Canad. Math. Bull. Vol. 59 (1), 2016 pp. 3–12
http://dx.doi.org/10.4153/CMB-2015-029-x

The Contraction Principle for Multivalued Mappings on a Modular Metric Space with a Graph

Monther Rashed Alfuraidan

Abstract. We study the existence of fixed points for contraction multivalued mappings in modular metric spaces endowed with a graph. The notion of a modular metric on an arbitrary set and the corresponding modular spaces, generalizing classical modulars over linear spaces like Orlicz spaces, were recently introduced. This paper can be seen as a generalization of Nadler and Edelstein's fixed point theorems to modular metric spaces endowed with a graph.

1 Introduction

Fixed point theorems for monotone single-valued mappings in a metric space endowed with a partial ordering have been widely investigated. These theorems are hybrids of the two most fundamental and useful theorems in fixed point theory: Banach's contraction principle [19, Theorem 2.1] and Tarski's fixed point theorem [15,33]. Generalizing the Banach contraction principle for multivalued mappings, Nadler [25] obtained the following result.

Theorem 1.1 ([25]) *Let (X,d) be a complete metric space. Denote by $\mathcal{CB}(X)$ the set of all nonempty closed bounded subsets of X. Let $F\colon X \to \mathcal{CB}(X)$ be a multivalued mapping. If there exists $k \in [0,1)$ such that*

$$H\big(F(x),F(y)\big) \le k\, d(x,y)$$

for all $x, y \in X$, where H is the Hausdorff metric on $\mathcal{CB}(X)$, then F has a fixed point in X.

A number of extensions and generalizations of Nadler's Theorem were obtained by different authors; see, for instance, [13,20] and references cited therein. The Tarski theorem was extended to multivalued mappings by different authors; see [5,14].

Ran and Reurings [31] initially investigated the existence of fixed points for single-valued mappings in partially ordered metric spaces. They proved the following result.

Received by the editors December 20, 2014; revised March 10, 2015.
Published electronically May 21, 2015.
The author acknowledges the support of King Fahd University of Petroleum and Minerals.
AMS subject classification: **47H09**, 46B20, 47H10, 47E10.
Keywords: fixed point theory, modular metric spaces, multivalued contraction mapping, connected digraph.

Theorem 1.2 ([31]) *Let $(X, \leq)$ be a partially ordered set such that every pair $x, y \in X$ has an upper and lower bound. Let d be a distance on X such that (X, d) is a complete metric space. Let $f: X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

(i) *There exists a $k \in (0,1)$ with*

$$d\big(f(x), f(y)\big) \leq k\, d(x, y), \quad \textit{for all } x \geq y.$$

(ii) *There exists an $x_0 \in X$ with $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$.*

Then f is a Picard Operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$, $\lim_{n\to\infty} f^n(x) = x^*$.

After this, various authors considered the problem of existence of a fixed point for contraction mappings in partially ordered sets; see [6, 11, 18, 27] and references cited therein. Nieto, Pouso, and Rodriguez-Lopez [27] proved the following theorem.

Theorem 1.3 ([27]) *Let (X, d) be a complete metric space endowed with a partial ordering $\leq$. Let $f: X \to X$ be an order preserving mapping such that there exists a $k \in [0,1)$ with*

$$d(f(x), f(y)) \leq k\, d(x, y), \quad \textit{for all } x \geq y.$$

Assume that one of the following conditions holds:

(i) *f is continuous, and there exists an $x_0 \in X$ with $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$;*
(ii) *$(X, d, \leq)$ is such that for any nondecreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \leq x$ for $n \in N$, and there exists an $x_0 \in X$ with $x_0 \leq f(x_0)$;*
(iii) *$(X, d, \leq)$ is such that for any nonincreasing $(x_n)_{n\in N}$, if $x_n \to x$, then $x_n \geq x$ for $n \in N$, and there exists an $x_0 \in X$ with $x_0 \geq f(x_0)$.*

Then f has a fixed point. Moreover, if $(X, \leq)$ is such that every pair of elements of X has an upper or a lower bound, then f is a PO.

J. Jachymski [16] extended the above theorem to metric spaces endowed with a graph. In [16, 22] one can find the generalization of the results of [11, 27, 29, 30] to single-valued mappings in metric spaces with a graph. Subsequently, Beg, Butt, and Radojević [7] tried to extend the results of [16] to multivalued mappings, but their extension was not carried correctly (see [4]).

Recently, the author in [3] studied the existence of fixed points for multivalued mappings in modular function spaces endowed with a graph. He proved the following theorem.

