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ARTICLE

How to be absolutely fair Part I: The Fairness formula

Stefan Wintein and Conrad Heilmann

Erasmus Institute for Philosophy and Economics (EIPE), Erasmus School of Philosophy (ESPhil), Erasmus University, Rotterdam, The Netherlands
Corresponding author: Stefan Wintein; Email: wintein@esphil.eur.nl

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Abstract
We present the first comprehensive theory of fairness that conceives of fairness as having two dimensions: a comparative and an absolute one. The comparative dimension of fairness has traditionally been the main interest of Broomean fairness theories. It has been analysed as satisfying competing individual claims in proportion to their respective strengths. And yet, many key contributors to Broomean fairness agree that 'absolute' fairness is important as well. We make this concern precise by introducing the *Fairness formula* and the *absolute priority rule* and analyse their implications for comparative fairness.

Keywords: Absolute fairness; comparative fairness; claims; proportionality; John Broome

1. Introduction

John Broome (1984, 1988, 1990) has developed an influential theory of fairness, which has generated a thriving debate about the nature of fairness. Although Broome's key contribution dates back a few decades, the Broomean fairness literature has taken off more recently than that.[1]

Broome's theory applies when some good has to be divided amongst people who have *claims* to the good and is compactly described by the following formula:

Broomean formula. Fairness requires that claims should be satisfied in proportion to their strength.

Claims are a specific type of reason as to why a person should receive a good. They are 'duties owed to the person herself', as Broome puts it. Agents may have claims to a divisible good, such as in Owing Money:

[1]See Hooker (2005), Saunders (2010), Tomlin (2012), Curtis (2014), Lazenby (2014), Henning (2015), Kirkpatrick and Eastwood (2015), Paseau and Saunders (2015), Sharadin (2016), Heilmann and Wintein (2017), Piller (2017), Vong (2015, 2018, 2020), Wintein and Heilmann (2018, 2020, 2021).

Owing Money. Romeo owes 20 to Abram and 60 to Benvolio but has only 40 left. How, in order to be fair, should Romeo divide the 40?

Intuitively, it is fair that Romeo gives 10 to Abram and 30 to Benvolio. Now, although the allocation $(10, 30)$ is fair according to Broome's theory,[2] it is not the *only* such allocation. As fairness is, according to Broome, strictly comparative, *any* allocation in which Benvolio receives three times as much as Abram is perfectly fair. For instance, $(5, 15)$ is just as fair as $(10, 30)$. It may very well be that, *all things considered*, Romeo should exhaust the good and realize $(10, 30)$. Doing so, however, is not required by *fairness*. Or so one must accept, were one to base fairness judgements exclusively on the *Broomean formula*.

However, many contributors to the Broomean fairness literature have expressed that a concern for exhausting the good is a natural requirement. First, Broome himself addresses the concern, albeit outside of his fairness theory: claims should be satisfied, but not as a matter of fairness. That is, fairness only requires the *proportional* satisfaction of claims, not their satisfaction as such. As Piller (2017: 218) puts it, 'Claims ought to be satisfied. This requirement belongs to Broome's general moral theory; it is not part of his theory of fairness'. That is the first perspective on the concern for good-exhaustion: to accommodate these concerns *outside* of the theory of fairness.

Second, the concern for exhausting the good can also be treated as a matter of fairness itself: on this second perspective, fairness is not strictly comparative but also has an *absolute* dimension. It is this latter, absolute, dimension which grounds the good-exhaustion concern as a fairness concern. Key contributors advocate for this position (e.g. Hooker 2005; Saunders 2010; Lazenby 2014; Curtis 2014; Vong 2018). For instance, Hooker (2005: 341) remarks that 'fairness requires the greatest possible proportionate satisfaction of claims'. Others, such as Saunders (2010: 47), who refers to exhausting the good to be distributed as 'efficiency', and Lazenby (2014: 332f.) mention the absolute fairness issue in passing. Finally, Vong (2018: 74) devotes his analysis to an absolute notion of fairness – which he calls 'individual' fairness. On his account, it is a matter of fairness to the agent that their claims are satisfied. However, neither Vong nor any of the other contributors analyse the relations between comparative and absolute fairness in any depth.

In this article, we also take the second perspective: fairness itself has an absolute dimension.[3] More importantly, we fully work out how the concern for absolute fairness can be accommodated within a comparative Broomean theory of fairness. In short, we present a theory of absolute and comparative fairness. We develop the theory in three steps. In a first step, we give a sophisticated analysis of the notion of 'claims' in Broomean theories of fairness, by introducing the notions of claim

[2]For, as discussed in detail below, the claims of Abram and Benvolio are equally *strong* so that fairness, according to Broome's theory, requires that these claims receive equal satisfaction.

[3]In this article, we assume (rather than argue for) that fairness itself has an absolute dimension. As discussed, this position is prevalent in the recent Broomean fairness literature, yet has not been analysed. Our article thus contributes by working out the implications of this widely endorsed assumption. As we will go on to demonstrate in detail in our article 'How to be absolutely fair, Part II: philosophy meets economics', the relevance of absolute fairness also carries over to economics. In particular, we show how the concern for absolute fairness can motivate the routine stipulation of *efficient* division rules in the economic literature.

strength and claim *amount*, of *notional* and *absolute* claims, and of *individual* and *group* claims. Providing clear definitions of these notions is also of independent merit: they make precise conceptual distinctions which have been implicitly present in the Broomean fairness literature. In a second step, we introduce a 'fairness formula' that uses the aforementioned claim notions to analyse the comparative and absolute dimension of fairness, and – importantly – how these interact. Our fairness theory can be summarized with this slogan.

Fairness formula (FF). Fairness requires one: (i) to satisfy absolute claims (of individuals and groups) to as large an extent as possible, subject to the constraint that no one receives more than they have a claim to; (ii) to satisfy (absolute and notional) individual claims in proportion to their strength; (iii) to prioritize requirement (i) over (ii) whenever these two conflict, but in such a way that one does as much as possible to respect (ii).

In a third step, we develop the implications of the Fairness formula and, in particular, we will study how to handle cases in which there is a conflict between satisfying the requirements of (i) absolute and (ii) comparative fairness. We show that the prima facie plausible *weighted proportional rule* cannot handle such cases and introduce and justify the *absolute priority rule* that can provide a definite answer for them.[4]

Our new theory capitalizes on and extends three main advantages of Broomean fairness theories. First, the Broomean fairness literature does something unique: it focuses on analysing the concept of fairness *itself* and for its own sake (as opposed to formulate a concept of fairness as a means to theorize about something else). Here, our extension enhances the conceptual reach of the Broomean fairness literature. It makes precise the popular 'absolute' fairness concern and shows how it interacts with comparative fairness. Our fairness theory is thus capable of a more comprehensive conceptual analysis of fairness. Second, the focus on fair division problems such as Owing Money in the Broomean fairness literature gives it great potential for practical application. Our clarification of the claims notion, and in particular our development of the notions of individual and group claims, significantly extends the practical reach of Broomean fairness theories. As we will show, our theory allows us to analyse a greater variety of fairness cases. This, in turn, increases the practical relevance of Broomean fairness theories. Third, Broomean fairness theories have interdisciplinary potential, owing to the close kinship with the 'bankruptcy' literature in economic theory (e.g. Aumann and Maschler 1985; Thomson 2003, 2019). However, this close kinship has hitherto been almost ignored by the Broomean fairness literature in philosophy. Partly, this is due to the mathematical nature of the economic theories. Partly, this is due to subtly differing assumptions in both fields. Our theory, owing to the claims distinctions we will

[4]One restriction of the present analysis is that we will only study and apply the Fairness formula with respect to problems with 'divisible good', such as in Owing Money. We have dealt with the allocation of 'indivisible goods' (such as horses, or seats in a parliament) from a Broomean perspective in earlier work (Wintein and Heilmann 2018). The Fairness formula can in principle be applied to such indivisible cases, and we will take up this issue in future work.

