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RIGIDITY THEOREMS FOR HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN A UNIT SPHERE

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Abstract. In this paper, we give a characterization of Clifford tori $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$ and $S^m(a) \times S^{n-m}(\sqrt{1-a^2})$ $(2 \le m \le n-2, 0 < a^2 < 1)$ in a unit sphere $S^{n+1}(1)$. Our results extend the results due to Cheng and Yau [**4**], and Wang and Xia [**11**].

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1. Introduction. Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Now let us state a theorem due to Cheng and Yau [**4**].

THEOREM 1.1 ([**4**]). *Let M be an n-dimensional compact hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If*
(1) $r \ge 1$,
(2) *the sectional curvature of M is nonnegative,*
then M is either a totally umbilical hypersurface or a Riemannian product

$$S^k(a) \times S^{n-k}(\sqrt{1-a^2}), \quad 1 \le k \le n-1,$$

where $S^k(a)$ denotes the sphere of radius a.

On the other hand, Wang and Xia [**11**] have proved the following theorem.

THEOREM 1.2 ([**11**]). *Let M be an n-dimensional $(n \ge 3)$ compact orientable hypersurface immersed in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1, respectively. Assume that $\lambda\mu \le -1$ holds on M. Then M is isometric to a Riemannian product $S^1(\sqrt{1-\frac{n-2}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$.*

In the proof of Theorem 1.1, the condition $r \ge 1$ is necessary. Moreover, the topological assumption that M is compact in Theorem 1.2 plays an important role in

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the proof of this theorem. However, it can be easily checked that many hypersurfaces with constant scalar curvature $n(n-1)r,(r<1)$ also have *nonnegative Ricci curvature*.

EXAMPLE. $M_{1,n-1} = S^1(a) \times S^{n-1}(\sqrt{1-a^2})$. Then $M_{1,n-1}$ has two distinct constant principal curvatures

$$\lambda_1 = \cdots = \lambda_{n-1} = \lambda = -\frac{a}{\sqrt{1-a^2}}, \quad \lambda_n = \mu = \frac{\sqrt{1-a^2}}{a} \tag{1.1}$$

and constant mean curvature $H = \frac{1}{n}\sum_{i=1}^{n}\lambda_i = \frac{1-na^2}{na\sqrt{1-a^2}}$.

From (1.1), first we know $nH = (n-1)\lambda + \mu$. Then, by the formula (2.5) in section 2 the Ricci curvatures are given as follows:

$$R_{00} = (n-1)(1+\lambda\mu) = 0, \quad R_{ii} = (n-1) + (n-2)\lambda^2 + \lambda\mu \geq n-2 \geq 0 \tag{1.2}$$

for $i = 1, \ldots, n-1$. Then it can be easily seen that all Ricci curvatures of $M_{1,n-1} = S^1(a) \times S^{n-1}(\sqrt{1-a^2})$ $(0 < a^2 < 1)$ are nonnegative. By a straightforward computation for $M_{1,n-1} = S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, we obtain

$$a^2 = \frac{nr+2-n}{nr} > 0, \quad i.e. \ r > \frac{n-2}{n}. \tag{1.3}$$

Hence $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$ has constant scalar curvature $n(n-1)r$ and the Ricci curvatures of $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$ are nonnegative.

Now, in this paper, we consider a complete hypersurface M in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and two distinct principal curvatures. Then, we assert the following results.

THEOREM 1.3. *Let M be an n-dimensional $(n \geq 3)$ complete connected hypersurface with constant scalar curvature $n(n-1)r$ in $S^{n+1}(1)$. If*

(1) *M has two distinct principal curvatures,*

(2) *the Ricci curvatures of M are nonnegative,*

then M is isometric either to the Riemannian product $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$ or to $S^m(a) \times S^{n-m}(\sqrt{1-a^2})$ $(2 \leq m \leq n-2, 0 < a < 1)$.