Theorem 1.4 ([3, Theorem 3.3]) *Let $\rho \in \mathfrak{R}$ be convex. Let $C \subset L_\rho$ a be nonempty ρ-closed subset that has the following property. For any sequence $\{f_n\}_{n\in\mathbb{N}}$ in C, if f_n ρ-converges to f and $(f_n, f_{n+1}) \in E(G)$ for $n \in \mathbb{N}$, then $(f_n, f) \in E(G)$. Assume that ρ satisfies the Δ_2-type condition and C is ρ-bounded. Let $T: C = V(G) \to \mathcal{C}(C)$ be a monotone increasing G-contraction mapping and let $C_T := \{f \in C : (f, g) \in E(G)$ for some $g \in T(f)\}$. If $C_T \neq \varnothing$, then the following statements hold:*

(i) *For any $f \in C_T$, $T|_{[f]_{\widetilde{G}}}$ has a fixed point.*

https://doi.org/10.4153/CMB-2015-029-x Published online by Cambridge University Press

(ii) *If $f \in C$ with $(\bar{f}, f) \in E(G)$, where $\bar{f}$ is a fixed point of T, then there exists a sequence $\{f_n\}$ such that $f_{n+1} \in T(f_n)$, for every $n \geq 0$, and $\{f_n\}$ ρ-converges to $\bar{f}$.*
(iii) *If G is weakly connected, then T has a fixed point in G.*
(iv) *If $C' := \bigcup\{[f]_{\widetilde{G}_\rho} : f \in C_T\}$, then $T|_{C'}$ has a fixed point in C.*

The aim of this paper is to discuss the existence of fixed points for multivalued Lipschitzian mappings defined on some subsets of modular metric spaces X endowed with a graph G. These modular metric spaces were introduced in [9,10]. However, the approach we take is identical to the one used by the authors in [1]. Indeed, we look at the modular metric spaces as the nonlinear version of the classical modular spaces as introduced by Nakano [26] on vector spaces and modular function spaces introduced by Musielack [24] and Orlicz [28]. In [1] the authors defined and investigated the fixed point property in the framework of modular metric spaces and introduced the analogue of Banach Contraction Principle theorem in modular metric spaces.

In 1961, Edelstein [12] generalized the Banach Contraction Principle to mappings satisfying a less restrictive Lipschitz inequality such as local contraction. This result has been generalized to multivalued version by Nadler [25]. On the other hand, Mizoguchi and Takahashi [23] improved Reich's result [32] and proved the existence of fixed points for multivalued mappings with closed bounded values.

In this work, we obtain a multivalued version of the result of [2, Theorem 4.1] to modular metric spaces endowed with a graph. We also extend the results of Nadler [25] and Mizoguchi and Takahashi [23] to modular metric spaces with a graph.

2 Preliminaries

Let X be a nonempty set. Throughout this paper we will write

$$\omega_\lambda(x, y) = \omega(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$ for a function $\omega\colon (0, \infty) \times X \times X \to (0, \infty)$.

Definition 2.1 ([9,10]) A function $\omega\colon (0, \infty)\times X\times X \to [0, \infty]$ is said to be a *modular* on X if it satisfies the following axioms:

(i) $x = y$ if and only if $\omega_\lambda(x, y) = 0$, for all $\lambda > 0$;
(ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$, for all $\lambda > 0$, and $x, y \in X$;
(iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$, for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (i), we have only the condition (i')

$$\omega_\lambda(x, x) = 0, \quad \text{for all } \lambda > 0 \text{ and } x \in X,$$

then ω is said to be a pseudomodular on X. A modular ω on X is said to be regular if the following weaker version of (i) is satisfied:

$$x = y \text{ if and only if } \omega_\lambda(x, y) = 0, \text{ for some } \lambda > 0.$$

Finally, ω is said to be convex if for $\lambda, \mu > 0$ and $x, y, z \in X$, it satisfies the inequality

$$\omega_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda + \mu}\omega_\lambda(x, z) + \frac{\mu}{\lambda + \mu}\omega_\mu(z, y).$$

Note that for a pseudomodular ω on a set X, and any $x, y \in X$, the function $\lambda \to \omega_\lambda(x,y)$ is nonincreasing on $(0,\infty)$. Indeed, if $0 < \mu < \lambda$, then

$$\omega_\lambda(x,y) \le \omega_{\lambda-\mu}(x,x) + \omega_\mu(x,y) = \omega_\mu(x,y).$$

Definition 2.2 ([9,10]) Let ω be a pseudomodular on X. Fix $x_0 \in X$. The two sets

$$X_\omega = X_\omega(x_0) = \{x \in X\colon \omega_\lambda(x,x_0) \to 0 \text{ as } \lambda \to \infty\},$$
$$X_\omega^* = X_\omega^*(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0 \text{ such that } \omega_\lambda(x,x_0) < \infty\}$$

are said to be modular spaces (around x_0).

