, with a q value of about 0.63%, with the possible existence of ~0.2–0.3% variations. No linear polarization was found.

A search for circular polarization in about 40 other white dwarfs has produced negative results.

Motivated by the fact that Kemp's prediction that $q \sim \lambda$ is at variance with the observations, Shipman (1971) extended Kemp's theory to take into account radiative transfer in the atmosphere of Grw + 70°8247 and found a λ dependence of q more in conformity with the observations. However, two important discrepancies remained (a) in the *infrared*, the observed (Kemp and Swedlund, 1970) large q values of 8.5 and 15%, at mean wavelengths of 1.15 and 1.25 respectively, are far greater than the theoretical predictions and (b) in the *ultra-violet* the observed drop in q is somewhat greater than the theoretical predictions. An updating of Shipman's work to take into account the most recent observations has led to the conclusion (Roussel and O'Connell, 1973) that the agreement in the optical region becomes worse. A similar analysis carried out for the linear polarization showed even less agreement between predictions and observations.

The work of our group at LSU has been motivated primarily by the need to develop a detailed theoretical model which would hopefully explain the many unusual features of the polarized continuum radiation, particularly the wavelength dependency of q. Our efforts have been essentially two-pronged – (a) calculation of an exact solution of Kemp's harmonic oscillator model and (b) development of a more physical model based on the behavior of atoms, ions, and electrons in a strong magnetic field.

In Section 2 we present a general development of radiation absorption and emission processes in the presence of both a magnetic field and an arbitrary central potential

$V(r)$. The results derived here form the basis for the development of the various radiation models discussed in subsequent sections.

Section 3 is devoted to Phase (a) of our program which is now essentially completed. The method used (Chanmugan *et al.*, 1972a, b) to obtain an exact solution of Kemp's model, valid for all values of the magnetic field and the temperature, is outlined and the results discussed. Predictions relating to the fractional linear polarization, q^* say, are also obtained. The latter results have been used as input to the ATLAS model atmosphere program (Kurucz, 1969) to take account of radiative transfer, by use of Shipman's method. Finally, we discuss the results of a comparison between the output results and observations.

Kemp (1970) has remarked that "... what is now very much needed is an exact calculation of the strong B-field levels of hydrogen." In Section 4 we present the results of such a calculation (Smith *et al.*, 1972). We also expand on our previous remark to the effect that these results are also necessary for obtaining accurate results for the quadratic Zeeman terms in strong fields, particularly for the higher excited states.

In Section 5 we discuss transition probabilities in a strong magnetic field. Bound-bound transitions (Smith *et al.*, 1973) are useful for quadratic Zeeman calculations. Bound-free and free-free transition probabilities we believe to be the basic ingredients of a realistic physical model, which will explain the polarized radiation from magnetic white dwarfs.

2. Radiation Absorption and Emission in a Magnetic Field

The Hamiltonian for a particle of mass μ and charge $-e$ in a central potential $V(r)$ and a magnetic field $\mathbf{B}(|\mathbf{B}|=B_z)$, and interacting with radiation, is

$$H = \frac{1}{2\mu}\left(\mathbf{P} + \frac{e}{c}\mathbf{A} + \frac{e}{c}\mathbf{A}_r\right)^2 + V(\mathbf{r}), \tag{2}$$

where $\mathbf{A}$ and $\mathbf{A}_r$ are the vector potentials of the external electromagnetic field and the radiation field, respectively. We choose

$$\mathbf{A} = \tfrac{1}{2}(\mathbf{B}\times\mathbf{r}), \tag{3}$$

so that $\nabla\cdot\mathbf{A}=0$. Then, dropping A_r^2 terms (two-photon processes), we may write

$$H = H_0 + H_I, \tag{4}$$

where

$$H_0 = \frac{P^2}{2\mu} + V(\mathbf{r}) + \Omega L_z + \tfrac{1}{2}\mu\Omega^2(x^2+y^2), \tag{5}$$

and

$$H_I = \frac{e}{\mu c}\mathbf{A}_r\cdot[\mathbf{P} + \mu\Omega(\hat{\mathbf{B}}\times\mathbf{r})]. \tag{6}$$

Here $\Omega=(eB/2\mu c)$ is the Larmor frequency and L_z is the z-component of the angular momentum.

https://doi.org/10.1017/S0074180900100087 Published online by Cambridge University Press

Since H_0 is invariant under rotations about the z-axis and under inversion, the eigenstates can be labelled by the eigenvalues of L_z and the parity. Thus, a general form of the eigenfunctions may be written

$$\psi_m^{\pm}(\mathbf{r}) \equiv \psi_t(\mathbf{r}) = \sum_{il} a_{il}^t f_{il}^t(r)\, Y_{lm}(\theta, \phi). \tag{7}$$