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introduce, allows us to easily translate between both literatures. We will also introduce a general framework of 'modelling' fair division problems that captures how and where different fairness theories are similar and where they differ. Our contribution thus opens up the potential for Broomean fairness informing economic fairness theories and vice versa. In the article 'How to be absolutely fair, part II: philosophy meets economics', we provide a detailed demonstration of this latter advantage of building bridges between Broomean fairness theories in philosophy and the 'bankruptcy' fairness literature in economics.[5]

We proceed as follows. Section 2 formulates a notion of claims fit for absolute and comparative fairness, by introducing the notions of claim strengths and claim amounts, of notional and absolute claims, and of individual and group claims. Section 3 presents the fairness formula which accommodates absolute fairness as a matter of priority over comparative fairness. It also shows how to generally model and analyse fair division problems, and justifies the absolute priority rule. Section 4 discusses the two-dimensional nature of fairness, and in particular how the comparative and absolute dimension interact. It also illustrates the advantages of our theory over other accounts in the literature. Section 5 concludes.

2. Claims for Absolute and Comparative Fairness

We concur with Piller (2017) that one can understand the notion of a (Broomean) claim as a 'duty owed to the agent' intuitively, and we will by and large rely on this understanding.[6] We will not present an account of, for instance, the sources of claims or of how claims work in moral deliberation.[7] We will, however, introduce and discuss various notions that are associated with this intuitive understanding of the notion of a claim. First, we make explicit the notion of a claim 'amount' that is implicit in Broome's account. Second, we introduce a distinction between notional and absolute claims. Third, we extend the Broomean claims to cover both individuals and groups. All three precisifications of the Broomean claim concept make it fit to deal with absolute and comparative fairness. Finally, we make explicit the notion of claim satisfaction that we adopt in this article and make precise: (a) what it means to say that one allocation satisfies claims to a larger extent than another, and (b) what it means to say that an allocation satisfies claims in proportion to their strength.

2.1. Claim strength and claim amounts

According to the Broomean formula, claims should be satisfied in proportion to their *strength*. Besides a strength, each claim has an *amount*. Although the

[5]Our article thus focuses on improving the internal coherence and the capacities of Broomean fairness theories, and we also join forces with the economic literature. The question of how our fairness theory finds its place in the broader moral landscape, in particular how it relates the value of fairness to other moral values, we will take up elsewhere.

[6]Broome contrasts claims with *teleological reasons* and *side-constraints* but does not offer a detailed account of the nature of claims. Hence, in this sense his theory of fairness is incomplete. However, as Piller (2017: 216) observes, 'this incompleteness might not matter … because we understand talk of claims pre-theoretically'.

[7]See Hooker (2005) and Kirkpatrick and Eastwood (2015) for discussing these aspects.

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Broomean formula does not explicitly mention the amount of a claim, the notion is presupposed by the notion of *satisfaction*, which *is* explicitly mentioned by the formula. For the extent to which a claim is satisfied depends on the good received and the amount of the claim: amounts are needed to determine satisfaction.

As an example, consider Owing Money (cf. section 1). Romeo is the *distributor* in Owing Money. He owes it to Abram to repay his debt: Abram has a claim to be reimbursed by Romeo. As it takes 20 to reimburse Abram, the claim of Abram has an amount of 20. If Abram receives 10, his claim to reimbursement is satisfied for $\frac{10}{20} \cdot 100\% = 50\%$.

Besides an amount, each claim has a *strength*. The *strength* of a claim specifies how strong the reason is, *as compared with the reasons for satisfying the claims of other agents*, for satisfying that particular claim. That is, claim strength is a *strictly comparative* notion. To illustrate this, consider Owing Money once more. Romeo has just as much reason to reimburse Abram as he has to reimburse Benvolio: both have been promised to be paid back. That is, the claims of Abram and Benvolio – to get reimbursed by Romeo – are equally strong. Thus, Abram and Benvolio have equally *strong* claims, with *amounts* of 20 and 60 respectively.

In Owing Money the claims are equally strong. But claims are not always equally strong, as the next example illustrates.

Investing Time. Anna and Beta have invested time in realizing a joint project. For a certain period of time, Anna has spent one day a week on the project, whereas Beta has spent three days a week on it. After some time, the value of their project is 20. Anna and Beta split apart and their fiduciary, Rachel, is responsible for the division of the 20. How, in order to be fair, should Rachel divide the 20?

As Anna has contributed to realizing the joint project, Rachel the distributor in Investing Time, owes it to Anna to give her a share of its value. That is, Anna has a claim to (a share) of the project's value. Similarly, Beta has a claim to (a share of) the project's value. As the value of their joint project is 20, we say that the amount of the respective claims of Anna and Beta is 20. So Anna and Beta have claims with equal amounts but clearly, Beta's claim is stronger. Indeed, (all else being equal) it is natural to say that Beta's claim is three times as strong as Anna's claim, in virtue of Beta's time commitment that is three times as large. We will compactly summarize the claims-lesson of this subsection in a bullet-point and we will continue this practice in further subsections.

- A claim has both an *amount* and *strength*.

This seemingly simple observation is not only important, but ignoring or not fully analysing it has the potential to hamper fairness discussions, as we will illustrate in section 4 of this article.

2.2. Notional and absolute claims

Investing Time illustrates that claims may come in different strengths. But, when compared with Owing Money, it also illustrates a further important distinction: that between *absolute* and *notional* claims.

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To illustrate the difference between what we will call *absolute claims* and *notional claims*, compare the following true statements about (1) Owing Money and (2) Investing Time respectively:

(1) Abram has a claim to 20 that is equally strong as Benvolio's claim to 60.
(2) Anna has a claim to 20 that is $\frac{1}{3}$ times as strong as Beta's claim to 20.

Whereas Abram and Anna both have claims with an amount of 20, these claims are claims of different *types*. To see this, let's discuss the claims of Abram and Anna in turn. Abram's claim to be reimbursed by Romeo is not determined by comparisons with any other claimant's claims or how others are treated. We call Abram's claim *absolute*. As Hooker (2005) puts it:

> We might say that what Broome's theory demands is that I treat each of [the agents] fairly in comparison with how I treat the other of them. This is comparative fairness. But if individual or absolute fairness towards anyone is a matter of whether I give him or her the response owed him or her, never mind how others have been treated, then I have not treated either fairly. (Hooker 2005: 340)

Gerard Vong (2018) agrees and proposes to distinguish between different types of claims:

> [Absolute][8] claims are a subset of all claims, namely those claims that are determined without comparison with the treatment or claims of one or more others. (Vong 2018: 68)

Absolute claims require full satisfaction. When they are not completely satisfied, the claimant does not get all they should get. For instance, when Romeo realizes allocation $(10, 30)$ he fulfils the requirements of the Broomean formula. And one might say that he has done all for promoting fairness that he was able to, given what Owing Money describes about the situation at hand: all he had left was 40. However, as Abram's absolute claim does not receive full satisfaction, something is wrong. What is wrong is that it is non-comparatively, i.e. *absolutely*, unfair to Abram that he remains 10 short. Absolute claims are, as a subset of all claims, duties owed to the agent. In particular:

- Agent A has an *absolute claim with an amount* a if the distributor owes it to A to allot her all of a, irrespective of what claims other agents may have.

Absolute claims are a subset of all claims. Any claim that is not absolute we call a *notional* claim. In Investing Time, Anna contributed to the realization of the joint project, in virtue of which she has a claim to its value of 20. Anna's claim is not absolute and hence notional. To see this, note that when Anna does not get the full 20 it does *not* follow that something is wrong. In particular, nothing is wrong with Anna receiving 5 which, intuitively, is her 'fair share'.

[8]Vong uses 'individual' for what we label 'absolute'. Since our theory also covers fairness for groups, we reserve the label 'individual' for clearly distinguishing between individual and group fairness.

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So, Abram has an absolute claim to 20: he should get all of the 20, irrespective of which other claimants are around and irrespective of how much money Romeo has left. Anna's claim to 20 is notional: she should get a part of the 20, a part which is determined by comparing her claim with that of others. More generally:

- Agent A has a *notional claim with an amount a* if the distributor owes it to A to allot her a part of a, a part which is determined by comparing her claim with that of others.