We obviously have $X_\omega \subset X_\omega^*$. In general this inclusion can be proper. It follows from [9,10] that if ω is a modular on X, then the modular space X_ω can be equipped with a (nontrivial) distance, generated by ω and given by

$$d_\omega(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le \lambda\},$$

for any $x, y \in X_\omega$. If ω is a convex modular on X, then according to [9,10] the two modular spaces coincide, *i.e.*, $X_\omega^* = X_\omega$, and this common set can be endowed with the distance d_ω^* given by

$$d_\omega^*(x,y) = \inf\{\lambda > 0 : \omega_\lambda(x,y) \le 1\},$$

for any $x, y \in X_\omega$. These distances will be called Luxemburg distances.

First attempts to generalize the classical function spaces of the Lebesgue type L^p spaces were made in the early 1930's by Orlicz and Birnbaum in connection with orthogonal expansions. Their approach consisted of considering spaces of functions with some growth properties different from the power type growth control provided by the L^p-norms. Namely, they considered the function spaces defined as follows:

$$L^\phi = \left\{ f\colon \mathbb{R} \to \mathbb{R};\ \exists\lambda > 0 : \rho(\lambda f) = \int_{\mathbb{R}} \phi\big(\lambda |f(x)|\big)\ dx < \infty \right\},$$

where $\phi\colon [0,\infty] \to [0,\infty]$ was assumed to be a convex function increasing to infinity, *i.e.,* the function, which to some extent, behaves similarly to power functions $\phi(t) = t^p$. Modular function spaces L^ϕ furnishes a wonderful example of a modular metric space. Indeed define the function ω by

$$\omega_\lambda(f,g) = \rho\Big(\frac{f-g}{\lambda}\Big) = \int_{\mathbb{R}} \phi\Big(\frac{|f(x)-g(x)|}{\lambda}\Big)\ dx,$$

for all $\lambda > 0$, and $f, g \in L^\phi$. Then ω defines a modular metric on L^ϕ. Moreover, the distance d_ω^* is exactly the distance generated by the Luxemburg norm on L^ϕ.

For more examples on modular function spaces, the reader can consult Kozlowski [21] and, for modular metric spaces, [9,10].

Definition 2.3 Let X_ω be a modular metric space.

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_ω is said to be ω-convergent to $x \in X_\omega$ if and only if $\omega_1(x_n, x) \to 0$, as $n \to \infty$. Then x will be called the ω-limit of $\{x_n\}$.

(ii) The sequence $\{x_n\}_{n\in N}$ in X_ω is said to be ω-Cauchy if $\omega_1(x_m, x_n) \to 0$, as $m, n \to \infty$.

(iii) A subset M of X_ω is said to be ω-closed if the ω-limit of a ω-convergent sequence of M always belong to M.
(iv) A subset M of X_ω is said to be ω-complete if any ω-Cauchy sequence in M is a ω-convergent sequence and its ω-limit is in M.
(v) A subset M of X_ω is said to be ω-bounded if we have

$$\delta_\omega(M) = \sup\{\omega_1(x,y); x,y \in M\} < \infty.$$

In general if $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for some $\lambda > 0$, then we cannot have $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for all $\lambda > 0$. Therefore, as it is done in modular function spaces, we will say that ω satisfies the Δ_2-condition if $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for some $\lambda > 0$ implies $\lim_{n\to\infty} \omega_\lambda(x_n,x) = 0$, for all $\lambda > 0$. In [9,10], one will find a discussion about the connection between ω-convergence and metric convergence with respect to the Luxemburg distances. In particular, we have

$$\lim_{n\to\infty} d_\omega(x_n,x) = 0 \ \text{ if and only if } \ \lim_{n\to\infty} \omega_\lambda(x_n,x) = 0, \ \text{for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$. And in particular we have that ω-convergence and d_ω convergence are equivalent if and only if the modular ω satisfies a the Δ_2-condition. Moreover, if the modular ω is convex, then we know that d^*_ω and d_ω are equivalent which implies

$$\lim_{n\to\infty} d^*_\omega(x_n,x) = 0 \text{ if and only if } \lim_{n\to\infty} \omega_\lambda(x_n,x) = 0, \ \text{for all } \lambda > 0,$$

for any $\{x_n\} \in X_\omega$ and $x \in X_\omega$ [9,10].