The sum on l in (7) over all even integers leads to the state with even parity $(+)$ and the sum over odd l, to the odd parity $(-)$ state. Here m is the eigenvalue of L_z. The $f^t(r)$ are suitably chosen functions of r and the a^t are parameters. In special cases (as for example when $V(r)$ refers to the potential of an oscillator – see Section 3) it may be possible to solve for ψ_t in closed form. However, in general (as for example when $V(r)$ is Coulombic – see Sections 4 and 5), ψ_t must be obtained numerically. For bound states ψ_t is obtained by variational techniques, whereas for continuum states ψ_t is obtained by a numerical solution of the Schrodinger equation. Thus, the probability per unit time of a spontaneous transition from a state t of energy E_t to a state t' of energy $E_{t'}$ (where t and t', of course, refer to eigenstates of H_0), with the emission of one photon into a solid angle $\mathrm{d}\Omega$, in the presence of an external magnetic field B is

$$A_{t't}\mathrm{d}\Omega = \frac{e^2\hbar\omega_{t't}}{2\pi\mu^2c^3}\,|M_{t't}^{\mathbf{k}}|^2\,\mathrm{d}\Omega\,, \tag{8}$$

where

$$M_{t't}^{\mathbf{k}} = \langle t'|e^{i\mathbf{k}\cdot\mathbf{r}}\left[\nabla\cdot\hat{e}_q + \frac{i\mu}{\hbar}\,\Omega(\hat{\mathbf{B}}\times\mathbf{r})\cdot\hat{e}_q\right]|\,t\rangle\,. \tag{9}$$

The propagation vector, polarization vector, and angular frequency of the photon are denoted by $\mathbf{k}$, $\hat{e}_q$ and $\omega_{t't}$, respectively, where

$$\hbar\omega_{t't} = E_t - E_{t'}\,, \tag{10}$$

and where the unit directions $\hat{e}_q$ are defined by

$$\hat{e}_{\pm} = \mp\frac{1}{\sqrt{2}}(\hat{e}_x \pm i\hat{e}_y)\,; \qquad \hat{e}_0 = \hat{e}_z\,. \tag{11}$$

From (9) it is clear we may write

$$|M_{t't}^{\mathbf{k}}|^2 = \left|\left[D_{t't}^{\mathbf{k}} + \frac{i\mu\Omega}{\hbar}\,F_{t't}^{\mathbf{k}}\right]\right|^2, \tag{12}$$

where

$$D_{t't}^{\mathbf{k}} = \langle t'|\,e^{i\mathbf{k}\cdot\mathbf{r}}\nabla\cdot\hat{e}_q\,|t\rangle \tag{13}$$

is the usual momentum matrix element, and where

$$F_{t't}^{\mathbf{k}} = \langle t'|\,e^{i\mathbf{k}\cdot\mathbf{r}}(\hat{\mathbf{B}}\times\mathbf{r})\cdot\hat{e}_q\,|t\rangle\,. \tag{14}$$

Now if $kr \ll 1$, which is generally true for transitions in the discrete spectrum, then we may rewrite (12) in terms of the familiar dipole length matrix element

$$\mathbf{R}_{m'm} = \langle m'|\,\mathbf{r}\,|m\rangle\,. \tag{15}$$

In the dipole approximation $\Delta m = \pm 1$ or 0 and the parity of the wave function must change in the transition and so with this understanding we now choose to label our initial and final states simply by m and m', respectively. Now using (5), we obtain the commutation relation

$$[\mathbf{r}, H_0]_{m'm} = \frac{\hbar^2}{\mu} \langle m' | \nabla | m \rangle - \hbar (m' - m) \Omega \langle m' | \mathbf{r} | m \rangle . \tag{16}$$

But for a transition between two eigenstates of H_0, with eigenvalues E_m and $E_{m'}$, we have

$$[\mathbf{r}, H_0]_{m'm} = (E_m - E_{m'}) \langle m' | \mathbf{r} | m \rangle . \tag{17}$$

Hence

$$\langle m' | \nabla | m \rangle = \frac{\mu}{\hbar} \omega_{m'm} (1 + f) \mathbf{R}_{m'm} , \tag{18}$$

where

$$f \equiv (m' - m)(\Omega / \omega_{m'm}) \equiv (m' - m) d . \tag{19}$$

In addition, we note that

$$\langle m' | (\hat{\mathbf{B}} \times \mathbf{r}) \cdot e_z | m \rangle = i (m' - m) \mathbf{R}_{m'm} \cdot \hat{e}_z . \tag{20}$$

Hence, we finally obtain for the transition probability

$$A_{m'm} \mathrm{d}\Omega = \frac{e^2}{2\pi\hbar c^3} \omega_{m'm}^3 |\mathbf{R}_{m'm} \cdot \hat{e}_q|^2 \, \mathrm{d}\Omega . \tag{21}$$

