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PRIME-UNIVERSAL QUADRATIC FORMS $ax^2 + by^2 + cz^2$ AND $ax^2 + by^2 + cz^2 + dw^2$

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Abstract

A positive-definite diagonal quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$ $(a_1, \ldots, a_n \in \mathbb{N})$ is said to be *prime-universal* if it is not universal and for every prime p there are integers $x_1, \ldots, x_n$ such that $a_1x_1^2 + \cdots + a_nx_n^2 = p$. We determine all possible prime-universal ternary quadratic forms $ax^2 + by^2 + cz^2$ and all possible prime-universal quaternary quadratic forms $ax^2 + by^2 + cz^2 + dw^2$. The prime-universal ternary forms are completely determined. The prime-universal quaternary forms are determined subject to the validity of two conjectures. We make no use of a result of Bhargava concerning quadratic forms representing primes which is stated but not proved in the literature.

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1. Introduction

Let $\mathbb{N}$ denote the set of positive integers. We use the notation

$$\langle a_1, \ldots, a_n\rangle := a_1x_1^2 + \cdots + a_nx_n^2, \quad a_1, \ldots, a_n \in \mathbb{N}.$$

The quadratic form $\langle a_1, \ldots, a_n\rangle$ is said to be prime-universal if it is not universal but represents every prime. If $\langle a_1, \ldots, a_n\rangle$ is prime-universal then so is $\langle a_{i_1}, \ldots, a_{i_n}\rangle$ for any permutation $\{i_1, i_2, \ldots, i_n\}$ of $\{1, 2, \ldots, n\}$, so we may assume without loss of generality that

$$a_1 \le a_2 \le \cdots \le a_n.$$

Our first objective is to determine all prime-universal ternary quadratic forms $\langle a_1, a_2, a_3\rangle$. In Section 2 we prove the following theorem.

Theorem 1.1. *Let $a, b, c \in \mathbb{N}$ satisfy $a \le b \le c$. Then $\langle a, b, c\rangle$ is prime-universal if and only if $\langle a, b, c\rangle$ is one of $\langle 1, 1, 2\rangle$, $\langle 1, 1, 3\rangle$, $\langle 1, 2, 3\rangle$, $\langle 1, 2, 4\rangle$ or $\langle 1, 2, 5\rangle$.*

As a consequence of this theorem, we deduce the following result.

TABLE 1. Prime-universal forms $\langle a, b, c, d\rangle$.

a	b	c	d	Conditions on d
1	1	2	d	$d \geq 15$
1	1	3	d	$d \geq 7$
1	2	3	d	$d \geq 11$
1	2	4	d	$d \geq 15$
1	2	5	5	
1	2	5	d	$d \geq 11$
2	2	3	d	$3 \leq d \leq 15, d \neq 8, 11$
2	3	3	5	
2	3	3	7	
2	3	4	d	$5 \leq d \leq 11, d \neq 6, 7, 10$
2	3	5	d	$5 \leq d \leq 43, d \neq 7, 19, 28, 34, 37, 39, 42$
2	3	7	8	

THEOREM 1.2. *Let $a, b, c \in \mathbb{N}$ satisfy $a \leq b \leq c$. Let*

$$S_{3p} := \{2, 3, 5, 7, 17, 43\}.$$

If $\langle a, b, c\rangle$ represents every integer in S_{3p} then $\langle a, b, c\rangle$ is prime-universal. Moreover, the set S_{3p} is minimal in the sense that if any integer is removed from S_{3p} then the resulting set no longer has this property.

We remark that the set S_{3p} is not unique as the set $\{2, 3, 5, 7, 41, 43\}$ also has the same property as S_{3p}.

Our second objective is to determine all possible prime-universal quaternary quadratic forms $\langle a_1, a_2, a_3, a_4\rangle$. In Section 3 we prove the following theorem.

THEOREM 1.3. *Let $a, b, c, d \in \mathbb{N}$ satisfy $a \leq b \leq c \leq d$. Suppose that $\langle a, b, c, d\rangle$ is prime-universal. Then $\langle a, b, c, d\rangle$ is one of the forms listed in Table 1.*

Theorem 1.3 does not assert that all of the infinitely many quaternary quadratic forms $\langle a, b, c, d\rangle$ listed in Table 1 are prime-universal. We can prove the prime-universality of all the forms in Table 1 except for the 27 forms

$$\begin{gathered}\langle 2,3,4,5\rangle, \quad \langle 2,3,4,11\rangle, \quad \langle 2,3,5,5\rangle, \quad \langle 2,3,5,11\rangle, \\ \langle 2,3,5,13\rangle, \quad \langle 2,3,5,14\rangle, \quad \langle 2,3,5,16\rangle, \quad \langle 2,3,5,17\rangle, \\ \langle 2,3,5,h\rangle \quad \text{for } h = 20,\ldots,27,29,\ldots,33,35,36,38,40,41,43.\end{gathered} \tag{1.1}$$

We now describe the difficulty in proving the prime-universality of these 27 forms. We recall that the ternary subforms of the quaternary form $\langle a, b, c, d\rangle$ are $\langle a, b, c\rangle$, $\langle a, b, d\rangle$, $\langle a, c, d\rangle$ and $\langle b, c, d\rangle$. We also recall that a ternary quadratic form $\langle a, b, c\rangle$ is said to be regular if and only if there exist a finite number of progressions of the form $A^k(B\ell + C)$ $(k, \ell = 0, 1, 2, \ldots)$, where A, B and C are positive integers with

