N-BODY SIMULATIONS OF DISKS

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Abstract. The methods used in the large-scale n-body simulations are discussed. However, the present review concentrates on the results already obtained in the n-body simulations using systems containing up to 200000 simulation stars. Results are presented which show that the stability criterion developed for flattened systems applies only to truly axisymmetric instabilities. Purely stellar disks acquire rather large velocity dispersions, generally two or more times the velocity dispersion required by Toomre for axisymmetric stability. In the computer simulations, the bar-forming instability can be prevented only by comparatively large velocity dispersions. However, simulations inclusing the effects of the galactic halo and core as a fixed background field show that bar formation can be prevented for fixed halo components as large or larger than the self-consistent disk component. Experiments performed to determine the collisional relaxation time for the large-scale gravitational n-body calculations show that these models are indeed 'collisionless'.

1. Introduction

Large-scale gravitational n-body simulations of disks of stars have been in progress for the past seven years. With the introduction of these efficient large-scale computer models for self-gravitating stellar systems, the field of experimental stellar dynamics is providing fresh insights into the structure and the dynamics of galaxies and other stellar systems. These simulations were started by two groups using somewhat different numerical models. The group of Richard Miller and Kevin Prendergast developed a completely discretized model where the forces, star positions, and velocities are allowed only discrete (integer) values less than some given maximum value. This model is described in Miller and Prendergast (1968). Our group developed a model where the forces, star positions, and velocities are not restricted to integer values. This model is described in Hohl and Hockney (1969). To obtain the gravitational potential or force in these models, an $n \times n$ array of cells is superposed over the galactic disk. The mass density in each of the $n \times n$ cells is used to obtain the gravitational force by means of convolution methods making use of fast Fourier transforms (Cooley and Tukey, 1965). Because of the periodic nature of finite Fourier transforms, Miller and Prendergast (1968) and Miller *et al.* (1970) used a doubly periodic configuration space for their earlier force calculations. Hohl and Hockney (1969) developed a modified Fourier transform method to obtain the potential for an isolated disk galaxy. In the simulations the dimension of the array for the force calculations is 128×128 or 256×256 and the number of simulation stars is generally near 100000. The results obtained with the two models are in general agreement; however, Miller *et al.* (1970) have concentrated their simulations on two-component disks containing both stars and a dissipative 'gas' component. Our own work has been concentrated on simulating purely stellar disks (Hohl, 1971a, 1972a). Some of the interesting problems investigated so far with these models are the 'Jeans instability' and the gravitational two-stream instability (Hohl, 1971b), development of spiral structure

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(Miller *et al.*, 1970; Quirk, 1971; Hohl, 1971a), evolution of purely stellar disks (Hohl, 1971a), and tests on stationary self-consistent disks (Hohl, 1972b).

Galaxies are essentially collisionless systems. Thus, the computer models should not cause heating of the 'stars' due to numerical or collisional effects. The collisional relaxation time of the model has been determined to be of the order of 1000 rotations by investigating the rate of energy equipartition of a system with a mass spectrum of stars (Hohl, 1973).

With the exception of computer time and storage, there are no difficulties in extending the simulations to three dimensions. Some initial results of such three-dimensional simulations have been obtained by Hockney and Brownrigg (1974).

2. Evolution of Purely Stellar Disks

Some of the initial simulations were performed for 'cold' (zero velocity dispersion) balanced disks of stars. As expected, such disks were found to be violently unstable. Figure 1 shows the evolution of an initially balanced uniformly-rotating disk con-

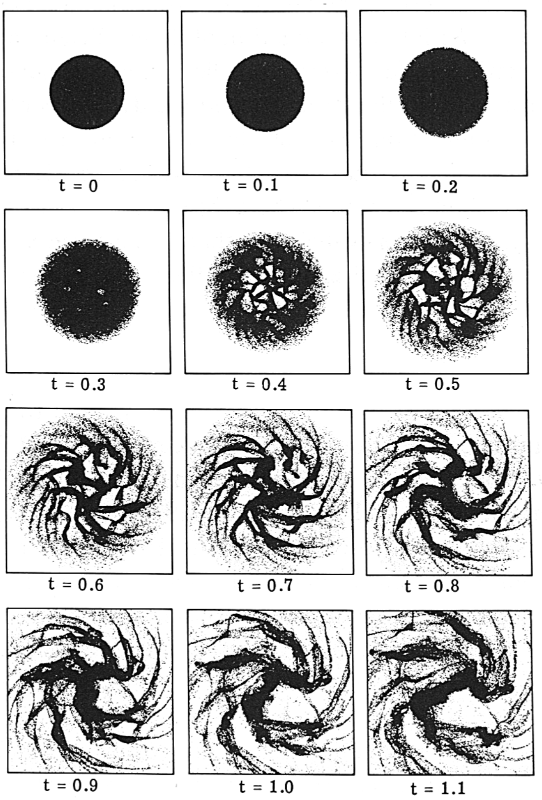

Fig. 1. Evolutions of an initially 'cold' (zero velocity dispersion) balanced disk containing 200000 stars. Time is shown in units of the rotational period of the cold balanced disk.

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sisting of 200000 'stars'. The initial surface mass density of the disk is given by

$$\mu(r)=\mu(0)\sqrt{1-r^2/R^2},$$

where R is the initial radius of the disk. The uniform angular velocity required to balance the cold (zero velocity dispersion) disk is

$$\omega_0=\tau\sqrt{G\mu(0)/2R},$$

where G is the gravitational constant. As can be seen in Figure 1, the disk quickly developes small-scale instabilities which later develop into a large-scale instability producing a bar with trailing spiral structure. The time shown in Figure 1 and subsequent figures is in units of the rotational period of the cold disk. Figure 2 displays the long-time evolution of another initially cold uniformly-rotating disk containing only 50000 stars. As can be seen, any spiral-like structure developed during the initial evolution quickly disappears as the system evolves further. The final state of such a

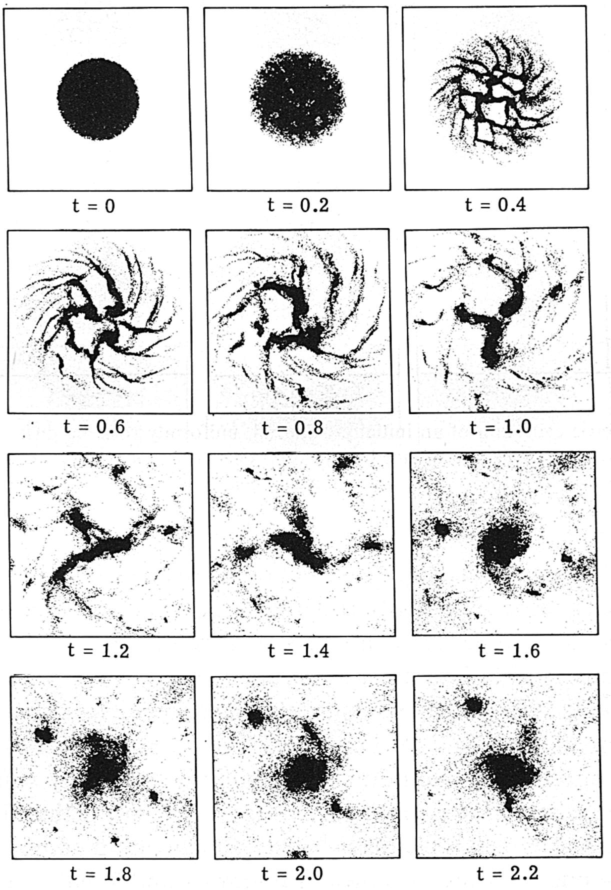

Fig. 2. Long time behavior of an initially cold balanced disk containing 50000 stars.