So whether an agent has an absolute or notional claim with amount a, the distributor never has the duty to allot more than a to the agent. Our distinction between notional and absolute claims is similar to Vong (2018) and others (e.g. Hooker 2005; Saunders 2010; Lazenby 2014; Curtis 2014). Indeed, it records, in precise fashion, a long-standing concern for analysing absolute fairness that is present in the Broomean fairness literature.

Next, we turn to the distinction between *individual* and *group* claims.

2.3. Individual and group claims

In Investing Time, the *individuals* Anna and Beta both have notional claims, with an amount of 20, grounded in the joint realization of their project.

However, Investing Time also involves an absolute claim. For, as Anna and Beta *jointly* realized the value of 20, Rachel owes it to *Anna and Beta together* to allot the 20, and all of the 20, to *them*. That is, the *group* consisting of Anna and Beta has a claim to 20 that requires full satisfaction: the *group* consisting of Anna and Beta has an absolute claim with an amount of 20.

We concur with Broome that a claim is 'a duty owed to the agent' herself, as is also apparent from our account of absolute and notional claims. But we add that it is important to distinguish between individual and group claims. *Individual claims* are duties owed to individual (receiving) agents. *Group claims* are duties owed to groups of individual (receiving) agents. As we will see later, this extension will significantly increase the conceptual reach and interdisciplinary potential of Broomean fairness.

- A claim is a duty owed to an *agent*, which can either be an (receiving) *individual* or a *group* of (receiving) individuals.

Next we discuss a few salient notions related to *claim satisfaction*.

2.4. Claim satisfaction: constrained

In Owing Money, Abram has a claim with an amount of 20. Suppose that Abram is allotted 30 by Romeo. Thus, Abram's claim to 20 is fully satisfied and he receives 'a gift on top of that' equal to 10. As a receipt of 20 *fully* satisfies his claim, we say that, although 30 amounts to 150% of Abram's claim amount, by receiving 30 the *satisfaction* of his claim of 20 is (only) 100%. More generally, when an agent has a claim with an amount a, that claim is *fully* satisfied when the agent receives a (or

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more). No doubt, Abram may *prefer* to receive more than the amount of his claim. But whereas receipts that exceed 20 may satisfy Abram's *preferences*, they do not count towards the satisfaction of his *claim*. That is, we take it that, in virtue of the very meaning of *claim* satisfaction, claim satisfaction is constrained:

- *The satisfaction of a claim is constrained by its amount a* : it has a maximum of 100%, realized by receiving any amount that is greater-than or-equal-to a.

When Abram receives 10, in Owing Money, his claim of 20 receives $\frac{10}{20} \cdot 100\% = 50\%$ satisfaction. For short, we write: $Sat(10,\ 20) = 50\%$. Likewise when, in Investing Time, Anna and Beta receive 10, their claims of 20 receive 50% satisfaction but in addition, the *group* consisting of Anna and Beta together receives $10 + 10 = 20$ so that its (absolute) group claim of 20 receives $\frac{10+10}{20} \cdot 100\% = 100\%$ satisfaction, or $Sat(20, 20) = 100\%$.[9]

2.5. Claim satisfaction: comparing allocations

When Abram is allotted 10 in Owing Money, his claim with an amount of 20 is satisfied to a larger extent than when he is allotted only 5 : indeed, Sat $(10,\ 20)$ is strictly greater than Sat $(5,\ 20)$. What we are interested in, however, is comparing *allocations* in terms of the extent to which they satisfy claims: does allocation $(10,\ 15)$, in which Abram receives 10 and Benvolio receives 15, satisfy claims to a larger extent than allocation $(9,\ 15)$? In order to answer that question and, more generally, to compare allocations in terms of the extent to which they satisfy claims, we will rely on the following criteria for individual and groups claims respectively.

- Allocation x satisfies the claims of the individuals to a larger extent than allocation y when the satisfaction afforded by x to *each* of the individuals is greater than or equal to that afforded by y and strictly greater for *at least one* individual.[10]
- Allocation x satisfies the claim of a group of individuals to a larger extent than y when the sum-total allotted to these individuals by x satisfies their group claim to a larger extent than the sum-total allotted to them by y.[11]

[9]In both Owing Money and Investing Time, claim satisfaction is *linear*. While in this article, we *only* consider fair division problems in which claim satisfaction is linear, we do not commit to the view that claim satisfaction is linear *tout court*, i.e. that claim satisfaction is linear in all fair division problems. To see why not, suppose that Luc needs 20 milligrams of a medicine in order to avoid that he becomes paralysed. Also, suppose that any intake of medicine less than 20 milligrams is fully ineffective. We owe it to Luc to save him from becoming paralysed so that Luc has a claim to be saved, for which we need 20 milligrams of medicine. Luc's (absolute) claim then, is conveniently represented as a claim with an amount of 20. But when Luc receives 10 milligrams, his claim to be saved is not satisfied at all so that, on the proposed representation of this situation, claim satisfaction is not linear. For more discussion of such cases, see also Hausman (2023: 134ff.).

[10]That is: $Sat(x_i, a_i) \geq Sat(y_i, a_i)$ for all i and $Sat(x_i, a_i) > Sat(y_i, a_i)$ for some i in N.

[11]So when group N has a claim with amount E the condition is that $Sat\left(\sum_{i \in N} x_i, E\right) > Sat\left(\sum_{i \in N} y_i, E\right)$.

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Note that our criteria of comparison is rather minimal: we compare allocations by comparing claim satisfaction in an ordinal and component-wise manner. For sure, in $(10,\ 20)$ for Owing Money, claims are satisfied to a larger extent than in $(9,\ 20)$. And, as satisfaction is constrained, in $(20,\ 20)$ claims are satisfied to a larger extent than in $(25,\ 15)$: in both allocations Abram's claim receives 100% satisfaction, whereas Benvolio's claim receives more satisfaction in the first allocation than in the latter. For Investing Time, the absolute claim of the group of Anna and Beta is satisfied to a larger extent by one allocation than another just in case the total sum that is allotted by the former is larger than that of the latter.

For quite a few allocations, though, our criteria do not declare one of them as satisfying claims to a larger extent than the other. For example, in $(5,\ 2)$ Abram's claim is satisfied to a larger extent than in $(4,\ 30)$, but for Benvolio it is just the other way around. Hence, in $(5,\ 2)$ claims are not satisfied to a larger extent than in $(4,\ 30)$ but neither are claims satisfied to a larger extent in $(4,\ 30)$ than in $(5,\ 2)$. Should we then say that, in $(5,\ 2)$ and $(4,\ 30)$, claims are satisfied to the same extent? Or should we say that $(5,\ 2)$ and $(4,\ 30)$ are incomparable with respect to the extent to which they satisfy claims? For the purposes of this article, we need not and will not answer this question: it will suffice to compare allocations by the extent to which they satisfy claims on the basis of the above, ordinal, criteria only. In particular, our criteria suffice to specify the notion of an allocation which *satisfies claims to as large an extent as possible.*

- Allocation x *satisfies claims to as large an extent as possible* just in case there is no allocation y available which satisfies claims to a larger extent than x does.

For Owing Money, e.g. allocations $(0,\ 40)$, $(10,\ 30)$ and $(20,\ 20)$ all satisfy the absolute individual claims to as large an extent as possible. But allocation $(25,\ 15)$ does not as e.g. allocation $(20,\ 20)$ satisfies claims to a larger extent than $(25,\ 15)$ does. For Investing Time, any allocation that allots 20 in total satisfies the absolute group claim to as large an extent as possible.