The corresponding expression for the intensity of emission $P_{m'm}$ is

$$P_{m'm} \mathrm{d}\Omega = \frac{e^2}{2\pi\hbar c^3} \omega_{m'm}^4 |\mathbf{R}_{m'm} \cdot \hat{e}_q|^2 \, \mathrm{d}\Omega . \tag{22}$$

Turning now to photon absorption processes in a magnetic field, we shall take as a prototype photoionization. The magnetic field affects the matrix elements in the same way as for photon emission, with the result that the cross section (Bethe and Salpeter, 1957) for photoionization σ from a bound state t' to a continuum state t is

$$\sigma \mathrm{d}\Omega = \frac{4\pi^2 e^2 \hbar^2}{\mu^2 c \omega_{t't}} |M_{t't}^{\mathbf{k}}|^2 \, \mathrm{d}\Omega . \tag{23}$$

In the dipole approximation

$$\sigma \mathrm{d}\Omega = \frac{4\pi^2 e^2}{c} \omega_{m'm} |\mathbf{R}_{m'm} \cdot \hat{e}_q|^2 \, \mathrm{d}\Omega . \tag{24}$$

The results of this section constitute the basic tools we need for a consideration of various models of polarized radiation.

3. Kemp's Harmonic Oscillator Model

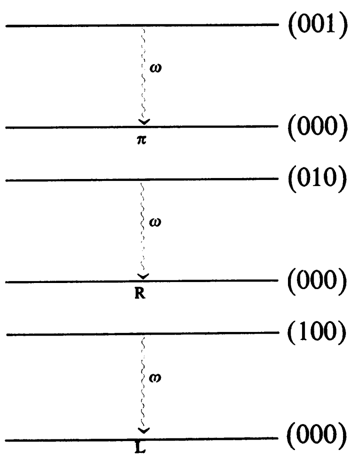
As a tractable model of the radiating system, Kemp (1970) chose a collection of electronic harmonic oscillators. A feature of Kemp's model is that one has to have a distribution of oscillators with essentially a continuum of natural frequencies to account for the continuous emission. Putting $V(r)=\mu\omega_0^2 r^2/2$ in (5), where ω_0^2 is the natural frequency of the oscillator, we obtain

$$H_0 = \frac{1}{2\mu}\left[P_x^2 + P_y^2 + \mu^2\omega_c^2(x^2+y^2)\right] + \frac{1}{2\mu}\left[P_z^2 + \mu^2\omega_0^2 z^2\right] + \Omega L_z . \quad (25)$$

An exact quantum-mechanical solution to this problem has been obtained. It was shown (Chanmugan *et al.*, 1972a) that this Hamiltonian is equivalent to that of a three-dimensional anisotropic oscillator with fundamental frequencies ω_0, $\omega_c-\Omega$ and $\omega_c+\Omega$ so that the energy eigenvalues are given by (with $\omega_c^2=\omega_0^2+\Omega^2$)

$$E = \hbar(\omega_c+\Omega)(n_+ + 1/2) + \hbar(\omega_c-\Omega)(n_- + 1/2) + \hbar\omega_0(n_z + 1/2), \quad (26)$$

where $(n_+ n_-, n_z)$ refer to the number of quanta of different frequencies. Introducing now the radiation terms H_I, and considering transitions of frequency ω leads to the



$$\omega_c^2 \equiv \omega_0^2 + \Omega_L^2$$

$$\omega = \omega_c - \Omega_L \qquad \omega_0 = (\omega^2 + 2\omega\Omega_L)^{1/2} \qquad |H_I|^2 \sim (\omega+\Omega_L)^{-1}$$

$$\omega = \omega_c + \Omega_L > 2\Omega_L \qquad \omega_0 = (\omega^2 - 2\omega\Omega_L)^{1/2} \qquad |H_I|^2 \sim (\omega-\Omega_L)^{-1}$$

$$q = \frac{R-L}{R+L} = \begin{cases} 1 & \text{if } 2\Omega_L \geqslant \omega \\ -(\Omega_L/\omega) & \text{if } 2\Omega_L < \omega \end{cases}$$

$$q^* = \frac{(\sigma_1+\sigma_2)-\pi}{(\sigma_1+\sigma_2)+\pi} = \begin{cases} -\dfrac{\omega+2\Omega_L}{\omega+3\Omega_L} & \text{if } 2\Omega_L \geqslant \omega \\ \dfrac{\Omega_L^2}{2\omega^2-\Omega_L^2} & \text{if } 2\Omega_L < \omega \end{cases}$$

Fig. 1. Emission of linear, right-circularly, and left-circularly polarized light of frequency ω, from a system of harmonic oscillators of charge $-e$ and mass μ, with a continuous range of natural frequencies ω_0, in a magnetic field B, with associated Larmor frequency $\Omega=(eB/2\mu c)$.