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disk has a surface density distribution closely approximated by an exponential variation with the hot core primarily pressure supported. Essentially identical results were obtained for disks with different initial surface density variations (Hohl, 1970).

Toomre (1964) developed a local criterion for the suppression of all axisymmetric instabilities which requires a minimum radial velocity dispersion given by

$$\sigma_{r,\,\min} = 3.36\, G\mu/\kappa,$$

where κ is the local value of the epicyclic frequency. To test Toomre's criterion, an initially balanced uniformly-rotating disk of stars with the velocity dispersion required by Toomre's criterion was investigated to determine whether any axisymmetric instabilities or other misbehavior was present (Hohl, 1971a). This was done by neglecting the azimuthal component of the gravitational field. The evolution of the disk containing 100000 stars is shown in Figure 3. It remains, of course, axisym-

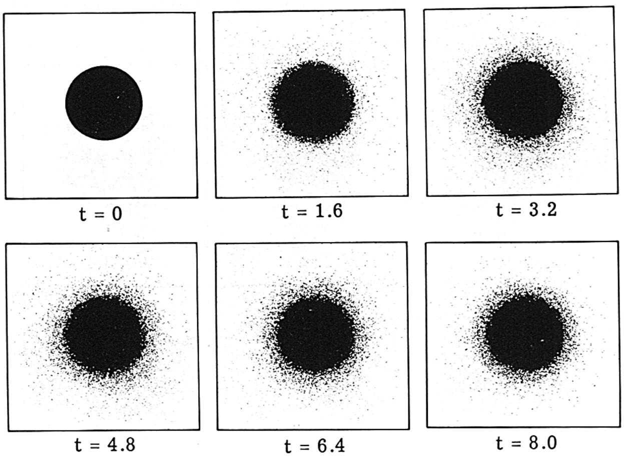

Fig. 3. Axisymmetric evolution of an initially balanced, uniformly rotating disk of 100000 stars. The stars have an initial velocity dispersion given by Toomre's criterion, and they move under a purely radial gravitational field.

metric. After a few small pulsations it settles down to an essentially steady state with the radial velocity dispersion nearly equal to $\sigma_{r,\,\min}$. Thus, any instabilities the disk may have are not axisymmetric in nature. Similar results by a different method were obtained by Miller (1974). However, when the system at $t = 8$ rotations in Figure 3 is allowed to evolve without the axisymmetry constrained, a large-scale bar instability quickly develops.

The totally unrestrained evolution of an initially uniformly rotating disk with Toomre's velocity dispersion is displayed in Figure 4. It shows that the disk is indeed stabilized against the small-scale disturbances which would completely disrupt the disk in less than one rotation, as shown in Figures 1 and 2. However, we again find that the disk is not stabilized against relatively slowly-growing large-scale distur-

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bances which cause the system to assume a very pronounced bar-shaped structure after two rotations. After about four rotations the disk population of stars constitutes a nearly axisymmetric distribution surrounding a dense central oval or bar-shaped core. The oval contains nearly two-thirds of the total mass of the disk and

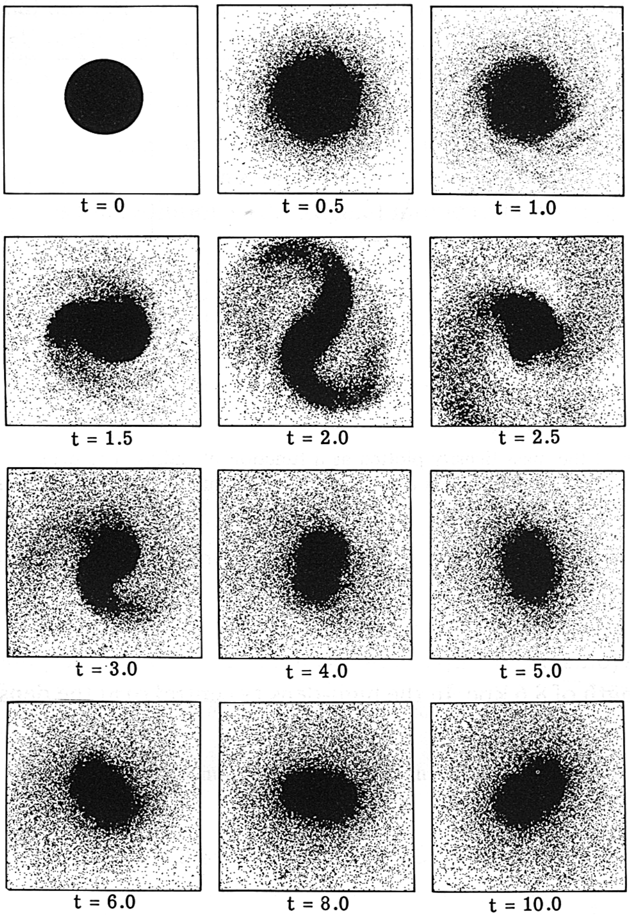

Fig. 4. Unconstrained evolution of the initially balanced uniformly rotating disk of 100000 stars. The stars have an initial velocity dispersion given by Toomre's criterion.

has an axis ratio of about 3:2. After about four rotations there is little change in the structure of the disk, and the bar rotates with a constant period approximately two times that of the cold balanced disk. To picture the radial variation of parameters describing the disk, the disk was divided into a number of concentric rings each of 0.5 kpc width. The radial dependence of various parameters averaged azimuthally over each ring was then obtained. Figure 5 shows the evolution of the surface mass density obtained in that manner. It can be seen from this figure that the central mass density increases by about a factor of 4 during the 11.6 rotations shown. After eight