2.6. Claim satisfaction: proportional to strength

The last notion that we discuss is that of an *allocation in which individual claims are satisfied in proportion to their strength.* For Owing Money, individual claims are satisfied in proportion to their strength in e.g. allocation $(10,\ 30)$. For, in this allocation, the equally strong claims of Abram and Benvolio receive equal satisfaction (both 50%). But also in $(5,\ 15)$ are claims satisfied in proportion to their strength: for here, the equally strong claims also receive equal satisfaction (both 25%). For Investing Time, individual claims are satisfied in proportion to their strength in e.g. $(5,\ 15)$ or $(1,\ 5)$: in these allocations, Beta's claim, which is three times as strong as Anna's claim, receives three times as much satisfaction. More generally, we say that:

- In allocation x *individual claims are satisfied in proportion to their strength* just in case, for any two individual agents i and j, the following holds. If i's claim is σ times as strong as j's claim then, in allocation x, the satisfaction of i's claim is σ times the satisfaction of j's claim, i.e. $\mathrm{Sat}(x_i, a_i) = \sigma \cdot \mathrm{Sat}\left(x_j, a_j\right)$.

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We have put forward a criterion which specifies *whether or not* an allocation satisfies claims in proportion to their strength. For the purposes of this article, we need not and will not define a measure which allows us to compare allocations in terms of the extent to which they satisfy claims in proportion to their strength.

3. Modelling Fairness

We now turn to the key elements of our fairness theory. We first present some general desiderata for fairness theories put forward in Wintein and Heilmann (2020). Building on this approach, we explain what is implicitly involved in any effort to analyse fair division problems and make recommendations for their resolution. We then make the notion of a 'Broomean problem' precise and introduce the Fairness formula. Thereafter, we introduce the weighted proportional rule and the absolute priority rule. Only the latter rule realizes the Fairness formula in all fair division cases, in particular in those in which the requirements of comparative and absolute fairness conflict.

3.1. Modelling fair division: fairness structures and functions

Consider the fair division problems Owing Money and Investing Time from the previous sections. Their description has been purely verbal. For analysing them, it is important to make explicit how the information in these descriptions is understood. This step in the analysis of fair division problems we call *modelling* a fair division problem. Importantly, this step is implicit in *all and any* fairness analysis. The following discussion makes precise how the description of a given fair division problem is used in the fairness analysis.

In a nutshell, to solve a fair division problem is to (1) model (represent) the problem as a *fairness structure* and to (2) recommend an allocation for the problem by applying *a fairness function* to that structure. The two-stage process of solving a fair division problem is summarized by Figure 1.

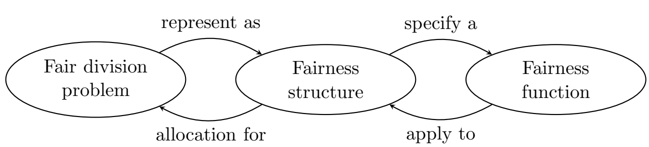

Figure 1. Solving a fair division problem.

A fairness structure is a formal structure which is interpreted in terms of theoretical fairness notions and on the basis of which, by applying a *fairness function* to it, fair divisions are obtained:

> By a [*fairness function*], we mean a function that assigns an allocation of the good-to-be-divided for each *fairness structure* that is within its domain. A fairness structure is obtained by *modelling* a fair division problem, that is by extracting the characteristics of the problem on the basis of which, according to the model, fair division should proceed. (Wintein and Heilmann 2020: 722)

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The fairness structure we use in this article are what we call *Broomean problems*, introduced in the next section.[12] The fairness functions associated with Broomean problems we call *division rules*. Many division rules exist but we will motivate the use of one specific division rule.

3.2. Broomean problems

We introduce *Broomean problems*, which are formal representations of fair division problems. A Broomean problem is a structure

$$\mathcal{B} = (E, N, a, s),$$

where the estate $E > 0$ specifies the amount of the good-to-be-divided amongst the individuals in $N = \{1, \ldots n\}$. An individual $i \in N$ has a claim (a_i, s_i) with amount $a_i \geq 0$ and strength $s_i > 0$, as specified by amounts-vector a and strengths-vector s.[13] Fair division problems Owing Money and Investing Time can be represented as Broomean problems, augmented with information about the type of claims involved, as follows.[14]

Table 1. $\mathcal{O}$wing Money, $\mathcal{I}$nvesting Time and their representations $\mathcal{O}$ and $\mathcal{I}$

Broomean problem	Individuals claims	Group claim
$\mathcal{O} = \left(40, \{A, B\}, (20,\ 60), \left(\frac{1}{2}, \frac{1}{2}\right)\right)$	absolute	none
$\mathcal{I} = \left(20, \{A, B\}, (20,\ 20), \left(\frac{1}{4}, \frac{3}{4}\right)\right)$	notional	absolute

The two representations $\mathcal{O}$ and $\mathcal{I}$ record the estate (good to be divided), the individuals involved, the respective amounts they have a claim to, and the respective strength of their claims.[15] For simple fair division problems like Owing Money and Investing Time, it is straightforward (and uncontroversial) how to represent them as Broomean problems. However, as we will see, some fair division problems may be plausibly represented by more than one Broomean problem. And so, what fairness requires in a given fair division problem crucially depends on its representation as a fairness structure.

[12]Other examples of fairness structures include apportionment problems (cf. Balinski and Young 2001; Wintein and Heilmann 2018), cooperative games (cf. Aumann and Maschler 1985; Wintein and Heilmann 2020) and weighted bankruptcy problems (cf. Casas-Méndez *et al.* 2011). Weighted bankruptcy problems will be discussed in detail in the article 'How to be absolutely fair, Part II: philosophy meets economics'.

[13]As claim-strengths are strictly comparative, they are only determined up to an arbitrary positive multiplicative constant ρ : if $s = \rho \cdot s'$ then vectors s and s' determine the same claim-strengths. However, for ease of exposition and analysis we will normalize claim strengths and assume that $\sum_{i \in N} s_i = 1$.

[14]Note that a Broomean problem $\mathcal{B} = (E, N, a, s)$ as such does not contain information about whether claims are absolute or notional and about whether or not group claims are involved. For sake of convenience, we have chosen not to represent this information in $\mathcal{B}$ but rather to display it separately, as in Table 1. As such, a Broomean problem captures, strictly speaking, only part of the salient fairness structure.

[15]For Investing Time, the group $\{A, B\}$ has a claim to 20.

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3.3. The Fairness formula

The essence of our theory of absolute and comparative fairness can be summarized by the following Fairness formula, which exploits all the notions that we defined and discussed in the previous section. The Fairness formula accommodates absolute fairness as a matter of priority over comparative fairness.

Fairness formula (FF). Fairness requires one: (i) to satisfy absolute claims (of individuals and groups) to as large an extent as possible, subject to the constraint that no one receives more than they have a claim to; (ii) to satisfy (absolute and notional) individual claims in proportion to their strength; (iii) to prioritize requirement (i) over (ii) whenever these two conflict, but in such a way that one does as much as possible to respect (ii).

The Fairness formula offers guidance for realizing *fair* divisions. How? In a nutshell, by modelling fair division problems as *Broomean problems* and applying the only division rule that is justified by the Fairness formula: the *absolute priority rule*. The absolute priority rule is a refinement of the so-called *weighted proportional rule* and it is instructive to consider the latter first, before introducing the *absolute priority rule*.

3.4. The weighted proportional rule

In Owing Money and Investing Time, there is not enough to go around: the sum of claim amounts exceeds the estate. Although that is typically the case in a fair division problem, it will be instructive to discuss, in section 4, cases of *abundant good*, i.e. cases in which the estate exceeds the sum of claims. That is why it is helpful to introduce the notion of a so-called *truncated estate* $\overline{E}$, which cannot be larger than the sum of all claim amounts. Formally, $\overline{E}$ is equal to either the estate E or the sum of claim amounts, whichever is smaller. So for cases such as Owing Money and Investing Time, where there is not enough to go around, the truncated estate $\overline{E}$ equals the estate E.