results

$$|\langle 100|\, H_{\mathrm{I}}\, |000\rangle|^2 \sim (\omega - \Omega)^{-1} \tag{27}$$

$$|\langle 010|\, H_{\mathrm{I}}\, |000\rangle|^2 \sim (\omega + \Omega)^{-1} \tag{28}$$

$$|\langle 001|\, H_{\mathrm{I}}\, |000\rangle|^2 \sim \omega^{-1} \tag{29}$$

for the emission of LCP, RCP, and linearly polarized radiation, respectively. We note (see Figure 1) that LCP will not occur for $2\Omega > \omega$. Similar conclusions hold when transitions between all possible levels are considered (Chanmugan *et al.*, 1972b). Taking into account the effect of the field on the distribution of oscillators, and setting

$$s \equiv t^{-1} \equiv (\omega / 2\Omega), \tag{30}$$

it follows that

$$q = (s^2 - 1)^{1/2} - s \quad \text{for} \quad s \geqslant 1 \tag{31}$$

and

$$q = 1 \quad \text{for} \quad 0 < s < 1, \tag{32}$$

in agreement with Kemp's classical results. In addition, we have also results for the

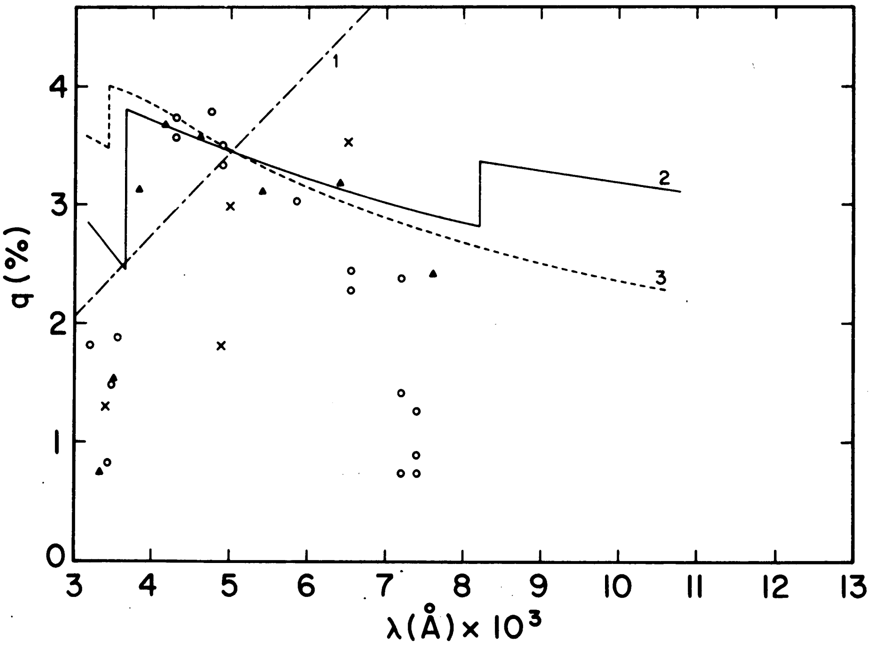


Fig. 2. The wavelength dependence of the predicted circular polarization is shown for the various models. Curve 1 corresponds to the optically thin model of Kemp (1970). Curve 2 corresponds to Model C with the parameters [$T = 12000$ K, $\log g = 8$, H = 0.9, He = 0.1, $B = 1.2 \times 10^7$ G]. Curve 3 corresponds to Model C with the parameters [$T = 14000$ K, $\log g = 8$, $H = 0.0$, He = 1.0, $B = 2 \times 10^7$ G]. The observed circular polarizations are indicated by the following: crosses indicate those of Kemp and Swedlund (1970), triangles those of Angel and Landstreet (1970b) and open circles those of Angel and Landstreet (1972).