Definition 2.4 Let (X,ω) be a modular metric space. We will say that ω satisfies Δ_2-type condition if for any $\alpha > 0$, there exists $C > 0$ such that

$$\omega_{\lambda/\alpha}(x,y) \le C\,\omega_\lambda(x,y),$$

for any $\lambda > 0$, $x,y \in X_\omega$, with $x \ne y$.

Note that if ω satisfies the Δ_2-type condition, then ω satisfies the Δ_2-condition. The above definition will allow us to introduce the growth function in the modular metric spaces as in the linear case.

Definition 2.5 ([2]) Let (X,ω) be a modular metric space. Define the growth function Ω by

$$\Omega(\alpha) = \sup\Big\{\frac{\omega_{\lambda/\alpha}(x,y)}{\omega_\lambda(x,y)}; \lambda > 0, x,y \in X_\omega, x \ne y\Big\},$$

for any $\alpha > 0$.

The following properties were proved in [2].

Lemma 2.1 ([2, Lemma 2.1]) *Let (X,ω) be a modular metric space. Assume that ω is a convex regular modular that satisfies the Δ_2-type condition. Then*

(i) $\Omega(\alpha) < \infty$, *for any* $\alpha > 0$,
(ii) Ω *is a strictly increasing function, and* $\Omega(1) = 1$,
(iii) $\Omega(\alpha\beta) \le \Omega(\alpha)\Omega(\beta)$, *for any* $\alpha,\beta \in (0,\infty)$,

(iv) $\Omega^{-1}(\alpha)\Omega^{-1}(\beta) \le \Omega^{-1}(\alpha\beta)$, *where* Ω^{-1} *is the function inverse of* Ω,
(v) *for any* $x, y \in X_\omega$, $x \ne y$, *we have*

$$d_\omega^*(x,y) \le \frac{1}{\Omega^{-1}(1/\omega_1(x,y))}.$$

The following technical lemma will be useful later on in this work.

Lemma 2.2 ([2]) *Let* (X,ω) *be a modular metric space. Assume that* ω *is a convex regular modular that satisfies the* Δ_2*-type condition. Let* $\{x_n\}$ *be a sequence in* X_ω *such that*

$$\omega_1(x_{n+1}, x_n) \le K\,\alpha^n, n = 1, \dots,$$

where K *is an arbitrary nonzero constant and* $\alpha \in (0,1)$. *Then* $\{x_n\}$ *is Cauchy for both* ω *and* d_ω^*.

Note that this lemma is crucial, since the main assumption on $\{x_n\}$ will not be enough to imply that $\{x_n\}$ is ω-Cauchy, since ω fails the triangle inequality.

Let us finish this section with the needed graph theory terminology, which will be used throughout.

A directed graph (digraph) G is called an oriented graph if whenever $(x,y) \in E(G)$, $(y,x) \notin E(G)$. If x, y are vertices of the digraph G, then a directed path from x to y of length N is a sequence $\{x_i\}_{i=0}^N$ of $N+1$ vertices of N such that

$$x_0 = x, \quad x_N = y \quad \text{and} \quad (x_i, x_{i+1}) \in E(G), \quad i = 0, 1, \dots, N.$$

Let (X,ω) be a modular metric space, and let M be a nonempty subset of X_ω. Let Δ denote the diagonal of the cartesian product $M \times M$. Throughout, we consider a directed graph G_ω such that the set $V(G_\omega)$ of its vertices coincides with M, and the set $E(G_\omega)$ of its edges contains all loops, *i.e.*, $E(G_\omega) \supseteq \Delta$. We assume G_ω has no parallel edges (arcs), so we can identify G_ω with the pair $(V(G_\omega), E(G_\omega))$. Our graph theory notations and terminology are standard and can be found in any graph theory books. For example, the reader can consult [8,17]. Moreover, we can treat G_ω as a weighted graph (see [17, p. 309]) by assigning to each edge the distance between its vertices.