The weighted proportional rule P proposes to divide the truncated estate in a Broomean problem $\mathcal{B}$ proportional to the *strength-weighted claim amounts* of the individuals, so that each individual i receives:

$$P(\mathcal{B})_i = \frac{s_i \cdot a_i}{\sum_{j \in N} s_j \cdot a_j} \cdot \overline{E} \tag{1}$$

For $\mathcal{O}$wing Money and $\mathcal{I}$nvesting Time, the recommendations of the weighted priority rule, $P(\mathcal{O}) = (10,\ 30)$ and $P(\mathcal{I}) = (5,\ 15)$, are consistent with the Fairness formula, as we will explain below.

For Owing Money, the Fairness formula demands the realization of allocation $(10, 30)$. For in $(10,\ 30)$, the equally strong (individual) claims of Abram and Benvolio receive equal satisfaction (of 50% each) so that realizing this allocation meets the requirements of comparative fairness. But also, in $(10,\ 30)$ no individual receives more than his claim amount and there is no other allocation of the estate of 40 in which the absolute claims of Abram and Benvolio are satisfied to a larger extent than in $(10,\ 30)$. Hence, by realizing $(10,\ 30)$ Romeo meets the requirements of absolute fairness as well. So, by realizing $(10,\ 30)$ Romeo lives up to the requirements of both absolute and comparative fairness.

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For Investing Time, the Fairness formula demands the realization of allocation $(5, 15)$, in which Beta's (individual) claim, which is three times as strong as that of Anna, receives three times as much satisfaction: realizing (5, 15) meets the requirements of comparative fairness. But also, in (5, 15) no individual receives more than their claim amount and there is no other allocation of the estate of 20 in which the absolute claim of the group consisting of Anna and Beta together is satisfied to a larger extent than in (5, 15). Hence, by realizing (5, 15) Rachel meets the requirements of absolute fairness as well. So, by realizing (5, 15), Rachel lives up to the requirements of both absolute and comparative fairness.

The recommendation of the weighted proportional rule, $P(\mathcal{B})$, does not depend on whether the (individual or group) claims in the $\mathcal{B}$ roomean problem to which P is applied are absolute or notional. However, the *justification* of that recommendation in terms of the Fairness formula *does* depend on whether we are dealing with individual claims that are *notional* or *absolute*, and whether there are *group claims*: for Owing Money, absolute fairness requires that the absolute *individual* claims receive maximal satisfaction whereas for Investing Time, it is the absolute *group* claim that should receive maximal satisfaction as a matter of absolute fairness. More generally, it readily follows that for any Broomean problem with claim types as in Owing Money or Investing Time, absolute fairness requires to allot as much of the (truncated) estate as possible in such a way that no individual receives more than they have a claim to. For Owing Money and Investing Time, this requirement of absolute fairness can be realized conjointly with the requirement of comparative fairness. However, as we will see in section 3.5, the joint realization of both requirements of fairness is not possible for all Broomean problems. The next proposition precisely delineates the Broomean problems for which this is possible.

Proposition 1. For any Broomean problem $\mathcal{B} = (E, N, a, s)$: it is possible to respect the requirements of both (i) absolute and (ii) comparative fairness as articulated by the Fairness formula if and only if $P(\mathcal{B})_i \leq a_i$ for each individual i. In that case, $P(\mathcal{B})$ respects both (i) and (ii).

Proof: See the article ‘How to be absolutely fair, Part II: philosophy meets economics’, section 3.1. □

So, the Fairness formula recommends to realize $P(\mathcal{B})$ whenever $P(\mathcal{B})_i \leq a_i$ for each individual i as for these cases, $P(\mathcal{B})$ respects the requirements of absolute and comparative fairness. However, when $P(\mathcal{B})_i > a_i$ for some individual, it is simply not possible to be both absolutely and comparatively fair. For such cases, the Fairness formula prioritizes absolute fairness. Next, we will illustrate this proposal and explain that it demands that we trade in the weighted proportional rule P for the *absolute priority rule* $P^\dagger$. Indeed, the absolute priority rule captures and operationalizes the content of the Fairness formula.

3.5. The absolute priority rule

Consider the following fair division problem.

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Needing Owed Money. Romeo owes 20 to Abram and 60 to Benvolio and has 80 left. Abram needs his money twice as strongly as Benvolio. Romeo is bound to care for the needs of Abram and Benvolio, such that Romeo's reason for reimbursing Abram is twice as strong as his reason for repaying Benvolio. How, in order to be fair, should Romeo divide the 80?

Needing Owed Money is naturally represented as:

Broomean problem	Individual claims	Group claim
$\mathcal{N} = \left(80, \{A, B\}, (20, 60), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$	absolute	none

The weighted proportional rule P recommends allocation $P(\mathcal{N}) = (32,\ 48)$ for $\mathcal{N}$eeding Owed Money, which is neither absolutely nor comparatively fair. Moreover, it follows from Proposition 1 that there is no allocation for $\mathcal{N}$eeding Owed Money which satisfies the requirements of both absolute and comparative fairness. What to do?

The response to Needing Owed Money seems straightforward: in order to be fair, Romeo just needs to fully reimburse Abram and Benvolio, i.e. he must realize allocation $(20,\ 60)$. We concur that this is, indeed, what fairness requires. But note that this judgement comes down to prioritising absolute fairness over comparative fairness.

For, as Romeo's reason for reimbursing Abram is twice as strong as it is for reimbursing Benvolio, Abram's claim with an amount of 20 is twice as strong as Benvolio's claim with an amount of 60. Comparative fairness, which requires that individual claims are satisfied in proportion to their strength, then requires that Abram's claim receives twice as much satisfaction as the claim of Benvolio. To ensure this, Romeo can realize, for instance, allocation $(20, 30)$ where Abram's claim receives 100% and where Benvolio's claim receives 50% satisfaction. Should Romeo then realize $(20,\ 30)$? No, as it just seems absurd to disrespect Benvolio's absolute claim in order to ensure that the claims of Abram and Benvolio are satisfied in proportion to their strength: fairness should not demand to level down.

In $(20,\ 60)$, the absolute claims of Abram and Benvolio are satisfied to as large an extent as possible and neither Abram nor Benvolio receives more than they have a claim to: realizing $(20,\ 60)$ fulfils the requirements of absolute fairness. Moreover, $(20,\ 60)$ is the *only* allocation which fulfils the requirements of absolute fairness. However, as the claims of Abram and Benvolio receive the same satisfaction in $(20,\ 60)$, their claims are not satisfied in proportion to their strength. Indeed, to say that fairness requires that Romeo realizes $(20,\ 60)$ is to prioritize the absolute dimension of fairness over the comparative one.

Whereas Needing Owed Money illustrates that the Fairness formula prioritizes absolute fairness, the example does not aptly illustrate what it means to do so while one does 'as much as one can to satisfy claims in proportion to their strength'. To illustrate the latter feature, consider:

More Needed Money. Romeo owes 20 to Abram, 60 to Benvolio, 40 to Capulet and has 80 left. Abram needs his money twice as strongly as both Benvolio and Capulet do. How, in order to be fair, should Romeo divide the 80?

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$\mathcal{M}$ore Needed Money is naturally represented as:

Broomean problem	Individual claims	Group claim
$\mathcal{M} = \left(80, \{A, B, C\}, (20,\ 60,\ 40), \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)\right)$	absolute	none

The weighted proportional rule P recommends allocation $(22.86,\ 34.29,\ 22.86)$ for $\mathcal{M}$ore Needed Money, which is neither absolutely nor comparatively fair.

The *absolute priority rule* does better. We first discuss the example and then define the rule. According to the absolute priority rule $P^{\dagger}$, we need to *reimburse* Abram, by allotting him 20, as his weighted proportional share of 22.86 exceeds his claim amount of 20. After reimbursing Abram, the *remaining estate* is 60 which, according to the absolute priority rule, has to be divided amongst the *remaining individuals* Benvolio and Capulet in accordance with the weighted proportional rule. Doing so yields 36 for Benvolio and 24 for Capulet so that $P^{\dagger}$ recommends $(20,\ 36,\ 24)$ for More Needed Money. In this allocation, Abram gets all of his claim reimbursed whereas the claims of Benvolio and Capulet are, as the reader may care to verify, satisfied in proportion to their strength.