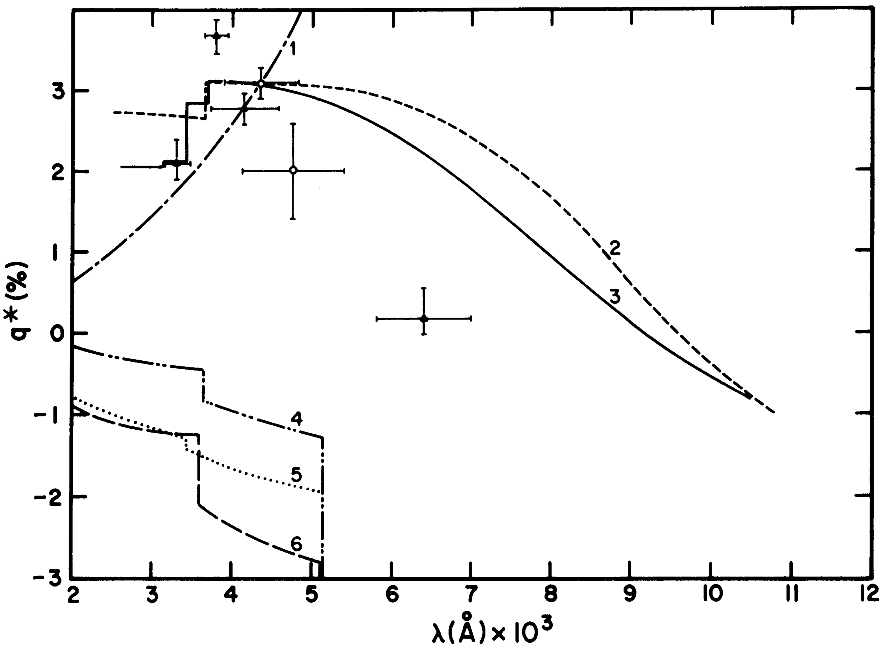


Fig. 3. The wavelength dependence of the predicted linear polarization is shown for the various models. Curve 1 corresponds to the optically thin model of Chanmugam *et al.* (1972a, b). Curve 2 corresponds to Model C with the parameters [$T = 12000$ K, $\log g = 8$, $H = 0.9$, $\mathrm{He} = 0.1$, $B = 5 \times 10^7$ G]. Curve 3 corresponds to Model C with the parameters [$T = 14000$ K, $\log g = 8$, $H = 0.0$, $\mathrm{He} = 1.0$, $B = 5.5 \times 10^7$ G]. Curve 4 corresponds to Model A with the parameters [$T = 12000$ K, $\log g = 8$, $H = 0.9$, $\mathrm{He} = 0.1$, $B = 2.1 \times 10^8$ G]. Curve 5 corresponds to Model B with the parameters [$T = 14000$ K, $\log g = 8$, $H = 0.0$, $\mathrm{He} = 1.0$, $B = 2.1 \times 10^8$ G]. Curve 6 corresponds to Model B with the parameters [$T = 12000$ K, $\log g = 8$, $H = 0.9$, $\mathrm{He} = 0.1$, $B = 2.1 \times 10^8$ G]. The observed linear polarizations are indicated as follows; the triangles are those of Angel and Landstreet (1970b) and the open circles those of Angel and Landstreet (1972).

linear polarization ratio

$$q^* = [(1+t)^{1/2} + (1-t)^{1/2} + \\ - 2(1-t^2)^{1/2}]/[(1+t)^{1/2} + (1-t)^{1/2} + 2(1-t^2)^{1/2}] \quad \text{for} \quad t \leqslant 1$$

and

$$q^* = [s^{1/2} - 2(1+s)^{1/2}]/[s^{1/2} + 2(1+s)^{1/2}] \quad \text{for} \quad 1 > s > 0. \tag{33}$$

For $2\Omega \ll \omega$, these results reduce to

$$q = -(\Omega/\omega) \tag{34}$$

and

$$q^* = \tfrac{3}{4}(\Omega/\omega)^2. \tag{35}$$

If we had assumed a constant density of states even in the presence of a magnetic field, the q^* in (35) would be smaller by a factor of $\frac{2}{3}$ but (34) for q would be unchanged. Kemp's model of course provided the motivation for the successful search for circular polarization by Kemp *et al.* (1970b). However, the wavelength dependence of the observed q did not agree with the predicted $q \sim \lambda$ behavior. Now Kemp's model applies literally only to an optically thin body. Shipman (1971) was thus motivated to consider radiative transfer in Kemp's model. Actually he used Kemp's bremsstrahlung model which gives the same predictions as the harmonic oscillator model except that the q value given by (34) comes out 8 times larger. As a result Shipman was able to explain the wavelength dependence of the circular polarization, for $3000\,\text{Å} \leqslant \lambda \leqslant 9000\,\text{Å}$, with a derived B field $\sim 10^7$ G. However, discrepancies between the model predictions and the observations occurred in the ultra-violet and the infrared. In addition, it has been pointed out by Roussel and O'Connell (1973) that if the more recent observations of 1972 are included, the agreement in the optical region becomes worse, as shown in Figure 2. The observed values of q beyond 6000 Å are much smaller than predicted.

Making use of the predicted q^* values, Shipman's model, and the Atlas white-dwarf model atmosphere program, a similar analysis has been carried out for the linear polarization in Roussel and O'Connell (1973). The results are displayed in Figure 3. The agreement here is even worse than in the circular case. As a result, we feel that it is desirable to develop a new model for the polarized radiation which will predict the behavior of atoms in magnetic fields.