At this point we introduce some notation that will be used through the reminder of this work. For a subset M of modular metric space X_ω, set

$$\mathcal{CB}(M) = \{C : C \text{ is nonempty } \omega\text{-closed and } \omega\text{-bounded subset of } M\}.$$

Definition 2.6 Let (X,ω) be a modular metric space, M be a nonempty subset of X_ω. A multivalued mapping $T\colon M \to \mathcal{CB}(M)$ is called:

- a G_ω*-contraction* if there exists a constant $k \in [0,1)$ such that for any $u, v \in M$ with $(u,v) \in E(G_\omega)$ and any $U \in T(u)$ there exists $V \in T(v)$ such that

$$(U,V) \in E(G_\omega) \quad \text{and} \quad \omega_1(U,V) \le k\,\omega_1(u,v).$$

- a $(\varepsilon,k) - G_\omega$*-uniformly locally contraction* if there exists a constant $k \in [0,1)$ such that for any $u, v \in M$ with $(u,v) \in E(G_\omega)$, $\omega_1(u,v) < \varepsilon$ and any $U \in T(u)$ there exists $V \in T(v)$ such that

$$(U,V) \in E(G_\omega) \quad \text{and} \quad \omega_1(U,V) \le k\,\omega_1(u,v).$$

A point $x \in M$ is called a *fixed point* of T whenever $x \in T(x)$. The set of fixed points of T will be denoted by $\mathrm{Fix}(T)$.

We will say that M satisfies property (P) if and only if

(P) for any sequence $\{x_n\}_{n\in\mathbb{N}}$ in M, if x ω-converges to z as $n \to \infty$ and $(x_n, x_{n+1}) \in E(G_\omega)$, then $(x_n, z) \in E(G_\omega)$, for all n.

The above property plays a very important role in the proofs of the main results of this work.

3 The Main Results

The following result, where the directed graph G_ω is defined on a subset M of the modular metric space (X, ω), can be seen as a generalization of Nadler's fixed point result [25] to modular metric spaces endowed with a graph.

Theorem 3.1 *Let (X, ω) be a modular metric space. Suppose that ω is a convex regular modular that satisfies the Δ_2- type condition. Assume that $M = V(G_\omega)$ is a nonempty ω-complete subset of X_ω satisfying property* (P)*. Let $T\colon M \to \mathcal{CB}(M)$ be a G_ω-contraction map and let $M_T := \{x \in M : (x, y) \in E(G_\omega)$ for some $y \in T(x)\}$. If $M_T \neq \varnothing$, then T has a fixed point.*

Proof Let $x_0 \in M_T$; then there exists an $x_1 \in T(x_0)$ with $(x_0, x_1) \in E(G_\omega)$. Since T is a G_ω-contraction, there exists $x_2 \in T(x_1)$ such that $(x_1, x_2) \in E(G_\omega)$ and

$$\omega_1(x_1, x_2) \leq k\omega_1(x_0, x_1).$$

Similarly, there exists $x_3 \in T(x_2)$ such that $(x_2, x_3) \in E(G_\omega)$ and

$$\omega_1(x_2, x_3) \leq k\ \omega_1(x_1, x_2).$$

By induction we build $\{x_n\}$ in M with $x_{n+1} \in T(x_n)$ and $(x_n, x_{n+1}) \in E(G_\omega)$ such that

$$\omega_1(x_{n+1}, x_n) \leq k\ \omega_1(x_n, x_{n-1}),$$

for every $n \geq 1$. Hence,

$$\omega_1(x_{n+1}, x_n) \leq k^n\ \omega_1(x_1, x_0),$$

for every $n \geq 0$. The technical Lemma 2.2 implies that $\{x_n\}$ is ω-Cauchy. Since M is ω-complete, $\{x_n\}$ ω-converges to some point $z \in M$. Since $(x_n, x_{n+1}) \in E(G_\omega)$, for every $n \geq 1$, $(x_n, z) \in E(G_\omega)$ by property (P). Since T is a G_ω-contraction, there exists $z_n \in T(z)$ such that

$$\omega_1(x_{n+1}, z_n) \leq k\ \omega_1(x_n, z),$$

for every $n \geq 1$. Hence

$$\omega_2(z_n, z) \leq \omega_1(z_n, x_{n+1}) + \omega_1(x_{n+1}, z) \leq k\ \omega_1(x_n, z) + \omega_1(x_{n+1}, z),$$

for every $n \geq 1$. Since $\{x_n\}$ ω-converges to z, we conclude that $\lim_{n\to\infty} \omega_2(z_n, z) = 0$. The Δ_2- type condition satisfied by ω implies that $\lim_{n\to\infty} \omega_1(z_n, z) = 0$, *i.e.*, $\{z_n\}$ ω-converges to z. Since $T(z)$ is ω-closed, we conclude that $z \in T(z)$, *i.e.*, z is a fixed point of T. This completes the proof of Theorem 3.1. ■

Edelstein [12] has extended the classical fixed point theorem for contractions to the case when X is a complete ε-chainable metric space, and the mapping $T: X \to X$ is an (ε, k)-uniformly locally contraction. This result was extended by Nadler [25] to multivalued mappings. Here we investigate Nadler's result in modular metric spaces endowed with a graph. First let us introduce the ε-chainable concept in modular metric spaces with a graph. Our definition is slightly different from the one used in the classical metric spaces, since the modular functions can fail the triangle inequality.