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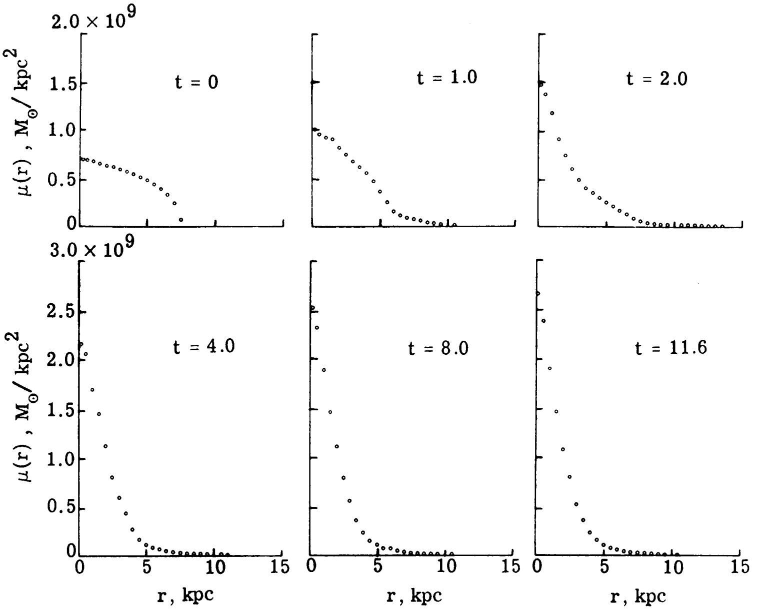

Fig. 5. Evolution of the mass density plotted as a function of radius. The density is given in units of solar masses per kpc^2. Note that each one of the 100000 simulation stars has a mass $0.84 \times 10^6\ M_\odot$.

rotations the radial variation in density changes only very little. To determine how closely the final density variation corresponds to an exponential variation the final density is plotted in Figure 6 on a semilog scale. The distribution of the disk population of stars (outside $r = 8$ kpc) is closely approximated by an exponential variation with a scale length of 8.6 kpc. In the high-density central oval the density also closely

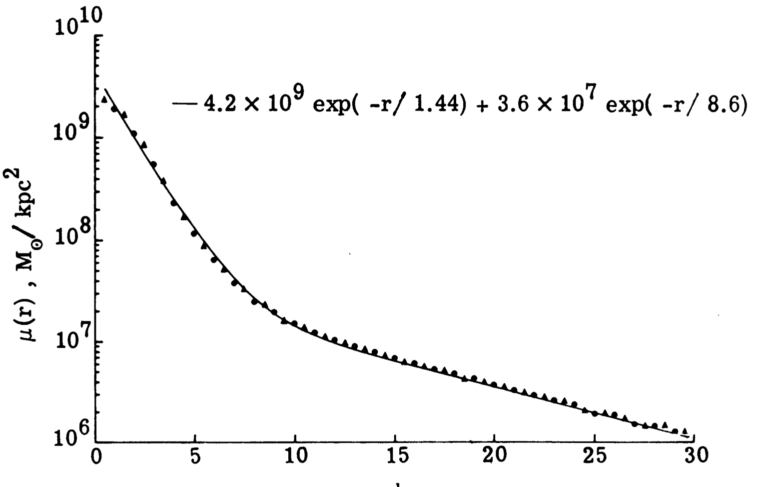

Fig. 6. Dependence of the azimuthally averaged density on radius for the disk after $t = 11.6$.

follows an exponential law, but with the much shorter scale length of 1.44 kpc. The structure of the disk changes very little during the last 5 rotations. Note that the sum of the two exponentials represented by the solid line in Figure 6 closely approximates the density variation.

An interesting end result for all the disks of stars investigated so far which go through the initial unstable evolution is that the final distribution in the radial direction for the 'disk population of stars' is closely approximated by an exponential variation of density. This result may be significant since it agrees with observational evidence (de Vaucouleurs, 1959; Freeman, 1970) which indicates that the luminosity in the outer regions of many spiral and SO galaxies seems also to decrease exponentially with radius.

Figure 7 illustrates the evolution of the angular momentum distribution of the disk.

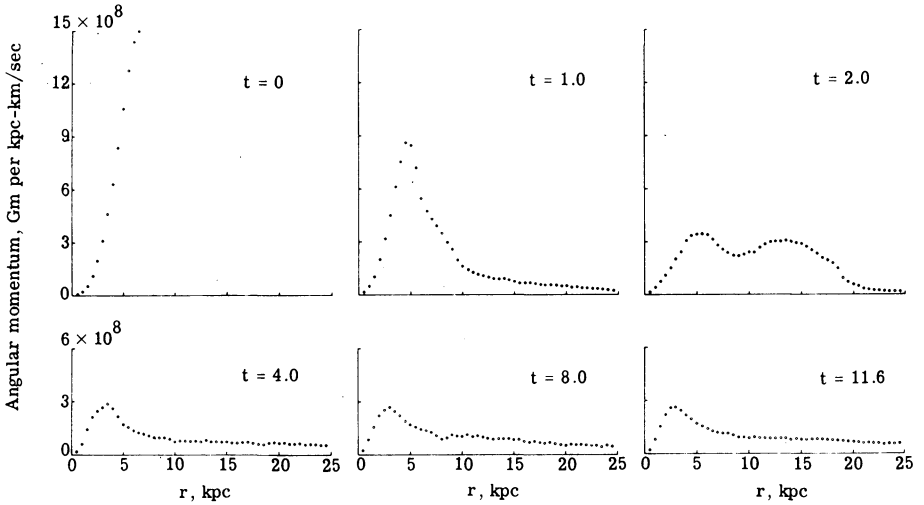

Fig. 7. Evolution of the azimuthally averaged angular momentum distribution.

As the bar-forming instability is developing at $t = 2$ rotations, much of the angular momentum is transferred to large radii as stars are flung outward from the ends of the bar.

The initial radial velocity dispersion is about 120 km s^{-1} at the center of the disk and goes to zero at the edge. The final velocity dispersion at the center is increased to about 180 km s^{-1} and drops quickly to about 60 km s^{-1} at the edge of the bar. The disk population has a nearly constant radial velocity dispersion of about 60 km s^{-1} for r larger than 8 kpc. The ratio of $\sigma_r/\sigma_{r,\,\mathrm{min}}$ for the final state of the evolved disk varies from about 1 in the center to about 7 at $r = 15$ kpc. Thus the system represents a rather hot disk. This is typical of computer generated galaxies. The radial dependence of the mean circular velocity of the stars $\langle V_\theta \rangle$, of $\omega = \sqrt{K_r/r}$ (K_r = radial gravitational field), and of $r\omega$ are shown in Figure 8. Note that in the central region ($r < 8$ kpc) the disk is primarily pressure supported while for large r, rotation is the