THEOREM 1.4. *Let M be an n-dimensional $(n \geq 3)$ complete connected hypersurface immersed in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and two distinct principal curvatures λ and μ of multiplicities $n-1$ and 1, respectively. Assume that $\lambda\mu \leq -1$ holds on M. Then M is isometric to a Riemannian product $S^1(\sqrt{1-\frac{n-2}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$.*

REMARK 1.1. Theorems 1.1 and 1.2 are different from Theorems 1.3 and 1.4. In the proof of Theorems 1.1 and 1.2, the topological assumption that M is compact is necessary. But in Theorems 1.3 and 1.4 we only assume that M is complete.

REMARK 1.2. When M is compact, our Theorem 1.4 reduces to Theorem 1.2.

2. Preliminaries.

Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of M with respect to the induced metric,

$\omega_1, \ldots, \omega_n$ their dual form. Let e_{n+1} be the local unit normal vector field. In this paper, we shall make use of the following convention on the ranges of indices:

$$1 \leq i, j, k, \ldots \leq n, \quad 1 \leq a, b, \ldots \leq m, \quad m+1 \leq \alpha, \beta, \ldots \leq n.$$

Then we have the structure equations

$$dx = \sum_i \omega_i e_i, \tag{2.1}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_j h_{ij} \omega_j e_{n+1} - \omega_i x, \tag{2.2}$$

$$de_{n+1} = -\sum_{i,j} h_{ij} \omega_j e_i, \tag{2.3}$$

where h_{ij} denotes the components of the second fundamental form of M.

By the equation of Gauss the curvature tensor, the Ricci tensor and the scalar curvature of M in $S^{n+1}(1)$ are respectively given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.4}$$

$$R_{ik} = (n-1)\delta_{ik} + nHh_{ik} - \sum_j h_{ij}h_{jk}, \tag{2.5}$$

$$n(n-1)r = n(n-1) + n^2H^2 - S, \tag{2.6}$$

where r denotes the normalized scalar curvature, $S = \sum_{i,j} h_{ij}^2$ the squared norm of the second fundamental form and H the mean curvature $H = \frac{1}{n}\sum_k h_{kk}$ respectively.

Now we assume that M has two distinct principal curvature λ and μ, that is,

$$h_{ij} = \lambda_i \delta_{ij}, \quad \lambda_1 = \lambda_2 = \cdots = \lambda_m = \lambda, \quad \lambda_{m+1} = \cdots = \lambda_n = \mu.$$

Then from (2.5) the Ricci curvatures are respectively given by the following (see also the notion of Ricci curvatures in the second author and Yang [**10**]),

$$\begin{aligned} R_{aa} &= (n-1) + (m-1)\lambda^2 + (n-m)\lambda\mu, \\ R_{\alpha\alpha} &= (n-1) + m\lambda\mu + (n-m-1)\mu^2. \end{aligned} \tag{2.7}$$

Now we must consider two cases.

Case 1: $2 \leq m \leq n-2$.

In [**9**], Otsuki proved the following result.

LEMMA 2.1 (Theorem 2 and Corollary of [**9**]). *Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ such that the multiplicities of principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than* 1*, then this principal curvature is constant on*

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each integral submanifold of the corresponding distribution of the space of principal vectors.

From Lemma 2.1. we can easily obtain the following result.

PROPOSITION 2.1 ([**3**]). *Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures. If the multiplicities of these two distinct principal curvatures are greater than 1, then M is isometric to the Riemannian product $S^m(a) \times S^{n-m}(\sqrt{1-a^2})$, $2 \leq m \leq n-2$.*

Case 2: $m = n-1$ or $m = 1$.