Definition 3.1 Let (X, ω) be a modular metric space and let $M = V(G_\omega)$ be a nonempty subset of X_ω. M is said to be finitely ε-chainable (where $\varepsilon > 0$ is fixed) if and only if there exists $N \geq 1$ such that for any $a, b \in M$ with $(a, b) \in E(G_\omega)$ there is an N, ε-chain from a to b (that is, a finite set of vertices $x_0, x_1, \dots, x_N \in V(G_\omega) = M$ such that $x_0 = a$, $x_N = b$, $(x_i, x_{i+1}) \in E(G_\omega)$ and $\omega_1(x_i, x_{i+1}) < \varepsilon$, for all $i = 0, 1, 2, \dots, N - 1$).

We have the following result.

Theorem 3.2 *Let (X, ω) be a modular metric space. Suppose that ω is a convex regular modular that satisfies the Δ_2-type condition. Assume that $M = V(G_\omega)$ is a nonempty ω-complete, ω-bounded subset of X_ω which satisfies the property (P) and is finitely ε-chainable, for some fixed $\varepsilon > 0$. Let $T: M \to \mathcal{CB}(M)$ be an $(\varepsilon, k) - G_\omega$-uniformly locally contraction map. If $M_T := \{x \in M : (x, y) \in E(G_\omega)$ for some $y \in T(x)\} \neq \varnothing$, then T has a fixed point.*

Proof Since M is finitely ε-chainable, there exists $N \geq 1$ such that for any $a, b \in M$ with $(a, b) \in E(G_\omega)$ there is a finite set of vertices $x_0, x_1, \dots, x_N \in M$ such that $x_0 = a$, $x_N = b$, $(x_i, x_{i+1}) \in E(G_\omega)$ and $\omega_1(x_i, x_{i+1}) < \varepsilon$, for all $i = 0, 1, 2, \dots, N - 1$. For any $x, y \in M$ with $(x, y) \in E(G_\omega)$, define

$$\omega^*(x, y) = \inf\Big\{ \sum_{i=0}^{i=N-1} \omega_1(x_i, x_{i+1}) \Big\},$$

where the infimum is taken over all N, ε-chains $x_0, x_1, \dots, x_N$ from x to y. Since M is N, ε-chainable, $\omega^*(x, y) < \infty$, for any $x, y \in M$ with $(x, y) \in E(G_\omega)$. Using the basic properties of ω, we get

$$\omega_N(x, y) \leq \omega^*(x, y),$$

for any $x, y \in M$ with $(x, y) \in E(G_\omega)$. Moreover, if $\omega_1(x, y) < \varepsilon$, then we have $\omega^*(x, y) \leq \omega_1(x, y)$, for any $x, y \in M$ with $(x, y) \in E(G_\omega)$. Fix $x \in M_T$. Set $z_0 = x$. Choose $z_1 \in T(z_0)$ with $(z_0, z_1) \in E(G_\omega)$. Let $x_0, x_1, \dots, x_N$ be an N, ε-chain from z_0 to z_1. Since T is an $(\varepsilon, k) - G_\omega$-uniformly locally contraction map, there exist $y_0, y_1, \dots, y_N$ in M such that:

(a) $y_i \in T(x_i)$, for any $i = 1, \dots, N$;
(b) $(y_i, y_{i+1}) \in E(G_\omega)$, for any $i = 0, \dots, N - 1$;
(c) $\omega_1(y_i, y_{i+1}) \leq k\, \omega_1(x_i, x_{i+1})$, for any $i = 0, \dots, N - 1$.