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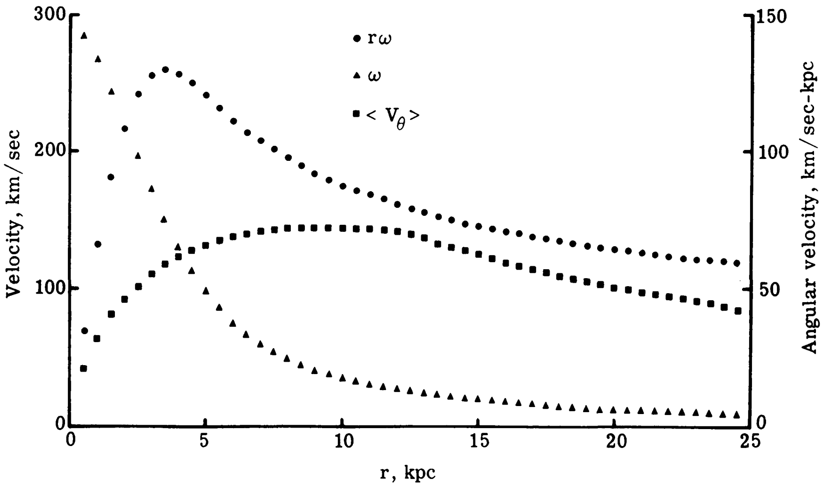

Fig. 8. Comparison of the azimuthally averaged angular velocity $\omega = \sqrt{K_r/r}$, the circular velocity $r\omega$, and the mean circular velocity of the stars $\langle V_\theta \rangle$.

main supporting factor. Very similar results are obtained for initially Gaussian or exponential surface mass density variations (Hohl, 1972). However, the evolution for the later two density distributions is not quite as violent as indicated by the final values of $\sigma_r/\sigma_{r,\,\min}$ of about 4 for the Gaussian variation and of about 2 for the exponential variation (Hohl, 1970). Miller (1971) obtains final states that appear even hotter than those presented here.

3. Cooling of Axisymmetric Disks

The stable axisymmetric disk that we were able to generate was rather hot, especially in the outer regions of the disk. We therefore determined the effects of cooling the disk. Various methods of cooling were tried. For example, during each rotation a certain percentage (we tried from 5 to 30%) of the stars, chosen at random, had a portion of their noncircular velocities removed in proportion to their present radii. Thus stars near the center kept nearly all their random motion whereas stars far away from the center were placed in more nearly circular orbits. Another method, somewhat more analogous to the 'gas collisions' of Miller *et al.* (1970), was simply to place a certain percentage of the stars during each rotation into purely circular orbits. All methods of cooling which we tried gave essentially the same results.

Figure 9 shows the effects of cooling when for each rotation 10% of the stars are selected at random and are placed in circular orbits. The initial disk has the same density variation and mean circular velocity as shown in Figures 6 and 8. During the first three rotations the appearance of the disk changes little. However, at $t=3$ there is already a pronounced bar structure at $r=2.5$ kpc. After $t=3$ a two-arm structure appears, which during the following 1.5 rotations displays a theta-like structure.

After $t=4.5$ the spiral structure opens up and rotates with a fairly constant speed of about 10 km s^{-1} kpc^{-1}. This compares with a mean circular velocity of the stars of about 15 km s^{-1} kpc^{-1} at $r=10$ kpc. At $t=6$ the value of $\sigma_r/\sigma_{r,\,min}$ for the disk was between 2 and 3. Any further cooling only caused the collective instabilities to heat up the disk as fast as it was being cooled.

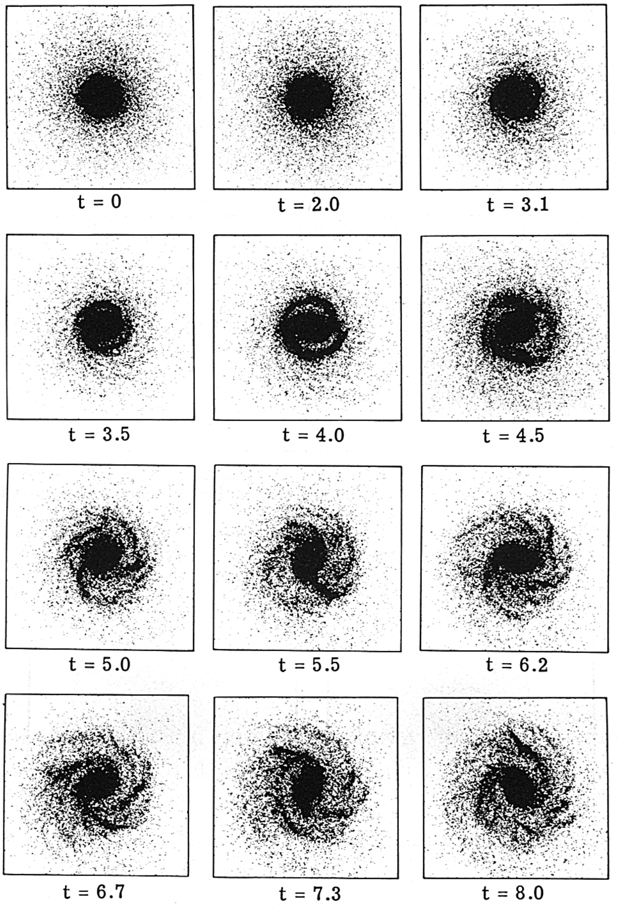

Fig. 9. Development of spiral structure in a disk of stars which is being cooled.

A much more detailed and somewhat more realistic simulation of spiral structure was performed by Miller *et al.* (1970) and by Quirk (1971) who used a two-component system of stars and a dissipative gas component. Again, this stellar component becomes rather hot while the dissipative 'gas' component is forced to remain cool. Spiral structure which persists for about 3 galactic rotations is found to develop, primarily in the cool gas component. They also find that the spiral pattern has some of the properties of a density wave in that stars move through the pattern at a higher velocity than the pattern speed.

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4. Effect of Satellite Galaxy

Toomre and Toomre (1972) have performed a number of simulations of close encounters of galaxies that produced spiral structure and tails and bridges often seen in multiple galaxies. In these calculations a number of massless points move under the influence of the two interacting galaxies which are represented as point masses.

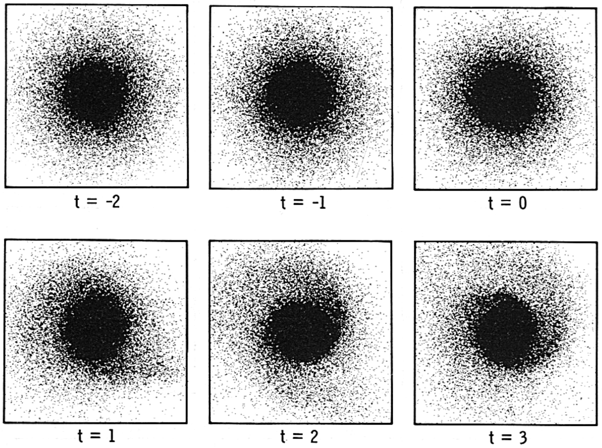

Fig. 10. Evolution of a relatively 'hot' galaxy under the influence of a companion galaxy.