So when there is no allocation that satisfies the requirements of both absolute and comparative fairness, the absolute priority rule $P^{\dagger}$ realizes an allocation that respects the requirements of absolute fairness while 'it does as much as possible to promote comparative fairness' by consecutive applications of the weighted priority rule P that reimburse all individuals whose weighted proportional share exceeds their claim amounts. More generally, and precisely, for any Broomean problem $\mathcal{B} = (E, N, a, s)$, the absolute priority rule $P^{\dagger}$ recommends an allocation for $\mathcal{B}$ by applying the following steps:

The absolute priority rule $P^{\dagger}$.

(0) Let E_R denote the *remaining estate*, i.e. that part of the truncated estate which has not been used for reimbursement thus far, and let N_R denote the set of *remaining individuals*, i.e. the individuals who have not been reimbursed thus far.
Initially, set $E_R = \overline{E}$ and $N_R = N$.

(1) Use the weighted proportional rule to divide the remaining estate E_R amongst the individuals in N_R and let x denote the resulting allocation.
If there is no individual for which $x_i > a_i$, allot x_i to each i in N_R.
Otherwise, move to step (2).

(2) Reimburse each individual for which $x_i > a_i$ by allotting them their claim amount a_i, update E_R and N_R accordingly, and revisit step (1).

When we apply $P^{\dagger}$ to $\mathcal{O}$wing Money or $\mathcal{I}$nvesting Time, we get that $P^{\dagger}(\mathcal{O}) = (10,\ 30)$ and $P^{\dagger}(\mathcal{I}) = (5,\ 15)$ without visiting step (2) of $P^{\dagger}$'s definition. Applying $P^{\dagger}$ to $\mathcal{M}$ore Needed Money yields $P^{\dagger}(\mathcal{M}) = (20,\ 36,\ 24)$, for which we

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need to visit step (2) once. To see that it may be necessary to visit step (2) more than once, consider a variant of More Needed Money which can be represented as Broomean problem $\mathcal{X}$:

$$\mathcal{X} = \left(50, \{A, B, C\}, (20, 10, 30), \left(\frac{1}{2}, \frac{2}{6}, \frac{1}{6}\right)\right)$$

When we apply the weighted proportional rule to $\mathcal{X}$, we find that $P(\mathcal{X}) = (27.27,\ 9.09,\ 13.64)$ so that only Abram's weighted proportional share exceeds his claim amount. So we need to reimburse Abram and divide the remaining 30 amongst Benvolio and Capulet in accordance with the weighted proportional rule. Doing so yields 12 for Benvolio and 18 for Capulet so that Benvolio's weighted proportional share exceeds his claim amount. Hence, we also need to reimburse Benvolio so that, after two rounds of reimbursement, the absolute priority rule recommends $(20, 10, 20)$ for $\mathcal{X}$.

4. Two-dimensional Considerations

In this section we further motivate and illustrate the two-dimensional conception of fairness that is articulated by the Fairness formula and substantiated by the absolute priority rule. Although the three specific topics that we will discuss in this section are more or less independent from one another, it is apparent from all three discussions that *modelling* is a crucial, and often neglected, step in any fairness analysis. More specifically, in this section we will discuss the following topics.

In section 4.1 we discuss the constraint that 'no one receives more than they have a claim to' that is part of the requirement of absolute fairness as articulated by the Fairness formula. As claim satisfaction can never exceed 100%, one may wonder whether this constraint is not in fact redundant. In section 4.1 we explain that it is not: sometimes the constraint plays a crucial role in determining the recommendations that follow from the Fairness formula.

In section 4.2 the distinction between claim strengths and amounts takes centre stage. We explain that previous authors have either (1) neglected the distinction between claims strengths and amounts altogether, such as Curtis (2014) or (2) failed to properly conceptualize the strength-amount distinction, such as Morrow (2017). As a consequence, the corresponding accounts of fairness are (1) of limited scope or (2) solve fair division problems in a way that cannot be properly justified by appealing to fairness. Our theory of fairness improves upon both (1) and (2).

In section 4.3 we turn to cases of abundant good and explain how Saunders (2010) suggests that these cases ought to be analysed by a Broomean theory of fairness. We first distinguish between fairness and *subjective equity* (which is defined in terms of *preferences* and for which *no-envy* is key). We then carefully distinguish between different types of Broomean problems. We use both distinctions to provide a normative motivation that can underpin Saunders's intuitive suggestions.

More generally, we demonstrate the fruitfulness of the claims-based notions and the two-dimensional conception of fairness that we have introduced in this article.

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4.1. The constraint of absolute fairness

Consider the following fair division problem.

Combi-Deal. Anton sells pizzas for 10 per piece and Bernard sells ice-creams for 6 per piece. They decide to join forces and set up a combi-deal: for 14 customers can buy a ticket, either at Anton's or Bernard's shop, which gets them a pizza at Anton's and an ice-cream at Bernard's. Anton sells twice as many tickets for the combi-deal as Bernard. How to fairly divide the joint revenues of 14 per combi-deal amongst Anton and Bernard?

The question of what fair division amounts to in Combi-Deal has a less clear-cut answer than in the previous problems that we considered.

What is clear-cut is that Anton and Bernard together have an absolute *group* claim with an amount of 14. For, as it is their *joint* combi-deal, the associated revenues of 14, and all of the 14, are owed to *them*. Also, it is clear that the individual claims of Anton and Bernard are notional. What is less clear, however, is how to represent the amounts and strengths of these individual claims. More than one way of doing so suggests itself, including:

Broomean problem	Individual claims	Group claim
$\mathcal{C}_1 = \left(14, \{A, B\}, (14,\ 14), \left(\frac{10}{13}, \frac{3}{13}\right)\right)$	notional	absolute
$\mathcal{C}_2 = \left(14, \{A, B\}, (10,\ 6), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$	notional	absolute

The rationale of representing Combi-deal as $\mathcal{C}_1$ is as follows.

Rationale of $\mathcal{C}_1$. Anton and Bernard have a claim to receive a part of the price of the combi-deal ticket. Thus, the amount of their (notional) claims is 14. The strength of their respective claims is a function of the price of their individual product and the fraction of combi-deal tickets that they sold. A plausible candidate for this function is the product function: the strength of Anton's claim is $10 \cdot \frac{2}{3}$ as the price of a pizza is 10 and as Anton sold $\frac{2}{3}$ of the tickets. Similarly, the strength of Bernard's claim is $6 \cdot \frac{1}{3}$. Normalising the claims strength yields a strength of $\frac{10}{13}$ for Anton and of $\frac{3}{13}$ for Bernard[16].

To be sure, we are not suggesting that one *should* represent Combi-Deal as $\mathcal{C}_1$. All we suggest is that it is not, at least not prima facie, implausible to represent it as such. When Combi-Deal is represented as $\mathcal{C}_1$, fairness requires that allocation $(10.77,\ 3.23)$ is realized. For, $(10.77,\ 3.23)$ satisfies the absolute group claim of Anton and Bernard to as large extent as possible while it does not allot more to the agents than their individual claim amount of 14. But also, in $(10.77,\ 3.23)$ the claims of Anton and Bernard are satisfied in proportion to their strength. Hence, $(10.77,\ 3.23)$ satisfies the requirements of both absolute and comparative fairness.

Still, one might argue that something is wrong with recommending $(10.77,\ 3.23)$ for Combi-Deal. For in this allocation, Anton receives *more* from selling a pizza via the combi-deal (10.77) than from just selling a pizza (10). And this, so one may

[16]See footnote 3.2: a Broomean problem records normalized claim strengths. Here Anton and Bernard's normalized strengths are, respectively, $\frac{20/3}{26/3} = \frac{10}{13}$ and $\frac{6/3}{26/3} = \frac{3}{13}$.