It is easy to check that $y_0 = z_0, y_1, \cdots, y_N$ is an N, ε-chain from z_0 to $y_N \in T(z_1)$. Set $y_N = z_2$. Using the fact that T is an $(\varepsilon, k) - G_\omega$-uniformly locally contraction map, we get

$$\omega^*(z_1, z_2) \le k\ \omega^*(z_0, z_1).$$

By induction, we construct a sequence $\{z_n\} \in M$ with $(z_n, z_{n+1}) \in E(G_\omega)$ such that

$$\omega^*(z_n, z_{n+1}) \le k\ \omega^*(z_{n-1}, z_n),$$

and $z_{n+1} \in T(z_n)$, for any $n \ge 1$. Obviously we have $\omega^*(z_n, z_{n+1}) \le k^n\ \omega^*(z_0, z_1)$, for any $n \ge 1$. Since ω satisfies the Δ_2-type condition, there exists $C > 0$ such that

$$\omega_1(z_n, z_{n+1}) \le C\ \omega_N(z_n, z_{n+1}) \le C\ \omega^*(z_n, z_{n+1}) \le C\ k^n\ \omega^*(z_0, z_1),$$

for any $n \ge 1$. Lemma 2.2 implies that $\{z_n\}$ is ω-Cauchy. Since M is ω-complete, then $\{z_n\}$ ω-converges to some $z \in M$. We claim that z is a fixed point of T. Indeed we first note that $(z_n, z) \in E(G_\omega)$ for any $n \ge 1$ by the property (P). Using the ideas developed above, there exists $v_n \in T(z)$ such that

$$\omega^*(z_{n+1}, v_n) \le k\ \omega^*(z_n, z),$$

for any $n \ge 1$. By ω properties, we have

$$\omega_{N+1}(v_n, z) \le \omega_1(z_{n+1}, z) + \omega_N(z_{n+1}, v_n) \le \omega_1(z_{n+1}, z) + k\ \omega^*(z_n, z),$$

for any $n \ge 1$. Since $\{z_n\}$ ω-converges to z, there exists $n_0 \ge 1$ such that for any $n \ge n_0$, we have $\omega_1(z_n, z) < \varepsilon$. Hence, $\omega^*(z_n, z) \le \omega_1(z_n, z)$, for any $n \ge n_0$, which implies

$$\omega_{N+1}(v_n, z) \le \omega_1(z_{n+1}, z) + k\ \omega_1(z_n, z),$$

for any $n \ge n_0$. Therefore, we have $\lim_{n\to\infty} \omega_{N+1}(v_n, z) = 0$. The Δ_2- type condition satisfied by ω implies that $\lim_{n\to\infty} \omega_1(v_n, z) = 0$, *i.e.,* $\{v_n\}$ ω-converges to z. Since $v_n \in T(z)$ and $T(z)$ is ω-closed, we conclude that $z \in T(z)$, *i.e., z* is a fixed point of T. This completes the proof of Theorem 3.2. ∎

References

[1] A. N. Abdou and M. A. Khamsi, *Fixed points results of pointwise contractions in modular metric spaces.* Fixed Point Theory Appl. **2013**, 2013:163.

[2] ———, *Fixed points of multivalued contraction mappings in modular metric spaces.* Fixed Point Theory Appli. **2014**, 2014:249 http://dx.doi.org/10.1186/1687-1812-2014-249

[3] M. R. Alfuraidan, *Fixed points of multivalued mappings in modular function spaces with a graph.* Fixed Point Theory Appli. **2015**, 2015:42.

[4] ———, *Remarks on monotone multivalued mappings on a metric space with a graph.* J. Inequal. Appl., to appear.

[5] I. Beg, *Fixed points of fuzzy multivalued mappings with values in fuzzy ordered sets.* J. Fuzzy Math. **6**(1998), no. 1, 127–131.

[6] I. Beg and A. R. Butt, *Fixed point for set valued mappings satisfying an implicit relation in partially ordered metric spaces*. Nonlinear Anal. **71**(2009), no. 9, 3699–3704. http://dx.doi.org/10.1016/j.na.2009.02.027

[7] I. Beg, A. R. Butt, and S. Radojević, *The contraction principle for set valued mappings on a metric space with a graph*. Comput. Math. Appl. **60**(2010), no. 5, 1214–1219. http://dx.doi.org/10.1016/j.camwa.2010.06.003

[8] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*. Fifth ed., CRC Press, Boca Raton, FL, 2011.