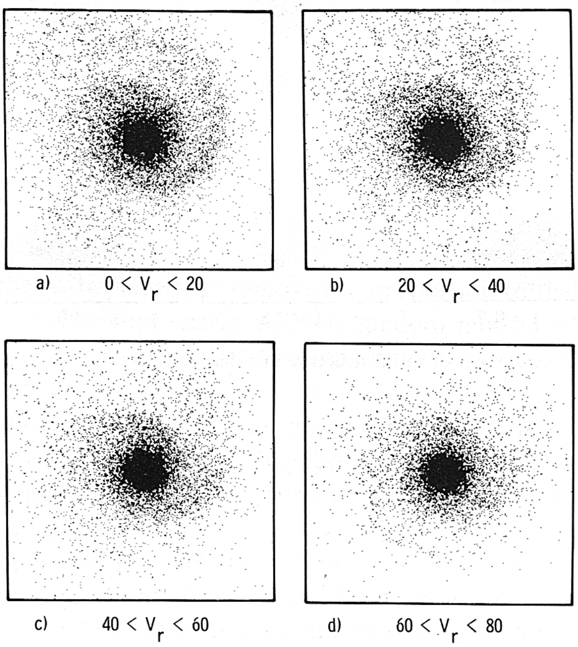

Fig. 11. Distribution of stars in four velcocity intervals at $t = 3$ for the disk shown in Figure 10.

A simulation of a close passage where the self-gravity of the disk stars is taken into account is of interest. For this purpose a stellar disk was perturbed by the passage of a companion galaxy having one-fourth the mass of the primary galaxy. The parameters for M51 as supplied by Alar Toomre were chosen for the calculations. The orbit was direct, the inclination of the companion's orbit plane to the plane of the primary galaxy was 75°, the angle between the node and periapsis was 20°, and periapsis was 25 kpc. The effect of the passage on the structure of the primary galaxy is shown in Figure 10. The time is given in rotational periods of the primary galaxy; $t=0$ corresponds to the time at periapsis. These results indicate that only a weak two-arm spiral structure developed. This was to be expected since the velocity dispersion of the primary galaxy is rather high. The value of $\sigma_r/\sigma_{r,\,\mathrm{min}}$ ranges from about 2 in the central region to 6 in the outer disk. The effect of the large random velocities on the formation of spiral structure can be determined by plotting the distribution of stars in various velocity intervals. This is done in Figure 11. As can be seen, the spiral structure is quite pronounced for the lower velocities, but can hardly be detected for the higher velocities. This was to be expected since the large random velocities have a dispersive effect on the formation of spiral structure. It thus appears that a much cooler primary galaxy is needed for a realistic study of the effects of a close passage.

5. Evolution of Stationary Disks

One of the few stationary solutions of the collisionless Boltzmann equation with velocity dispersion is that for the uniformly rotating disk given by (Hohl, 1972b),

$$f(r, v_r, v_\theta)=\frac{\mu(0)}{2\pi R\sqrt{\omega_0^2-\omega^2}}\left[(\omega_0^2-\omega^2)(R^2-r^2)-v_r^2-(v_\theta-r\omega)^2\right]^{-1/2},$$

where v_r and v_θ are the radial and azimuthal velocity components, ω_0 is the rotational velocity of the cold balanced disk, and ω is the actual rotational velocity of the disk. Kalnajs (1972) has performed a normal-mode stability analysis and finds that the simple mode corresponding to the bar disturbance becomes unstable for $\omega>0.508$ ω_0. A series of numerical experiments was performed to test this result.

The evolution of four disks of stars corresponding to equation (1) with (a) $\omega=0.8\,\omega_0$, (b) $\omega=0.6\,\omega_0$, (c) $\omega=0.4\,\omega_0$, and $\omega=0$ is presented in Figure 12. Each of the 100000 stars in the simulation represents $0.84\times10^6\,M_\odot$ so that the total mass of the disk galaxy is $0.84\times10^{11}\,M_\odot$. The rectangular border enclosing the disks represents the active 128×128 array of cells used in the calculations. The initial radius of the disks is 16 kpc. Since the disks become progressively more stable as the initial velocity dispersion is increased (or ω is decreased), the evolution of the more stable systems is investigated for longer times. Again the times shown are in units of the rotational period of the cold (zero velocity dispersion) disk $\tau_r=2\pi/\omega_0$. Figure 12(a) shows that, for $Q=1$ (or $\omega=0.8\,\omega_0$), the system is unstable and within two rotations it has formed a bar-shaped structure. After three rotations this structure remains essentially un-

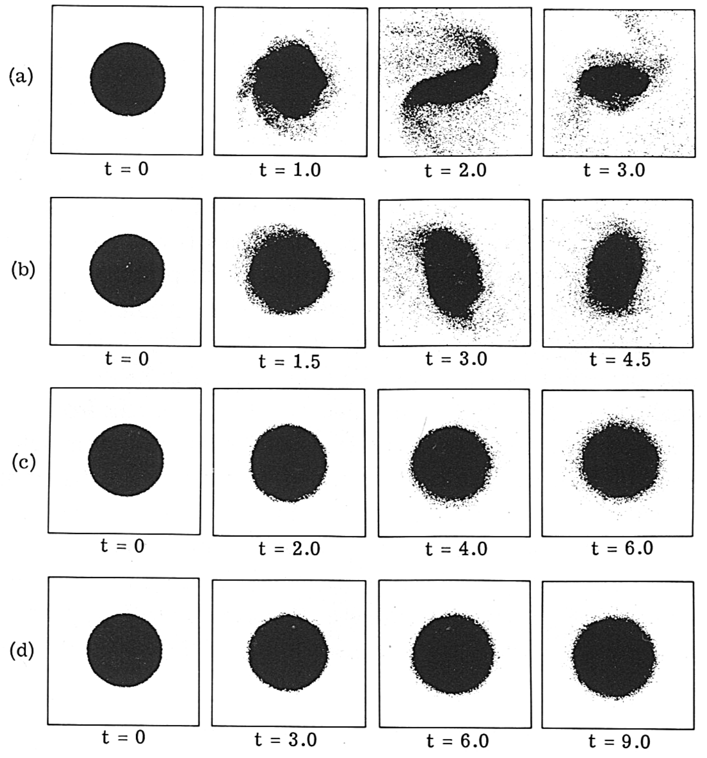

Fig. 12. Evolution of an initially rotating and stationary disk galaxy for four values of the angular velocity given by (a) $\omega = 0.8\ \omega_0$, (b) $\omega = 0.6\ \omega_0$, (c) $\omega = 0.4\ \omega_0$, and (d) $\omega = 0$, where ω_0 is the angular velocity of the cold (zero velocity dispersion) disk.