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argue, is not how it should be. For by its nature, the combi-deal yields lower joint revenues (14) from selling a pizza plus ice-cream than from selling these products separately ($10 + 6 = 16$). It seems reasonable that the lower joint revenues from selling a combi-deal translate into associated *lower* revenues for the pizza and ice-cream that are sold as part of the deal: any fair allocation of the 14 should be such that Anton is allotted not more than 10 while Bernard is allotted not more than 6. Recommending $(10.77,\ 3.23)$ for Combi-Deal is not fair! One may very well concur with this argument and advocate the Fairness formula at the same time. Indeed, the recommendation $(10.77,\ 3.23)$ depends as much on the Fairness formula as it does on one's representation of Combi-Deal as $\mathcal{C}_1$. For a proponent of the argument against $(10.77,\ 3.23)$ just given, it is natural to represent Combi-Deal as $\mathcal{C}_2$, with the following rationale.

Rationale of $\mathcal{C}_2$. Anton and Bernard have claims to receive a part of the regular revenues of their individual products that are part of the combi-deal. Hence, the amounts of the (notional) claims of Anton and Bernard are 10 and 6 respectively. The strength of their respective claim is determined by the fraction of combi-deal tickets that they sold so that Anton's claim is 2 times as strong as Bernard's claim.

Applying the Fairness formula to $\mathcal{C}_2$ yields recommendation $(10,\ 4)$, as the reader may care to verify. In $(10,\ 4)$ no agent receives more from selling their product via the combi-deal than from only selling their own product: no agent receives more than he has a claim to.

The constraint of absolute fairness that 'no one receives more than they have a claim to' plays a crucial role in deriving recommendation $(10,\ 4)$ on the basis of $\mathcal{C}_2$. Let's suppose that we delete the constraint of absolute fairness from the Fairness formula. In that case, the resulting formula would, on the basis of $\mathcal{C}_2$, recommend allocation $(11,\ 3)$ in which Anton receives more than he has a claim to! To see this, observe that in $(11,\ 3)$ the absolute claim of {Anton, Bernard} is satisfied to as large an extent as possible (as it is in any allocation that allots 14 in total). But also, in $(11,\ 3)$ Anton's claim with an amount of 10 is fully satisfied, and so receives 100% satisfaction, whereas Bernard's claim with an amount of 6 receives 50% satisfaction, so that the individual claims of Anton and Bernard are satisfied in proportion to their strength. Hence, representation $\mathcal{C}_2$ of Combi-Deal illustrates the need to explicitly add the absolute constraint to the Fairness formula: without the constraint, the formula may yield recommendations that allot more to an individual than they have a claim to, which conflicts with the idea of a claim amount as an upper-bound on the receipts of the claimant.

4.2. On claim strengths and amounts

The Broomean formula only explicitly mentions claim strengths, but not amounts. Yet, as explained in section 2, the Broomean formula does mention the *satisfaction* of claims explicitly, which in turn depends on the good received and the amount of the claim: amounts are needed to determine satisfaction.

While there are some contributions that distinguish between claim strengths and amounts (e.g. Wintein and Heilmann 2018; Hausman 2023), many contributors

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have neglected the distinction between claims strengths and amounts altogether, such as Curtis (2014). He develops a Broomean theory of fairness that identifies claims with their amounts. As a consequence, his theory is of limited scope and, in particular, cannot be used to analyse problems such as Investing Time.[17] Or take Lazenby (2014), who discusses his Broomean account of fairness only in relation to problems where all individuals have claims, of varying strengths, to the same (amount of) good.[18] Indeed, a problem like Owing Money cannot be dealt with by Lazenby's account. Moreover, the fair division problems in section 3 involve agents with claims of varying amounts *and* strengths: these problems are outside the scope of Curtis's, Lazenby's and in fact any theory of fairness that we are aware of. In short, neglecting the strength-amount distinction restricts the scope of some accounts of fairness.

A second, more subtle and complex, issue is the failure to properly conceptualize the strength-amount distinction in some fairness accounts. To appreciate what is at stake, recall that the modelling of a fair division problem is a crucial step in solving it on the basis of the Fairness formula. This is illustrated by the two different representations of Combi-Deal just discussed, which lead to different recommendations. But it is also illustrated by an alternative representation of Owing Money that is sometimes given in the literature which, on our account, is flawed. This alternative representation has been worded[19] most explicitly by David Morrow (2017), who writes that:[20]

> For instance, if Romeo owes Benvolio and Abram sixty ducats and twenty ducats, respectively, but has only forty ducats with which to repay them, [fairness requires] that Romeo should give thirty ducats to Benvolio and ten to Abram, since Benvolio's claim on Romeo's money was three times as strong as Abram's. (Morrow 2017: 671)

Thus, on Morrow's analysis, Abram and Benvolio have claims to Romeo's money, which is 40, with the *strength* of their claims depending on the amount of money owed. In term of Broomean problems, Morrow proposes the following *alt*ernative model of $\mathcal{O}$wing Money:

$$\mathcal{O}^{alt} = \left(40, \{A, B\}, (40, 40), \left(\frac{1}{4}, \frac{3}{4}\right)\right)$$

Morrow writes that '[fairness requires] that Romeo should give thirty ducats to Benvolio and ten to Abram', i.e. that Romeo should realize $(10,\ 30)$ on the basis of

[17]See Wintein and Heilmann (2018) for a detailed development of this critique.

[18]More generally, the part of the Broomean fairness literature that discusses the fairness of (weighted) lotteries only seems to discuss problems where all individuals have claims to the same (amount of indivisible) good.

[19]Besides Morrow, also Sharadin (2016) analyses cases such as Owing Money along the lines of Morrow. Further, Saunders (2010) seems to advocate a similar analysis given that he writes that he is 'assuming, as Broome seems to, that the strength of the claim is simply dependent upon the amount of money owed'.

[20]In fact, Morrow writes that the 'Proportional Claims account', which he ascribes to Broome (1990), Rescher (2002) and Hooker (2005), entails that Romeo should give 30 ducats to Benvolio and 10 to Abram. As Broome (1990) understands fairness to be a strictly comparative notion, whereas Rescher (2002) and Hooker (2005) do not, summarizing their conceptions of fairness under one account, as Morrow does, is unhelpful. However, this need not distract us.

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$\mathcal{O}^{alt}$. This recommendation coincides with that of the absolute priority rule: $P^{\dagger}\left(\mathcal{O}^{alt}\right) = (10,\ 30)$.

However, Morrow does not tell us *why* fairness requires that Romeo should realize $(10, 30)$. On our account, the representation of Owing Money as $\mathcal{O}^{alt}$ is at odds with the recommendation that Romeo should realize $(10, 30)$. For, as we will demonstrate below, the $\mathcal{O}^{alt}$ representation fails to capture the absolute claims correctly. Without those, however, the requirements of absolute fairness, which only deals with absolute claims, cannot be invoked to argue that Romeo has to realize $(10, 30)$ rather than, say, $(5,\ 15)$.

The individual claims ascribed to Abram and Bevolio by $\mathcal{O}^{alt}$ are clearly notional and not absolute: Romeo owes it to Abram and Benvolio to give them a part of 40, and there's nothing wrong with one of them receiving less than 40. Nor does it make sense to say that the group {Abram, Benvolio} has an absolute claim to 40. Romeo owes 20 to Abram and 60 to Benvolio, for sure. But from this it does not follow that Romeo owes anything to the *group* consisting of Abram and Benvolio. Indeed, Abram and Benvolio may have never met, never interacted and be unaware of one another's existence: they do not form a group in any substantive sense. In sharp contrast, Anna and Beta *jointly* produced the value of 20 in Investing Time, so *the group* consisting of Anna and Beta has an absolute claim to receive all of the jointly produced 20. But in Owing Money, there is no such joint production and it does not make sense to ascribe an absolute claim to {Abram, Benvolio}.

But then, on Morrow's analysis, there are no absolute claims in Owing Money at all: $\mathcal{O}^{alt}$ can neither be understood as capturing absolute claims by the individuals nor by the group. Hence, absolute fairness has nothing to say about the allocation that Romeo should realize for Owing Money and, in particular, cannot explain why Romeo should realize $(10,\ 30)$ rather than $(5,\ 15)$. Although Morrow asserts that fairness requires that Romeo realizes $(10,\ 30)$ in Owing Money, he cannot justify this assertion owing to his representation of Owing Money as $\mathcal{O}^{alt}$. As such, Morrow's analysis of Owing Money is flawed. Or so we argue.