[9] V. V. Chistyakov, *Modular metric spaces. I. Basic concepts.* Nonlinear Anal. 72(2010), no. 1. 1–14. http://dx.doi.org/10.1016/j.na.2009.04.057

[10] ———, *Modular metric spaces. II. Application to superposition operators.* Nonlinear Anal. **72**(2010), no. 1, 15–30. http://dx.doi.org/10.1016/j.na.2009.04.018

https://doi.org/10.4153/CMB-2015-029-x Published online by Cambridge University Press

[11] Z. Drici, F. A. McRae, and J. Vasundhara Devi, *Fixed point theorems in partially ordered metric space for operators with PPF dependence.* Nonlinear Anal. **67**(2007), no. 2, 641–647. http://dx.doi.org/10.1016/j.na.2006.06.022
[12] M. Edelstein, *An extension of Banach's contraction principle.* Proc. Amer. Math. Soc. **12**(1961), 7–10.
[13] Y. Feng and S. Liu, *Fixed point theorems for multivalued contractive mappings and multivalued Caristi type mappings*. J. Math. Anal. Appl. **317**(2006), no. 1, 103–112. http://dx.doi.org/10.1016/j.jmaa.2005.12.004
[14] T. Fujimoto, *An extension of Tarski's fixed point theorem and its application to isotone complementarity problems*. Math. Programming **28**(1984), no. 1, 116–118. http://dx.doi.org/10.1007/BF02612716
[15] A. Granas and J. Dugundji, *Fixed point theory.* Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.
[16] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*. Proc. Amer. Math. Soc. **136**(2008), no. 4, 1359–1373. http://dx.doi.org/10.1090/S0002-9939-07-09110-1
[17] R. Johnsonbaugh, *Discrete mathematics*. Prentice-Hall, Inc., New Jersey, 1997.
[18] J. Harjani and K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*. Nonlinear Anal. **72**(2010), no. 3–4, 1188-1197. http://dx.doi.org/10.1016/j.na.2009.08.003
[19] W. A. Kirk and K. Goebel, *Topics in metric fixed point theory.* Cambridge Studies in Advanced Mathematics, 28, Cambridge University Press, Cambridge, 1990.
[20] D. Klim and D. Wardowski, *Fixed point theorems for set-valued contractions in complete metric spaces*. J. Math. Anal. Appl. **334**(2007), no. 1, 132–139. http://dx.doi.org/10.1016/j.jmaa.2006.12.012
[21] W. M. Kozlowski, *Modular function spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 122, Dekker, New York, 1988.
[22] G. G. Lukawska and J. Jachymski, *IFS on a metric space with a graph structure and extension of the Kelisky-Rivlin theorem*. J. Math. Anal. Appl. **356**(2009), no. 2, 453–463. http://dx.doi.org/10.1016/j.jmaa.2009.03.023
[23] N. Mizoguchi and W. Takahashi, *Fixed point theorems for multivalued mappings on complete metric spaces.* J. Math. Anal. Appl. **141**(1989), no. 1, 177–188. http://dx.doi.org/10.1016/0022-247X(89)90214-X
[24] J. Musielak, *Orlicz spaces and modular spaces.* Lecture Notes in Mathematics, 1034, Springer-Verlag, Berlin, 1983.
[25] S. B. Nadler, Jr., *Multi-valued contraction mappings*. Pacific J. Math. **30**(1969), 475–488. http://dx.doi.org/10.2140/pjm.1969.30.475
[26] H. Nakano, *Modulared semi-ordered linear spaces.* Maruzen Co., Ltd., Tokyo, 1950.
[27] J. J. Nieto, R. L. Pouso, and R. Rodriguez-Lopez, *Fixed point theorems in ordered abstract spaces*. Proc. Amer. Math. Soc. **135**(2007), no. 8, 2505–2517. http://dx.doi.org/10.1090/S0002-9939-07-08729-1
[28] W. Orlicz, *Collected Papers, part I, II.* PWN–Polish Scientific Publishers, Warsaw, 1988.
[29] D. O'Regan and A. Petrusel, *Fixed point theorems for generalized contraction in ordered metric spaces*. J. Math. Anal. Appl. **341**(2008), no. 2, 1241–1252. http://dx.doi.org/10.1016/j.jmaa.2007.11.026
[30] A. Petrusel and I. A. Rus, *Fixed point theorems in ordered L-spaces*. Proc. Amer. Math. Soc. **134**(2006), no. 2, 411–418.
[31] A. C. M. Ran and M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*. Proc. Amer. Math. Soc. **132**(2004), no. 5, 1435–1443. http://dx.doi.org/10.1090/S0002-9939-03-07220-4
[32] S. Reich, *Some fixed point problems*. Atti. Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **57**(1974), no. 3–4, 194–198.
[33] A. Tarski, *A lattice theoretical fixed point and its application*. Pacific J. Math. **5**(1955), 285–309. http://dx.doi.org/10.2140/pjm.1955.5.285

Department of Mathematics & Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
e-mail: monther@kfupm.edu.sa

https://doi.org/10.4153/CMB-2015-029-x Published online by Cambridge University Press