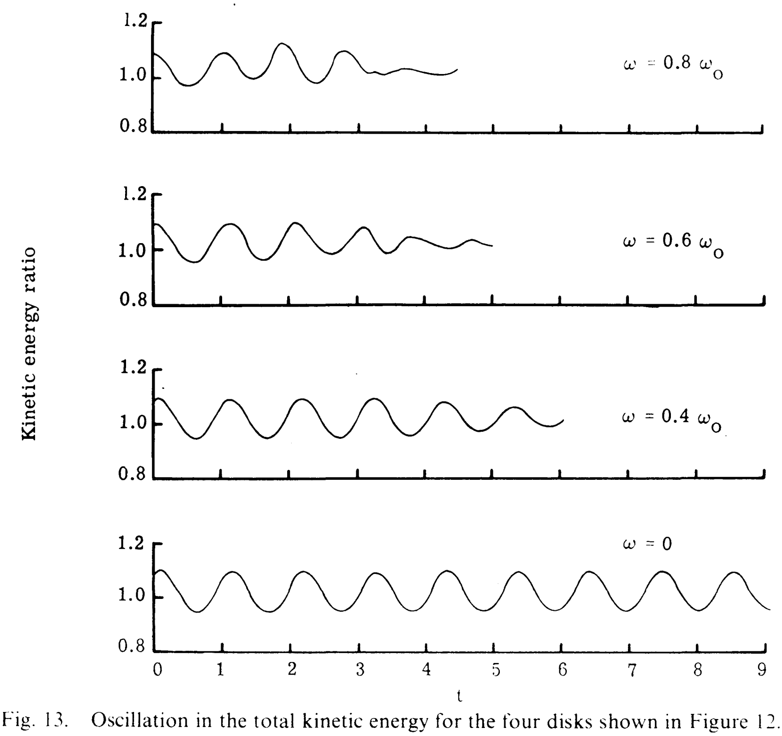

Fig. 13. Oscillation in the total kinetic energy for the four disks shown in Figure 12.

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changed. It should be noted that all small-scale instabilities which occurred in the cold disk have been stabilized. Only the large-scale 'bar-making' instability is present. A similar result is shown in Figure 12(b) for $Q = 1.35$. However, the bar structure is now much less pronounced. For $Q = 1.55$, Figure 12(c), the system is essentially stable. Some of the stars near the edge of the disk tend to diffuse to larger radii. This is to be expected since the distribution function $f(r, v_r, v_\theta)$ is singular at the edge and star orbits tend to be unstable there. Similar results are obtained for the nonrotating disk shown in Figure 12(d). Figure 12 indicates that the disk becomes stable for values of Q somewhere between 1.35 and 1.55 or for values of ω between 0.4 ω_0 and 0.6 ω_0. These results are in agreement with the normal mode analysis of Kalnajs. Since a pseudo-random number generator is used to obtain the initial conditions for the disk, small deviations from the stationary solution which result in initial oscillations of some parameters are to be expected. Figure 13 shows the variation of the kinetic energy for the four disks shown in Figure 10. For the two unstable disks these oscillations are quickly damped. For the case $\omega = 0.4\ \omega_0$ the oscillations show a slow rate of damping whereas for the nonrotating case no damping of the oscillations is noticed.

6. Effects of Fixed Halo Component

A number of simulations have been performed with a model modified to include a fixed central force in addition to the self-consistent disk population of stars (Hohl, 1970). The radial field is taken to represent the hot halo population and central core of the galaxy. Figure 14 gives the evolution of a 50000-star system with a fixed central potential corresponding to the Schmidt model of the galaxy. The disk stars contain 20% of the total mass of the system and the initial radius of the disk is 15 kpc. The system quickly develops spiral structure which remains quite pronounced up to about four rotations. After this time the spiral structure becomes quite diffuse due to the buildup of random velocities. The buildup of random velocities can be seen in Figure 15 where the evolution of the radial velocity dispersion is shown. After two rotations the velocity dispersion shows little further increase and the value of $\sigma_r/\sigma_{r,\,\mathrm{min}}$ is about 2 throughout the system. For systems containing a smaller percentage of disk stars the heating proceeds at a slower rate and the spiral structure remains for a longer time.

The heating rate due to collective effects (collisionless or violent relaxation) can be greatly reduced by modifying the interaction potential of the stars. Thus by using the interaction potential corresponding to a uniform density sphere with a radius of 4 cell dimensions, a system similar to that shown in Figure 14 but with an initial $\sigma_r = \sigma_{r,\,\mathrm{min}}$ shows very little further heating (Hohl, 1974). The use of such a modified interaction potential with the resulting low heating rate will be desirable for the study of large-scale galactic stability or density wave propagation since collective heating effects will no longer mask the phenomena under study.

Computer simulations of purely stellar disks have shown that such systems require rather large velocity dispersions to prevent the large-scale bar-making instability. In fact, these experiments indicate that only when the energy in rotation is less than 28% of the total kinetic energy of the stars is the system prevented from developing a bar structure. This result is in agreement with a new stability criterion proposed by Ostriker and Peebles (1973) which states that for stability the total kinetic energy in rotation must not exceed $14 \pm 2\%$ of the absolute value of the potential energy of the system. Indications are that the K or M dwarfs in the solar neighborhood constitute most of the known mass in this region, but their random radial motions are

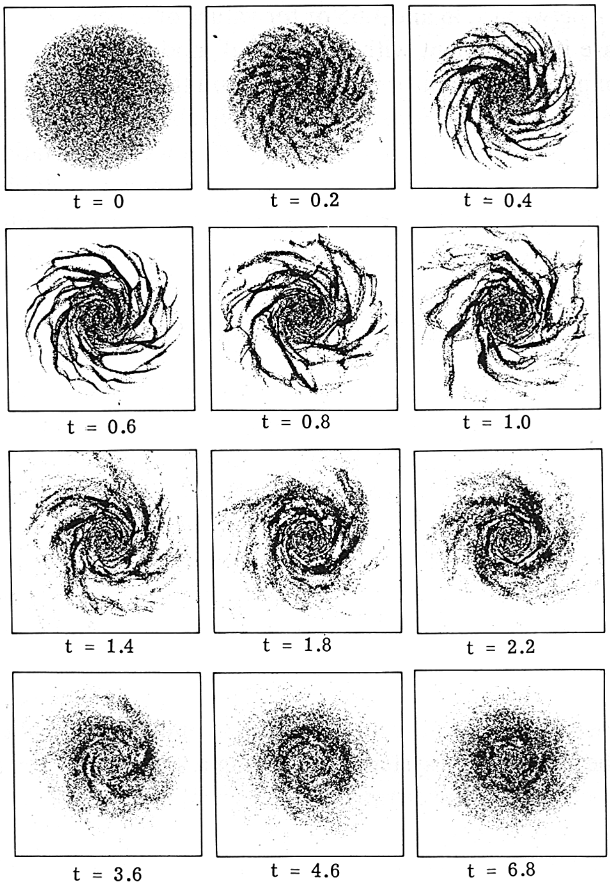

Fig. 14. Evolution of a disk of stars under the influence of a fixed central field corresponding to the Schmidt model of the Galaxy. The 50000 disk stars contain 20% of the total mass of the system. Time is in rotational periods of the initial balanced disk at $r = 10$ kpc.