4. Energy Spectrum of the Hydrogen Atom in a Strong Magnetic Field

The behavior of atoms in low fields can be adequately described using hydrogenic wave functions in perturbation theory. On the other hand for super-strong magnetic fields ($B \gtrsim 3 \times 10^{10}$ G) the Coulomb interaction becomes negligible in comparison with the magnetic energy so that the wave functions are essentially oscillator-like (Cohen *et al.*, 1970). It is clearly of importance to understand at what field strengths perturbation theory using hydrogenic wave functions breaks down and to devise a scheme for analyzing the system in fields of intermediate strength. Kemp (1970) has emphasized the importance of this difficult intermediate case for magnetic white dwarfs. Thus, a program was initiated to study the behavior of atoms in fields of *any* strength, with emphasis on B values from about 10^4 G to 10^{12} G.

Our natural starting point is the hydrogen atom, for which (neglecting spin)

$$H_0 = \frac{P^2}{2\mu} - \frac{e^2}{r} + \Omega L_z + \tfrac{1}{2}\mu\Omega^2 r^2 \sin^2\theta\,, \tag{36}$$

where θ is the polar angle.

To assess the relative magnitudes of the various terms, it is useful to choose units $\hbar = c = \mu = 1$, so that $e^2 = \alpha = a_0^{-1}$, where α is the fine-structure constant and a_0 is the Bohr radius. Thus

$$H_0 = \frac{P^2}{2} + \frac{\alpha^2}{2}\left[-\left(\frac{r}{2a_0}\right)^{-1} + \left(\frac{B}{B_0}\right) L_z + \tfrac{1}{4}\left(\frac{B}{B_0}\right)^2 \left(\frac{r}{a_0}\right)^2 \sin^2\theta\right], \tag{37}$$

where

$$B_0 \equiv \frac{\mu^2 c e^3}{\hbar^3} \equiv \alpha^2 B_c = 2.350 \times 10^9\ \text{G}\,. \tag{38}$$

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Let

$$\psi_{nlm}(\mathbf{r}) = R_{nl}(r)\, P_{lm}(\theta)\, e^{im\phi} \tag{39}$$

be the normalized solution of the Schrödinger equation when $B=0$. Then, it is well known (Bethe and Salpeter, 1957, p. 206) that this wave function is also a solution with the B term (but not the B^2 term) included in H_0. The corresponding energy eigenvalue is

$$E = E_0 + \frac{\alpha^2}{2}\left(\frac{B}{B_0}\right) m, \tag{40}$$

where E_0 is the energy without the magnetic field.

This is the basic theory of the normal Zeeman effects and Paschen-Bach effects. The quadratic Zeeman effect (Schiff and Snyder, 1939) is obtained by treating the B^2 term using first order perturbation theory, with the result that the total energy is

$$E \equiv E_0 + E_1 + E_2 \equiv \equiv \frac{\alpha^2}{2n^2}\left[-1 + mn^2\left(\frac{B}{B_0}\right) + F_{nlm}n^6\left(\frac{B}{B_0}\right)^2\right], \tag{41}$$

where

$$F_{nlm} = \frac{5}{4}\frac{\left[1 + \frac{1}{5n^2}(1 - 3l(l+1))\right]\left[l(l+1) + m^2 - 1\right]}{(2l+3)(2l-1)}. \tag{42}$$

Thus, we expect perturbation theory to be valid only for $B \ll B_H \equiv (B_0/n^4)$. For large n and $l=1$, we have $F=(1+m^2)/4$, so that (in Rydbergs)

$$E_2 = \tfrac{1}{4}n^4(1 + m^2)\left(\frac{B}{B_0}\right)^2 \mathrm{Ry}, \tag{43}$$

which is the formula used by Preston (1970). However, since $B_H \sim n^{-4}$, we see that, for a particular B value, the perturbation results are less reliable for the higher states. For example, when $n=10$, we get $B_H = 2.350\times10^5$ G. It is thus clear that the deduction of B field values from an analysis of the Balmer absorption spectra from magnetic white dwarfs can be made considerably more quantitative by determining the exact energy eigenvalues of the hydrogen atom in a magnetic field. This analysis is now being carried out by Surmelian, using the results below. As already mentioned a further motivation for obtaining the spectrum is to obtain the effect of the B field on the opacities of the atmospheres of magnetic white dwarfs and, in particular, to explain the λ dependence of the circularly and linearly polarized light.

Our initial effort (Rajagopal *et al.*, 1972) concentrated on the ionization energies of hydrogen in magnetic white dwarfs and the essence of the calculation was the use of a trial wave function which was *hydrogen*-like, in contrast to the *oscillator*-like trial wave function used by Cohen *et al.* (1970). Using merely a 4-parameter trial function,

$$\psi = c_1\psi_1(\beta_1 r) + c_2\psi_2(\beta_2 r), \tag{44}$$

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the values of the ionization energy obtained were significantly better than those of Cohen *et al.* (1970) for fields $\geqslant 3 \times 10^{10}$ G (see Figure 4). A general procedure for carrying out a multi-parameter calculation was also outlined. Using standard computing techniques, it was then possible to obtain (Smith *et al.*, 1972) the energy spectrum for the 14 lowest states of the hydrogen atom. The essence of the method is to write a general trial solution of the form given in (7) with the radial function $a^t_{il} f^t_{il}$ chosen to be

$$(d^t_{il} r^l + b^t_{il} r^{l+1})\, e^{-\beta_{il} t}, \tag{45}$$

where d and b are parameters.