In this case, we assume

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \lambda \quad \text{and} \quad \lambda_n = \mu,$$

where λ_i denotes the principal curvature of M. From Gauss equation (2.6), we get

$$n(n-1)(r-1) = (n-1)(n-2)\lambda^2 + 2(n-1)\lambda\mu. \tag{2.8}$$

If $\lambda = 0$ at some point p, then $r = 1$ at this point. Since the scalar curvature $n(n-1)r$ is constant, we obtain $r \equiv 1$ on M. We see from (2.8) that $\lambda((n-2)\lambda + 2\mu) \equiv 0$, then we have $\lambda \equiv 0$ on M. In fact, let $N = \{x \mid x \in M, \lambda(x) \neq 0\}$, $Q = \{y \mid y \in M, (n-2)\lambda(y) + 2\mu(y) = 0\}$. Since these principal curvatures λ and μ are continuous on M, we know that N is an open set, Q is a close set and $N \neq M$ (since $\lambda(p) = 0$). Next we prove $N = Q$. On one hand, if $x \in N$, then $\lambda(x) \neq 0$. By $\lambda((n-2)\lambda + 2\mu) \equiv 0$, we obtain $(n-2)\lambda(x) + 2\mu(x) = 0$, that is, $x \in Q$. Hence $N \subseteq Q$. On the other hand, if $y \in Q$, then $(n-2)\lambda(y) + 2\mu(y) = 0$. Since λ and μ are two distinct principal curvatures of M, we have $\lambda(y) \neq \mu(y)$. We see from $(n-2)\lambda(y) + 2\mu(y) = 0$ that $\lambda(y) \neq 0$. (If $\lambda(y) = 0$, then $\mu(y) = 0 = \lambda(y)$. This is a contradiction). Thus $y \in N$, and we then have $Q \subseteq N$. Therefore $N = Q$. We see that N is not only an open set but also a closed set. Combining M connected with $N \neq M$, we deduce that N is an empty set. It follows that $\lambda \equiv 0$ on M. By (2.5), we have that the sectional curvature of M is not less than 1. Hence M is compact by use of Bonnet-Myers Theorem. We see from Theorem 1.1 that M is a totally umbilical hypersurface. Thus $\lambda \neq 0$.

From (2.8), we have

$$\mu = \frac{n(r-1)}{2\lambda} - \frac{n-2}{2}\lambda, \quad \lambda - \mu = n\frac{\lambda^2 - (r-1)}{2\lambda} \neq 0 \tag{2.9}$$

and it follows that $\lambda^2 - (r-1) \neq 0$. If $\lambda^2 - (r-1) < 0$, from (2.9) we deduce that

$$r > 1 \quad \text{and} \quad \lambda^2 - \lambda\mu = \frac{n}{2}[\lambda^2 - (r-1)] < 0.$$

Therefore $\lambda\mu > \lambda^2$. We obtain the sectional curvature of M is greater than 1 from (2.5). Then M is a totally umbilical hypersurface by use of Theorem 1.1. As a result, we get $\lambda^2 - (r-1) > 0$. Put $w = [\lambda^2 - (r-1)]^{-1/n}$. Cheng [**3**] proved the following.

PROPOSITION 2.2 (Theorem 2.1 of [**3**]). *Suppose that $n \geq 3$. If M is an n-dimensional hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ and with two distinct principal curvatures, and the space of principal vectors corresponding to one of them is of one dimension, then M is a locus of the moving $(n-1)$-dimensional submanifold*

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$M_1^{n-1}(s)$ along which the principal curvature λ of multiplicity $n-1$ is constant and which is locally isometric to an $(n-1)$-dimensional sphere $S^{n-1}(c(s)) = E^n(s) \cap S^{n+1}(1)$ of constant curvature and $w = [\lambda^2 - (r-1)]^{-1/n}$ satisfies the ordinary differential equation of order 2

$$\frac{d^2w}{ds^2} - w\left(\frac{n-2}{2}\frac{1}{w^n} - r\right) = 0, \tag{2.10}$$

where $E^n(s)$ is an n-dimensional linear subspace in the Euclidean space R^{n+2} which is parallel to a fixed E^n.