much too low to satisfy the stability criterion of Ostriker and Peebles. How then can we account for apparently stable spiral galaxies such as own own? At present the actual mass of the core and/or halo component of our and other spiral galaxies is entirely unknown. Also, there appear to be no difficulties with the assumption that a large portion of the galactic mass is contained in the halo, likely consisting of high-

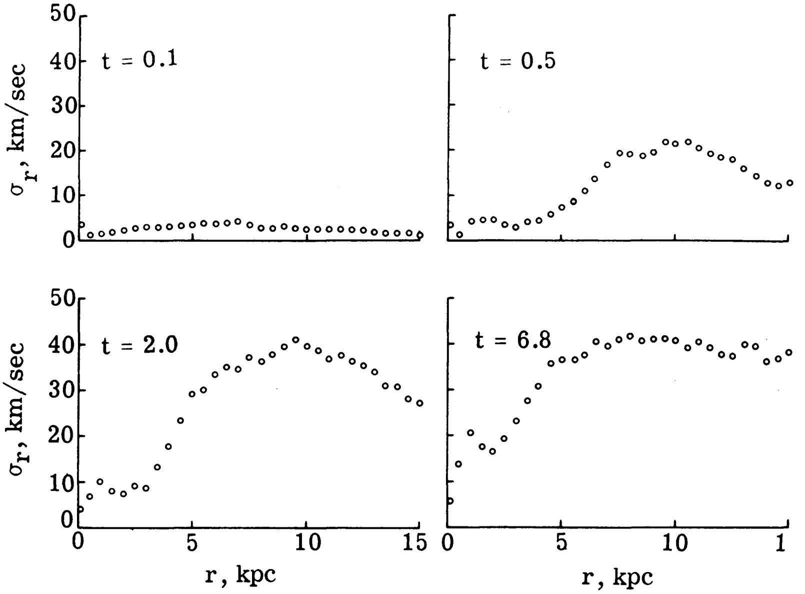

Fig. 15. Evolution of the radial velocity dispersion for the disk shown in Figure 14.

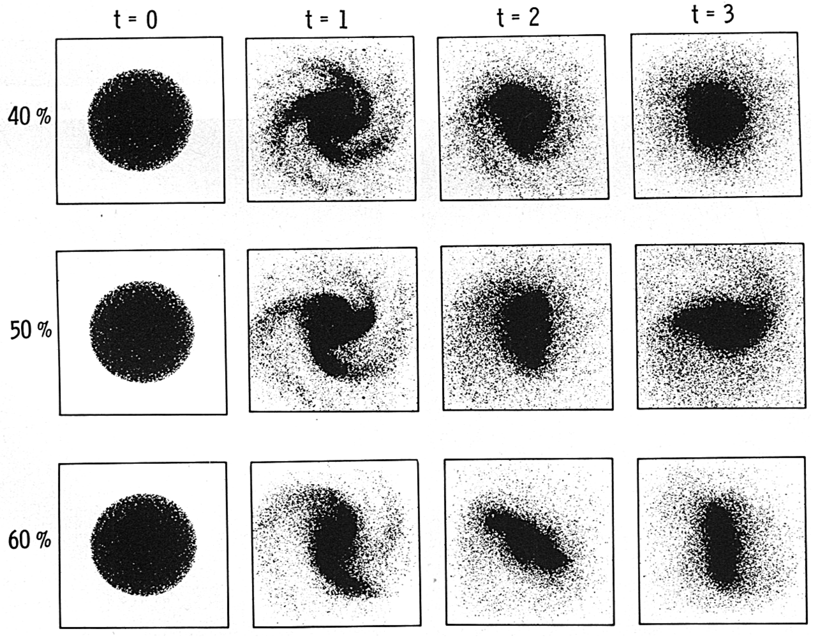

Fig. 16. Evolution of the stellar disk for three different fractions of the total mass contained in the disk stars. Time is in units of the rotational period at $r = 10$ kpc.

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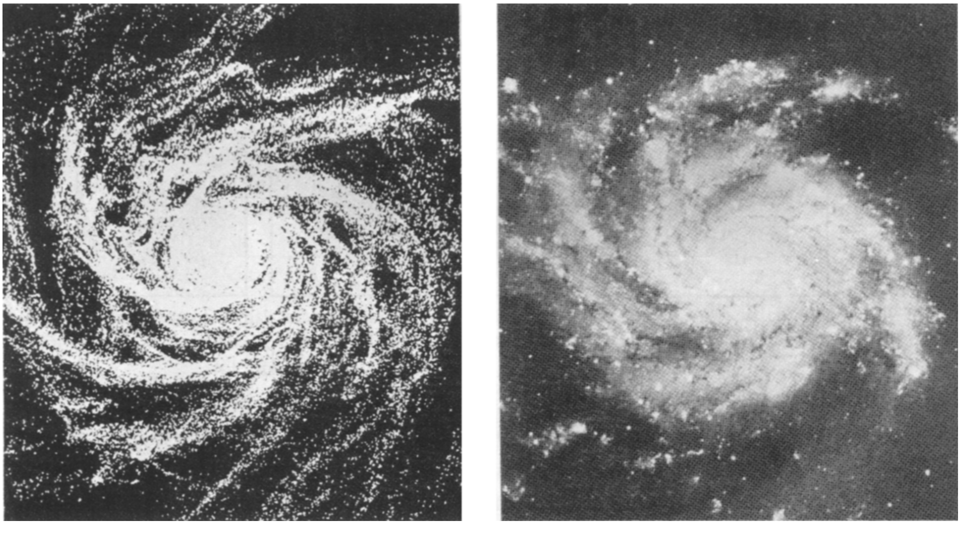
velocity dispersion objects. The result of Figure 14 shows that a sufficient large fixed halo component indeed does stabilize a disk galaxy against the bar-making instability. Similar stabilizing effects were found by Ostriker and Peebles (1973). Therefore simulations were performed to determine the amount of halo or fixed mass required to prevent bar formation. Again an axisymmetric fixed radial field corresponding to the Schmidt model was added to simulate the effects of a hot and time-independent core or halo component.