For superstrong B fields we used a partial wave expansion with values of l up to $20 + |m|$ included in summation (45). This was necessary in order to obtain convergence of the expansion of $\psi^{\pm}_m$. Up to nine Slater type orbitals were employed in the descrip-

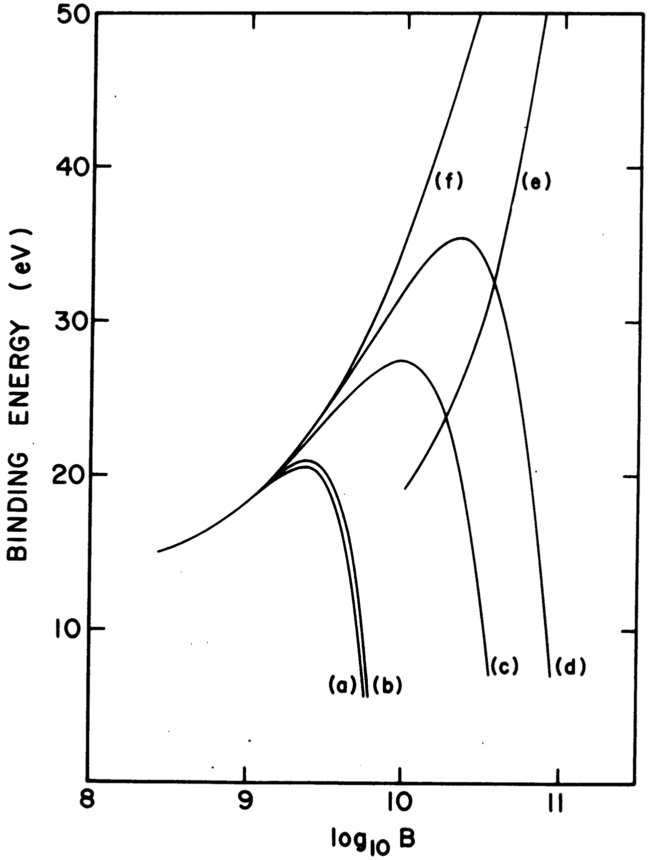

Fig. 4. The ionization energy of the ground state of hydrogen as a function of the magnetic field B calculated using (a) perturbation theory, (b) 2 linear parameter (c_1 and c_2) variational calculation, (c) 1 non-linear (β_1) parameter variational calculation, (d) 2 linear (c_1 and c_2) and 2 non-linear (β_1 and β_2) parameter variational calculation [see Equation (44)], (e) the work of Cohen *et al.* (1970), and (f) the work of Smith *et al.* (1972).

https://doi.org/10.1017/S0074180900100087 Published online by Cambridge University Press

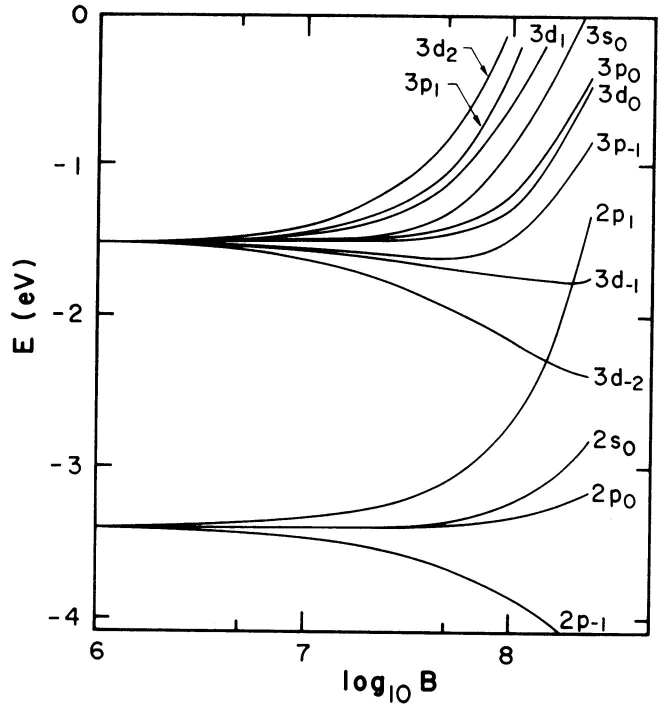

Fig. 5. The energy spectrum of hydrogen in a magnetic field for the 13 lowest states above the ground state.

tion of the radial function for the ground state and twelve for the excited states. Figure 4 gives the ionization energy E_1 (in eV) of the ground state of hydrogen as a function of the magnetic field B, calculated using (a) perturbation theory (b) two linear parameter (c_1 and c_2) variational calculation (c) one non-linear (β_1) parameter variational calculation (d) two linear (c_1 and c_2) and two non-linear (β_1 and β_2) parameter variational calculation [see Equation (44)] (e) the work of Cohen *et al.* (1970) (f) the work of Smith *et al.* (1972). The latter results are clearly superior.

