On the Cardinal Function of Interpolation Theory.

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1. The cardinal function is the interpolation function

$$C(x) = \sum_{r=-\infty}^{\infty} \frac{a_r \sin\left\{\frac{\pi}{w}(x-a-rw)\right\}}{\frac{\pi}{w}(x-a-rw)}$$

which takes the values a_r at the points $a + rw$. Its principal properties were discovered by Professor Whittaker,[1] amongst others that

(A) *When* $C(x)$ *is analysed into periodic constituents by Fourier's integral-theorem, all constituents of period less than* $2w$ *are absent.*

This statement requires some modification. Thus in Professor Whittaker's Example 2, in which a_r is always either 0 or 1, the cardinal series converges to the sum

$$\frac{2}{\sqrt{3}} \sin \frac{\pi(x-a)}{3w}$$

and this function cannot be analysed by Fourier's Theorem. The reason for this appears from Theorem 2 below, wherein it is shown that (roughly) the cardinal series converges and has the property (A) when and only when $\Sigma\, a_r^2$ converges. The remainder of the paper is concerned with questions of the convergence and summability of the series defining the cardinal function.

2. There is no loss of generality in taking $a = 0$, $w = 1$, and this will be done in the sequel. The cardinal function is then defined to be

$$C(x) = a_0 v_0(x) + \sum_{r=1}^{\infty} a_r v_r(x) + \sum_{r=1}^{\infty} a_{-r} v_{-r}(x) \tag{1}$$

where

$$\left. \begin{aligned} v_r(x) &= \frac{\sin \pi(x-r)}{\pi(x-r)}, \qquad & x \neq r \\ &= \quad 1 & x = r \end{aligned} \right\}.$$

[1] *Proc. Roy. Soc. Edin.*, XXXV (1915), p. 181.

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It has been shewn by Ferrar[1] that if $\sum_{r=1}^{\infty}(-1)^r a_r$, $\sum_{r=1}^{\infty}(-1)^r a_{-r}$ both oscillate finitely, the cardinal series converges and represents an analytic function. Theorem 1 deals with a similar set of conditions, namely that

$$\Sigma(-1)^r \frac{a_r}{r}, \; \Sigma(-1)^r \frac{a_{-r}}{r}$$

be convergent, and it is shewn that these conditions are also necessary. It is convenient to regard convergence as summability $(C0)$ and to discuss the summability (Ck) of the cardinal series for a positive integral k. With this understanding we prove

THEOREM 1. *If the cardinal series* (1) *is summable* (Ck) *for any (complex) value of* x *other than* $x = 0, \pm r$, *it is uniformly summable* (Ck) *in any finite region of the* x*-plane and its sum is an integral function.*

The series (1) *will be summable* (Ck) *to an integral function if and only if*

$$\sum_{r=1}^{\infty}(-1)^r \frac{a_r}{r}, \; \sum_{r=1}^{\infty}(-1)^r \frac{a_{-r}}{r}$$

are summable (Ck).

Assume in the first place that the series

$$\Sigma(-1)^r \frac{a_r}{r}$$

is summable (Ck). Let

$$u_r(x) = \frac{r}{x-r} \sin \pi x.$$

Then

$$\sum_{r=1}^{\infty}(-1)^r \frac{a_r}{r} u_r(x) = \pi \sum_{r=1}^{\infty} a_r \frac{\sin \pi (x-r)}{\pi (x-r)}.$$

Thus by a theorem due to Hardy[2] the series on the right will be uniformly summable (Ck) if for a fixed k and in any finite region of the x-plane

(2) $$\sum_{r=0}^{\infty} r^k \,|\Delta^{k+1} u_r| < K, \text{ independent of } x.$$

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[1] *ibid.* XLV (1925), p. 275. See also a later paper by the same author (*ibid.* XLVI (1926), p. 323) where further references to the literature are given.

[2] *Proc. Lond. Math. Soc.* (2) 6, 255. Theorem A.

Now

$$\Delta u_r = \left(\frac{r+1}{x-r-1} - \frac{r}{x-r}\right) \sin \pi x = \frac{x \sin \pi x}{(x-r-1)(x-r)}$$

and generally, as is easily verified by induction

$$\Delta^k u_r = \frac{k!\, x \sin \pi x}{(x-r)(x-r-1)\dots(x-r-k)}$$

so that in a finite region of the x-plane

$$|\Delta^{k+1} u_r| = O\left(\frac{1}{r^{k+2}}\right)$$

the constant implied in O being independent of x, r. Thus the r^{th} term of the series (2) is $O\left(\frac{1}{r^2}\right)$ and so the series is bounded uniformly with respect to x. The other parts of the theorem are all proved in the same way.