Figure 16 shows the evolution of the disks for three different fractions of total mass contained in the disk stars. The initial radius of the disks is 20 kpc and the time shown is in rotational periods at $r = 10$ kpc. The evolution of the system for 40% is that for a system with the fixed radial field representing 60% of the mass and the 100000 disk stars containing 40% of the mass. The mass of the disk stars is increased while the fixed mass remains constant to obtain the evolution for systems with larger disk components. The results show that the bar-forming instability is prevented for systems with up to 40% of the mass in the disk stars. For the system with 40% of the mass in the disk stars there is an initial tendency to form a bar-like spiral structure which, however, disappears and the systems become essentially axisymmetric. However, when the fraction of the mass contained in the disk stars becomes 50% or greater, a bar-structure quickly forms and remains for the duration of the calculations.

Additional computer experiments were performed for a more centrally located fixed mass component; namely, a uniform mass density sphere of 6 kpc diameter located at the center of the disk. Again it was found that the system quickly formed a bar when 50% of the mass was contained in the disk stars and the system was stabilized when the disk stars contained 40% of the total mass.



COMPUTER SIMULATION M 101, Sc-TYPE REGULAR SPIRAL GALAXY

Fig. 17. Comparison of computer generated disk with a normal spiral galaxy.

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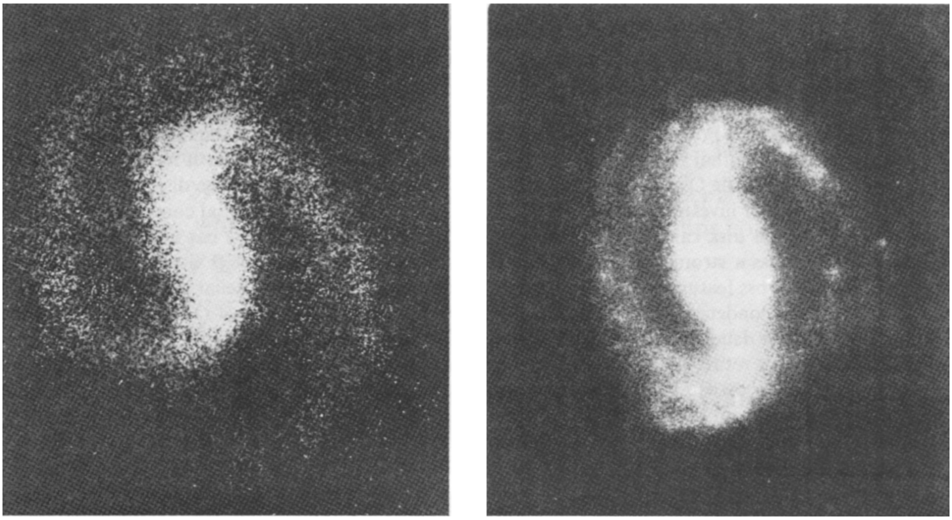
As was shown in previous simulations (Hohl, 1971a) a purely stellar disk could be stabilized only with very high velocity dispersions such that more than 70% of the total kinetic energy is in random motion with the remaining kinetic energy in rotation. The present computer experiments show that in order to stabilize the large-scale bar-making instability for an initially relatively cool disk with velocity dispersions near Toomre's $\sigma_{r,\,\mathrm{min}}$, a central core and/or halo component containing at least 60% of the total galactic mass is required.

In conclusion I would like to show a couple of comparisons of computer simulated galaxies with actual galaxies. This is done in Figures 17 and 18 for a regular and a



COMPUTER SIMULATION NGC-175, SBab-TYPE BARRED SPIRAL GALAXY

Fig. 18. Comparison of computer generated disk with a barred spiral galaxy.

barred spiral galaxy. Such a comparison should not be taken seriously, since for any computer generated galaxy one can find a similar looking object in the various collections of galaxy photographs. Note that the computer generated galaxies considerably change their structure during the following rotation whereas we expect the actual galaxy to preserve its overall structure.

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DISCUSSION

Biermann: If stars evolve, shed gas and this gas dissipates and turns into new stars, doesn't this correspond to a cooling in your calculations already?

Hohl: This process corresponds to the formation of cool young stars and has been included in the simulation of Miller and Prendergast. Of course the problem is that these cool stars quickly heat up.

Lynden-Bell: I notice that all your models have low Q at the middle. I wonder if models with large Q at the centre could satisfie Ostriker's criterium and yet keep Q small far out in the disk as it is observed.

Hohl: I have not yet investigated any models with a large central Q as an initial condition.

Bardeen: In my gas disk calculations a large Q in the center can stabilize the bar mode, but without a substantial halo mass a strong spiral instability remains unless the minimum Q is substantially greater than one. In their gross features the gas disk results are similar to Hohl's for stellar disks; for instance, in that a less centrally condensed halo seems most effective in reducing the value of Q required for stability. I would hope a more detailed comparison will be possible in the future.

Lecar: Before you retire your computer, while you have shown that a massive halo suppresses bar modes, do you think you can kill the bar mode and still retain the small-wavelength (Lin) modes?

Hohl: As you saw from the simulation, even though the bar mode was suppressed by a massive halo, the system still acquired a large velocity dispersion. Thus we still have problems in amplifying the short-wavelength (Lin) modes.

Contopoulos: You said your systems are collisionless. How does this result compare with Rybicki's calculation of the relaxation time, which turns out to be very short for two-dimensional systems?

Hohl: Rybicki's result does not apply to the disk simulation since our forces are cut off at a cell width. Both analytical predictions and computer simulations give relaxation times of several hundred rotations.

Freeman: When you get a persistent bar, is it a large amplitude wave or a material bar?

Hohl: It appears to be more a large amplitude wave since stars in the bar have highly elliptical orbits.

Hunter: This question concerns your numerical simulations of the simple disk models whose stability properties were discussed theoretically by Kalnajs. The slide you showed seemed to indicate that these disks are stable once t is sufficiently small, and the bar instability has been suppressed. According to the theory however, all Kalnajs' elementary disks are unstable to some kind of mode. Did your computations show signs of these other instabilities?

Hohl: No such instabilities were detected in the model. For example, there were small fluctuations in the total kinetic energy for the disks because of the random-number generator used in generating the initial conditions. For a disk with rotation these fluctuations were damped, for the case $\omega = 0$ they showed no damping. Thus any remaining instabilities must have very long growth time and did not show up in the simulations.

Miller: An appeal to the theoreticians present – it could be most useful to have more exact self-consistent models (preferably with stability analysis) along the lines of Kalnajs' models. An exponential disk could be especially helpful. Approximate models won't do, and the entire disk must be self-consistent (including the center).

Hohl: I agree, exact self-consistent model can easily be tested on the computer for stability.

Kalnajs: I can answer Hunter's question: the growth rates of the other unstable modes are small and their spatial structure is less obvious on plots than that of the bar mode.