We have illustrated and motivated the Fairness formula in some detail. We have done so exclusively for fair division problems in which there was not enough to go around: the amount to be divided was less than the sum of claims. However, in the literature we also find some discussions of what (Broomean) fairness requires in cases of abundant good. To further illustrate the Fairness formula, we will now apply it to such cases.

4.3. Fairness and abundant good

In *Fairness between competing claims*, Ben Saunders (2010) criticizes Broome's idea that fairness is strictly comparative. Saunders (2010: 45) does so by considering a case where he owes '£20 on one friend and £10 to another' and remarks that 'if I have the money that I promised to pay you, then it is unfair of me either to keep it or burn it and to repay less than I can', hinting at the absolute dimension of fairness. With respect to his example, Saunders then further remarks the following.

> I assume it would still be fair if I was to pay them £40 and £20, respectively, thereby giving each twice what they had a claim to, though Broome is not

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> explicit on this point. If we were dividing business profits proportionately to initial investment, this is what we would do, though if I am merely giving my friends a gift on top of repaying what I owe then perhaps I should give each an equal amount extra. (Saunders 2010: 45)

So, Saunders speculates about what fairness requires in cases of abundant good. In fact, he does so with respect to two different fair division problems.

Abundant Money. Marta owes £20 to Alice and £10 to Bob and has £60 in his pocket. How, in order to be fair, should Marta divide her money?

Profit. Ali and Benji have invested respectively £20 and £10 in a joint project. After some time, Ali and Benji want to split apart and the question arises how the value of their joint project, £60, should be divided. How, in order to be fair, should Ben divide the £60?

$\mathcal{A}$bundant Money and $\mathcal{P}$rofit are represented as follows:

Broomean problem	Individual claims	Group claim
$\mathcal{A} = (60, \{A, B\}, (20,\ 10), (\frac{1}{2}, \frac{1}{2})$	absolute	none
$\mathcal{P} = (60, \{A, B\}, (60,\ 60), (\frac{2}{3}, \frac{1}{3})$	notional	absolute

When applied to $\mathcal{A}$, the absolute priority rule recommends $(20,\ 10)$ in which both claims receive the maximal satisfaction of 100%. Hence, $(20,\ 10)$ satisfies claims to as large an extent as possible and, as the claims are equally strong, in proportion to their strength. Now, *any* allocation in which Alice and Bob receive the full amount of their claim *or more*, such as $(25,\ 15)$ or $(30,\ 30)$, satisfies claims to as large an extent as possible and in proportion to their strength. But $(20,\ 10)$ is the unique allocation which has these two features and in which no agent receives more than he has a claim to: the absolute constraint also plays a crucial role in the derivation of $(20, 10)$ from the Fairness formula on the basis of $\mathcal{A}$.

For sure, fairness does not *require* that Marta allots more to Alice and Bob than they have a claim to. But suppose that Marta decides to use all of her £60 in order to give Alice and Bob a 'gift on top of repaying what [he owes]' (Saunders 2010: 45). Saunders suggests that in such a situation, Marta should 'give each an equal amount extra', i.e. realize allocation $(20,\ 10) + (15,\ 15) = (35,\ 25)$. We concur. However, this requirement is not grounded in *fairness*, but rather in what Nicholas Rescher calls *subjective equity*. As Rescher (2002: ix) explains, fairness is based on *claims* so that 'fairness is therefore something quite different from subjective equity as based on the personal evaluation of distributive goods by the claimants involved'. An important criterion of subjective equity is that of *no-envy*: an allocation of goods is called envy-free when no individual *prefers* the share of another individual over her own share.[21] Now, as Alice and Bob both prefer to receive more money over less, the only division of the gift that is envy-free is the equal division. Hence, when Marta

[21]See e.g. Tinbergen *et al.* (1930/2021), Varian (1975), Dworkin (1981a, 1981b), Brams and Taylor (1996), Olson (2018, 2020), Heilmann and Wintein (2021) for different literatures in which the notion of envy-freeness plays a prominent role.

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realizes (22, 38), or (40, 20) for that matter, Alice can complain that the realized allocation is not envy-free. So, we concur with Saunders that *when* Marta decides to use her £60 to provide a gift to Alice and Bob on top what she owes them, there is a sense in which she 'should give each an equal amount extra', resulting in allocation (35, 25). However, we submit that this sense of 'should' is grounded in subjective equity, and not in (claims-based) fairness. When all claims are fully reimbursed, the requirements of fairness are fully met: as there are no *claims* on 'gifts on top of full reimbursement' fairness offers, in contrast to subjective equity, no guidance in how to allocate abundant good.

So, by distinguishing between fairness and subjective equity, we can provide a normative underpinning for Saunders's intuitive suggestion for problems such as Abundant Money. Moreover, our conceptual framework allows us to account for the *different* recommendations for Abundant Money and Profit that Saunders alludes to. Indeed, Abundant Money gives rise to $\mathcal{A}$, in which the individuals have absolute claims, whereas Profit is naturally represented as $\mathcal{P}$, with individual claims being notional and invested money affecting the strength of the claims. With respect to $\mathcal{P}$, the absolute priority rule recommends allocation $(40, 20)$ which accords with Saunders's suggestion.

5. Conclusions

We have made three key contributions. First, we have provided a new theory of fairness – summarized by the Fairness formula – that accommodates a concern for both comparative and absolute fairness. This theory improves the conceptual and practical reach of Broomean fairness theory. The conceptual distinctions we have introduced to formulate the theory are also of independent merit. They make explicit several conceptual concerns present in the preceding literature on Broomean fairness. It is also worth pointing out that the Broomean formula and the Fairness formula have an ancient ancestor:

Aristotelian formula. Fairness requires that equals should be treated equally, and unequals unequally, in proportion to relevant similarities and differences. (Aristotle 2009, *Nicomachean Ethics*)

On Aristotle' s view, fairness is thus equal treatment of equals, as well as the demand to take into account the 'relevant' similarities and differences in a proportional way. The Broomean concept of a claim can thus be seen as a more precise, and demanding, way to spell out the requirement in the Aristotelian formula. Likewise, the Fairness formula is also a precisification of the Aristotelian formula.

Second, we have justified the absolute priority rule that gives an algorithmic solution to any fair division problem that can be suitably represented as a Broomean problem. We have also used our Fairness formula and absolute priority rule to critically engage with recent contributions in the Broomean fairness literature.

Third, and generally, we have introduced a framework of modelling fair division with *fairness structures* and *fairness functions* which makes modelling choices in the analysis of fair division problems explicit. This general perspective has motivated the *Broomean problems* we have employed to justify the absolute priority rule. What is more, formulating our theory in terms of Broomean problems facilitates translation and interaction between theories in economic theory and Broomean fairness in philosophy. A full demonstration will be provided in the article 'How to be absolutely fair, Part II: philosophy meets economics'.

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Stefan Wintein is an Associate Professor in Theoretical Philosophy at Erasmus School of Philosophy, where he is also a member of the Erasmus Institute for Philosophy and Economics (EIPE). He believes that formal methods can help to shed light on philosophical problems and this belief is reflected in his research, which mainly concentrates on ethics (fairness) and philosophical logic (self-referential truth, many-valued logics). URL: https://www.eur.nl/people/stefan-wintein

Conrad Heilmann is an Associate Professor of Philosophy at Erasmus School of Philosophy, Co-Director of the Erasmus Institute for Philosophy and Economics (EIPE), and Core Faulty of the Erasmus Initiative 'Dynamics of Inclusive Prosperity' at Erasmus University Rotterdam. He co-edited the *Routledge Handbook of Philosophy of Economics* (2021). His research covers fairness, the methodology of financial economics, and rational choice theory. Email: heilmann@esphil.eur.nl; URL: https://www.eur.nl/people/conrad-heilmann

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