3. Proofs of Theorems 1.3 and 1.4. In order to give complete proofs of Theorems 1.3 and 1.4 we must verify the following.

LEMMA 3.1. *Equation* (2.10) *is equivalent to its first order integral*

$$\left(\frac{dw}{ds}\right)^2 + rw^2 + \frac{1}{w^{n-2}} = C, \tag{3.1}$$

where C is a constant; for a constant solution equal to w_0, one has that $r > 0$ and $w_0^n = \frac{n-2}{2r}$, so

$$C_0 = \frac{n}{2}\left(\frac{2r}{n-2}\right)^{(n-2)/n}. \tag{3.2}$$

Moreover, the constant solution of (2.10) *corresponds to the Riemannian Product $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$.*

Proof. From [**3**], we have $\nabla_{e_n} e_n = 0$. Hence, we know that any integral curve of the principal vector field corresponding to μ is a geodesic. Then we can deduce that $w(s)$ is a function defined in $(-\infty, +\infty)$ since M is complete and any integral curve of the principal vector field corresponding to μ is a geodesic.

The left hand side of equation (2.10) multiplied by $2\frac{dw}{ds}$ is precisely the derivative of the left hand side of equation (3.1). Let $w(s) = w_0$ in (2.10) and (3.1), so that $w_0^n = \frac{n-2}{2r}$ and the corresponding value of the constant C is C_0. Combining these with $w = [\lambda^2 - (r-1)]^{-1/n}$ and (2.9), we obtain $\lambda^2 = \frac{n(r-1)+2}{n-2}$ and $\mu^2 = \frac{n-2}{n(r-1)+2}$. Hence we get from Cartan's result in [**2**] that the constant solution of (2.10) corresponds to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$ and $\lambda^2 = \frac{n(r-1)+2}{n-2} = \frac{a^2}{1-a^2}$. That is, the constant solution of (2.10) corresponds to $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$. This completes the proof of Lemma 3.1.

Proof of Theorem 1.3. We assume that M has two distinct principal curvatures λ (multiplicity m) and μ (multiplicity $n-m$).

Case 1. $2 \le m \le n-2$.

By Proposition 2.1, we have M is isometric to the Riemannian product $S^m(a) \times S^{n-m}(\sqrt{1-a^2})$. From (2.7), we can get

$$R_{aa} = (n-1) + (m-1)\lambda^2 + (n-m)\lambda\mu = (m-1)(1+\lambda^2) \ge 0,$$

$$R_{\alpha\alpha} = (n-1) + m\lambda\mu + (n-m-1)\mu^2 = (n-m-1)\mu^2 \ge 0.$$

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Hence the Riemannian product $S^m(a) \times S^{n-m}(\sqrt{1-a^2})$ $(2 \le m \le n-2, 0 < a < 1)$ has nonnegative Ricci curvature.

Case 2. $m = n - 1$.
From (2.9), we have

$$1 + \lambda\mu = 1 + \frac{n(r-1)}{2} - \frac{n-2}{2}\lambda^2. \tag{3.3}$$

Since M has nonnegative Ricci curvature, by using (2.7), we obtain

$$R_{\alpha\alpha} = (n-1)(1+\lambda\mu) \ge 0, \tag{3.4}$$

that is

$$1 + \lambda\mu \ge 0. \tag{3.5}$$

Combining (3.5) with (3.3), we see that

$$\frac{n-2}{2}[\lambda^2 - (r-1)] \le r,$$

and it follows that

$$\frac{n-2}{2}\frac{1}{w^n} - r \le 0. \tag{3.6}$$

From (2.10), we know that

$$\frac{d^2w}{ds^2} = w\left\{\frac{(n-2)}{2}\frac{1}{w^n} - r\right\}.$$

A direct calculation then gives

$$\frac{d^2w}{ds^2} \le 0. \tag{3.7}$$

Thus $\frac{dw}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, $w(s)$ must be monotonic when s tends to infinity.