In Figure 5, we present the energy spectrum for the 13 lowest states above the ground state for B values from 10^6–10^8 G. The labelling of the curves corresponds to the usual labels for the hydrogenic energy levels in the absence of a magnetic field. Thus, for example, $3d_2$ is an even-parity state with $m=2$, and, as $B \to 0$, it also has $n=3$ and $l=2$.

5. Bound-Bound and Bound-Free (Photoionization) Transition Probabilities in a Strong Magnetic Field

Bound-bound transition probabilities have been calculated (Smith *et al.*, 1973) in the electric dipole approximation, using (21), between all states shown in Figure 5. Our results are presented in tabular form for some representative B values and are being used by Surmelian in his work on the quadratic Zeeman effect.

Henry and O'Connell are now calculating photoionization probabilities using (24), as we feel the results should provide us with an explanation of the spectral dependence of the polarized radiation. However, for magnetic fields for which $E_2 \ll E_1$ [see Equation (41)], analytic results may be obtained (Henry and O'Connell, 1972).

For $B=0$, the cross section for ionization of the hydrogen atom in a state with principal quantum number n may be written (Karzas and Latter, 1961)

$$\sigma_0(\omega) = 2.82 \times 10^{29} n^{-5} (\omega/2\pi)^{-3} g(\omega, n), \tag{46}$$

where $g(\omega, n)$, the Gaunt factor, is of order unity and is a slowly varying function of ω and n, For our present purposes we will neglect the ω dependence in g (i.e. we essentially use the Kramers expression, while at the same time emphasizing that, apart from its complexity, there is no difficulty in principle in carrying it along).

Let $\omega_{m'm}$ be the frequency of the light absorbed for B non-zero. Then (19) and (40) tell us that

$$E_0 - E_0^1 = E - E^1 + \frac{\alpha^2}{2}\left(\frac{B}{B_0}\right)(m' - m) = \\ = \omega_{m'm}(1 + f) = \omega_{m'm}(1 - d\Delta m). \tag{47}$$

Thus, from (24) and (46), we obtain

$$\sigma_{\pm} \mathrm{d}\Omega = (1 + f)^{-4} \sigma_0 \mathrm{d}\Omega \tag{48}$$

which is similar to that found by Lamb and Sutherland (1972). Hence

$$(\sigma_{\pm 1}/\sigma_0) = (1 \mp d)^{-4}. \tag{49}$$

We now define the circular polarization ratio for the photoionization model,

$$q_p \equiv (\sigma_+ - \sigma_-)/(\sigma_+ + \sigma_-). \tag{50}$$

For small d, we have

$$(\sigma_{\pm 1}/\sigma_0) = (1 \pm 4d), \tag{51}$$

and

$$q_p = 4(\Omega/\omega_{m'm}). \tag{52}$$

The latter result is the same as that obtained from Kemp's model [see (34)], except for the factor of 4. We anticipate that inclusion of the dependence of the Gaunt factor, as well as B^2 terms in the energy eigenvalues may lead to more interesting predictions with respect to the λ dependence of q_p.

Acknowledgements

Most of the ideas presented here arose from continued collaboration and discussions with G. Chanmugan, R. J. W. Henry, A. K. Rajagopal, K. M. Roussel, E. R. Smith, and G. L. Surmelian, to whom the author would like to express his thanks. The author would like to thank Mrs Gladys Wiggins for excellent work with the manuscript, and Dr Henry for comments on same. It has come to our notice that many of the topics discussed above have also been investigated by Lamb and Sutherland (1972).

Note Added After the Conference

The energy spectrum of He II in a strong magnetic field, as well as bound-bound transition probabilities have now been obtained (Surmelian and O'Connell, 1973). The use of similar techniques in the study of multi-electron atoms is now being pursued. The photoionization calculation has been extended to include Gaunt factors (Roussel *et al.*, 1973). With regard to the detection of strong magnetic fields by using quadratic Zeeman displacements (see Preston, 1970), it is important to know, for the various *nlm* states, at what fields perturbation theory breaks down. This analysis has now been carried out (O'Connell, 1971). Free-free transitions in a magnetic field has also been suggested as a model for the polarized radiation (O'Connell, 1974).

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