3.[1] The general nature of Theorem 2 has been discussed in the introduction. The precise statement is

THEOREM 2. *If $\{a_r\}$ is a sequence of real numbers such that*

$$(\alpha) \qquad \sum_{r=-\infty}^{\infty} a_r^2 \text{ is convergent}$$

then the cardinal series is absolutely convergent and its sum is of the form

$$(\beta) \quad C(x) = \int_0^1 \{\phi(t) \cos \pi x t + \psi(t) \sin \pi x t\} dt, \text{ with } \phi, \psi \text{ each } L^2.$$

There is at most one function $C_1(x)$ of the form (β) which takes the values a_r at the points $x = r$ and if there is such a function then the condition (α) is satisfied and the cardinal series converges absolutely to $C_1(x)$.

We have interpreted the property (A) to mean only that $C(x)$ can be put in the form (β), without insisting that this analysis into periodic components can be effected by applying Fourier's integral theorem to $C(x)$. Now

$$\sum_{r=-\infty}^{\infty} a_r^2 = a_0^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r + a_{-r})^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r - a_{-r})^2$$

and thus (α) implies the convergence of the series

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[1] This section has been rewritten in accordance with the valuable suggestions of Mr W. L. Ferrar, who kindly read the paper in manuscript.

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$$(\alpha') \qquad a_0^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r + a_{-r})^2, \ \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r - a_{-r})^2.$$

Define

$$\phi_p(t) = a_0 + \sum_{r=1}^{p}(a_r + a_{-r})\cos \pi r t, \ \psi_p(t) = \sum_{r=1}^{p}(a_r - a_{-r})\sin \pi r t$$

Then, as in the proof of the Riesz-Fischer theorem,[1] the condition (α') implies the existence of functions $\phi(t)$, $\psi(t)$ of integrable square such that

$$\int_0^1 \left\{\phi(t) - \phi_p(t)\right\}^2 dt \to 0, \ \int_0^1 \left\{\psi(t) - \psi_p(t)\right\}^2 dt \to 0, \text{ as } p \to \infty.$$

Thus if x is confined to a finite region of the x-plane in which the upper bound of $|\cos \pi x t|$, $|\sin \pi x t|$ is K, $(0 \leqslant t \leqslant 1)$, we have

$$\left|\int_0^1 \left\{\phi(t)\cos \pi x t + \psi(t)\sin \pi x t\right\} dt - a_0 v_0(x) - \sum_{r=1}^{p}\left\{a_r v_r(x) + a_{-r}v_{-r}(x)\right\}\right|$$

$$= \left|\int_0^1 \left\{\phi(t) - \phi_p(t)\right\}\cos \pi x t \, dt + \int_0^1 \left\{\psi(t) - \psi_p(t)\right\}\sin \pi x t \, dt\right|$$

$$\leqslant \left|\int_0^1 \left\{\phi(t) - \phi_p(t)\right\}\cos \pi x t \, dt\right| + \left|\int_0^1 \left\{\psi(t) - \psi_p(t)\right\}\sin \pi x t \, dt\right|$$

$$\leqslant K\left\{\int_0^1 \left\{\phi(t) - \phi_p(t)\right\}^2 dt\right\}^{\frac{1}{2}} + K\left\{\int_0^1 \left\{\psi(t) - \psi_p(t)\right\}^2 dt\right\}^{\frac{1}{2}}$$

$\to 0$ uniformly with respect to x, as $p \to \infty$.

Thus the series

$$(3) \qquad a_0 v_0(x) + \sum_{r=1}^{\infty}\{a_r v_r(x) + a_{-r}v_{-r}(x)\}$$

is uniformly convergent in any finite region of the x-plane and its sum is the integral (β).