We see from (3.1) that the positive function $w(s)$ is bounded. Since $w(s)$ is bounded and is monotonic when s tends to infinity, we find that both $\lim_{s\to-\infty} w(s)$ and $\lim_{s\to+\infty} w(s)$ exist and then we have

$$\lim_{s\to-\infty}\frac{dw(s)}{ds} = \lim_{s\to+\infty}\frac{dw(s)}{ds} = 0. \tag{3.8}$$

By the monotonicity of $\frac{dw}{ds}$, we see that $\frac{dw}{ds} \equiv 0$ and $w(s)$ is a constant. Then, according to Lemma 3.1, it is easily seen that M is isometric to the Riemannian product $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$. This completes the proof of our Theorem 1.3. □

Proof of Theorem 1.4. Combining (3.3) with $\lambda\mu \le -1$, we deduce that

$$\frac{n-2}{2}[\lambda^2 - (r-1)] \ge r,$$

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and it follows that

$$\frac{n-2}{2}\frac{1}{w^n} - r \geq 0. \tag{3.9}$$

From (2.10) and (3.9), we have

$$\frac{d^2w}{ds^2} \geq 0. \tag{3.10}$$

We see from (3.1) that the positive function $w(s)$ is bounded. Combining $\frac{d^2w}{ds^2} \leq 0$ with the boundedness of $w(s)$, we see that $w(s)$ is a constant. Then, according to Lemma 3.1, it is easily seen that M is isometric to the Riemannian product $S^1(\sqrt{\frac{nr+2-n}{nr}}) \times S^{n-1}(\sqrt{\frac{n-2}{nr}})$. From this we complete the proof of our Theorem 1.4. □

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REFERENCES

1. Luis J. Alias, Sebastião C. de Almeida and Aldir Brasil Jr., Hypersurfaces with constant mean curvature and two principal curvatures in $\mathbb{S}^{n+1}$, *An. Acad. Brasil. Ciênc.* **76** (2004), 489–497.
2. E.Cartan, Familles de surfaces isoparametriques dans les espaces a courvure constante, *Ann. of Math.* **17** (1938), 177–191.
3. Q. M. Cheng, Hypersurfaces in a unit sphere $S^{n+1}(1)$ with constant scalar curvature, *J. London Math. Soc. (2)* **64** (2002), 755–768.
4. S. Y. Cheng and S. T. Yau, Hypersurfaces with constant scalar curvature, *Math. Ann.* **225** (1977), 195–204.
5. S. S. Chern, M. do Carmo and S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, *Functional analysis and related fields* (Proc. Conf. for M. Stone, Univ. of Chicago, Chicago 1968), (Spinger-Verlag, 1970), 59–75.
6. H. B. Lawson, Local rigidity theorems for minimal hypersurfaces, *Ann. of Math.* **89** (1969), 179–185.
7. H. Li, Hypersurfaces with constant scalar curvature in space forms, *Math. Ann.* **305** (1996), 665–672.
8. H. Li, Global rigidity theorems of hypersurface, *Ark. Mat*. **35** (1997), 327–351.
9. T. Otsuki, Minimal hypersurfaces in a Riemannian manifold of constant curvature, *Amer. J. Math.* **92** (1970), 145–173.
10. Y. J. Suh and H. Y. Yang, Kaehler submanifolds with Ricci curvature bounded from below, *Michigan Math. J.* **53** (2005), 545–552.
11. Q. L. Wang and C. Y. Xia, Rigidity theorems for closed hypersurfaces in space forms, *Quart. J. Math. Oxford Ser. (2)* **56** (2005), 101–110.
12. G. Wei, Complete hypersurfaces with constant mean curvature in a unit sphere, *Monatsh. Math.* **149** (2006), 251–258.
13. G. Wei, Rigidity theorems for hypersurfaces in a unit sphere, *Monatsh. Math.* **149** (2006), 343–350.