Now the convergence of $\sum a_r^2$ implies that of $\sum |a_r/r|$ and this series majorises the cardinal series. Thus the latter is absolutely and uniformly convergent, and since it can be rearranged as the series (3) its sum is the integral (β). Again, let

$$C_1(x) = \int_0^1 \{\phi_1(t)\cos \pi x t + \psi_1(t)\sin \pi x t\}\, dt$$

with $\phi_1(t)$, $\psi_1(t)$ each L^2 and $C_1(r) = a_r$. Then

$$a_0 = \int_0^1 \phi_1(t)\, dt, \ \frac{a_r + a_{-r}}{2} = \int_0^1 \phi_1(t)\cos \pi r t\, dt, \ \frac{a_r - a_{-r}}{2} = \int_0^1 \psi_1(t)\sin \pi r t\, dt$$

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[1] *Cf.* Hobson. *Functions of a Real Variable* (2nd Ed.), p. 576. That (α) implies the existence of a $C(x)$ of the form (β) such that $C(r) = a_r$ is in fact the Riesz-Fischer theorem.

and by Parseval's theorem the series

$$a_0^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r + a_{-r})^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r - a_{-r})^2 = \sum_{r=-\infty}^{\infty} a_r^2$$

is convergent, its sum being the sum of the integrals of $\phi_1^2(t)$, $\psi_1^2(t)$. Thus by the first part of Theorem 2, which has just been proved, the cardinal series converges absolutely to a function

$$C(x) = \int_0^1 \{\phi(t)\cos \pi xt + \psi(t)\sin \pi xt\}\, dt.$$

Then $C_1(r) = C(r)$ for all integral values of r, and on applying Parseval's theorem to the function $C_1(x) - C(x)$ we deduce that $\phi_1(t) - \phi(t)$, $\psi_1(t) - \psi(t)$ are equivalent to zero, and thus that $C_1(x) = C(x)$ for all values of x.

Added 19*th Feb.* 1927. 4. It was shown by Professor Whittaker that if $\{a_r\}$ is bounded as $r \to \pm\infty$, the cardinal series is equivalent to the Gauss interpolation series. This connection was afterwards investigated by Ferrar who proved that in all cases the convergence of the cardinal series (in the form (3)) implies that of the Gauss series (with the terms bracketed in pairs) to the same sum; while that the convergence of the Gauss series implies that the cardinal series is summable by the method of de la Vallée Poussin, to the same sum. Professor Whittaker's theorem is a corollary of this. For if the cardinal series is summable $(V.P.)$ it is also summable by Abel's limit.[1] But if $\{a_r\}$ is bounded it is clear that

$$|a_r v_r(x) + a_{-r} v_{-r}(x)| = O\left(\frac{1}{r}\right)$$

and so by Littlewood's converse of Abel's theorem,[2] the series (3) is convergent. Moreover we have

$$a_r v_r(x) + a_{-r} v_{-r}(x) = \frac{\sin \pi x}{\pi}(-1)^r\left(\frac{a_r}{x-r} + \frac{a_{-r}}{x+r}\right)$$

$$= \frac{\sin \pi x}{\pi}(-1)^{r+1}\left\{\frac{a_r - a_{-r}}{r} + O\left(\frac{1}{r^2}\right)\right\}$$

so that the series (3) converges and diverges with

$$\sum_{r=1}^{\infty}(-1)^{r+1}\frac{a_r - a_{-r}}{r}.$$

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[1] By a theorem due to Gronwall. *Cf.* de la Vallée Poussin in *The Rice Institute Pamphlet* XII (1925), p. 117.

[2] Hobson. *loc. cit.*, p. 184.

If this series converges its sum is $C'(0)$, as may be verified by differentiating the series (3) and putting $x = 0$ in the result.

Combining this result with Professor Whittaker's we have

THEOREM 3. *If $\{a_r\}$ is bounded, the series*

(i) $\sum_{r=1}^{\infty} (-1)^{r+1} \dfrac{a_r - a_{-r}}{r}$

(ii) $a_0 v_0(x) + \sum_{r=1}^{\infty} \{a_r v_r(x) + a_{-r} v_{-r}(x)\}$

(iii) $$a_0 + \left\{ x\delta a_{\frac{1}{2}} + \frac{x(x-1)}{2!}\delta^2 a_0 \right\} + \left\{ \frac{(x+1)x(x-1)}{3!}\delta^3 a_{\frac{1}{2}} + \frac{(x+1)x(x-1)(x-2)}{4!}\delta^4 a_0 \right\} + \ldots$$

are either all divergent (for non integral values of x), or else all convergent.

In the latter case the series (ii), (iii) *have the same sum $C(x)$ and the sum of* (i) *is $C'(0)$.*

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