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$1 \le C < B$, such that $n \in \mathbb{N}$ is represented by $ax^2 + by^2 + cz^2$ if and only if n does not belong to any of these progressions. The classic example of a regular ternary quadratic form is $\langle 1, 1, 1\rangle$, as Legendre showed that n is represented by $\langle 1, 1, 1\rangle$ if and only if $n \neq 4^k(8\ell + 7)$ for any nonnegative integers k and ℓ. Dickson and Jones showed that there are exactly 102 regular quadratic forms $\langle a, b, c\rangle$, and these are listed in [2], together with a precise description of the integers they represent. Two of the 27 forms listed in (1.1) have $\langle 2, 3, 4\rangle$ as a ternary subform and the remaining 25 have $\langle 2, 3, 5\rangle$ as a ternary subform. These two forms are neither regular [2] nor spinor-regular [4], and for such forms it is extremely difficult to determine precisely which positive integers they represent. In fact, at present, there is no general algorithm for determining the positive integers represented by a positive-definite ternary quadratic form [11, page 1695]. If n is a large positive integer, the number of representations of n by $\langle a, b, c\rangle$ is approximated by an expression involving the class number of an imaginary quadratic field depending on n. Bounds for such class numbers are closely tied to whether or not a quadratic Dirichlet L-function has a Siegel zero, and this is an unsolved problem in number theory.

For example, Ono and Soundararajan [9, Theorem 3, page 419] made two assumptions about the location of the nontrivial zeros of both Dirichlet L-functions and Hasse–Weil L-functions in order to prove that the only positive integers not excluded by congruence conditions which are not represented by Ramanujan's ternary quadratic form $\langle 1, 1, 10\rangle$ are

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391, 679, 2719,$$

thereby illustrating the difficulty in determining the integers represented by a ternary quadratic form.

Another example involves the three ternaries

$$x^2 + 2y^2 + 5z^2 + xz, \quad x^2 + 3y^2 + 6z^2 + xy + 2yz, \quad x^2 + 3y^2 + 7z^2 + xy + xz.$$

Kaplansky [6, page 207] described these ternaries as 'plausible candidates' for representing all odd positive integers. Rouse [11] formally made the conjecture that they do. It is known that they represent every odd positive integer less than 2^{14}. This conjecture remains open and appears to be very difficult to prove.

A numerical study of the positive integers represented by the two ternaries of interest to us, namely $\langle 2, 3, 4\rangle$ and $\langle 2, 3, 5\rangle$, suggested the following two conjectures.

Conjecture 1.4. *The quadratic form* $\langle 2, 3, 4\rangle$ *represents all odd integers* $n > 647$.

Conjecture 1.5. *The quadratic form* $\langle 2, 3, 5\rangle$ *represents all prime numbers* $p \neq 43$.

Conjecture 1.4 has been verified for all odd positive integers n which satisfy $647 < n < 10^5$. Conjecture 1.5 has been verified for all prime numbers p satisfying $2 \le p < 10^5$, $p \neq 43$. Due to the difficulty in determining the integers represented by nonregular ternary quadratic forms $\langle a, b, c\rangle$, it does not seem unreasonable to make these two conjectures. Based on these two conjectures, we prove the following theorem in Section 3.

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THEOREM 1.6. *Let $a, b, c, d \in \mathbb{N}$ satisfy $a \le b \le c \le d$. Assuming that Conjectures 1.4 and 1.5 are valid, $\langle a, b, c, d\rangle$ is prime-universal if and only if $\langle a, b, c, d\rangle$ is listed in Table 1.*

As a consequence of Theorem 1.6, we obtain the following result.

THEOREM 1.7. *Let $a, b, c, d \in \mathbb{N}$ satisfy $a \le b \le c \le d$. Let*

$$S_{4u} := \{1, 6, 10, 14, 15\},$$
$$S_{4p} := \{2, 3, 5, 7, 13, 17, 23, 41, 43\}.$$

Assume that Conjectures 1.4 and 1.5 hold. If $\langle a, b, c, d\rangle$ represents every integer in S_{4p} but not every integer in S_{4u}, then $\langle a, b, c, d\rangle$ is prime-universal, and S_{4p} is minimal in the sense explained in Theorem 1.2.

In proving our results, we make use of the following simple result known as the 'bounding lemma', which is proven in [12, pages 532–533].

LEMMA 1.8 (Bounding lemma). *Let $k \in \mathbb{N}$. Let $a_1, \dots, a_k \in \mathbb{N}$ with $a_1 \le a_2 \le \dots \le a_k$. Set*

$$q := \langle a_1, \dots, a_k\rangle, \quad q_1 := 0,$$
$$q_i := \langle a_1, \dots, a_{i-1}\rangle, \quad i = 2, \dots, k.$$

If q represents a positive integer n but q_i does not represent n, for some $i \in \{1, 2, \dots, k\}$, then $a_i \le n$.

We note that in proving our results, we have not used a result of Bhargava (see [5, Theorem C, page 674]) which asserts that if a quadratic form represents a certain finite set of prime numbers then the form represents all prime numbers, as its proof does not appear to be available in the literature.

We close this introduction by remarking that the ternary quadratic form $\langle a, b, c\rangle$ ($a, b, c \in \mathbb{N}, a \le b \le c$) cannot be universal (see [2, Theorem 95, page 104]) and that Ramanujan [10] determined all the quaternary quadratic forms $\langle a, b, c, d\rangle$ with $a, b, c, d \in \mathbb{N}$, $a \le b \le c \le d$, which are universal (see also [2, page 105]).

2. Prime-universal forms $\langle a, b, c\rangle$

In this section we present the proofs of Theorems 1.1 and 1.2. As we often need to consider several forms at the same time, we write, for example,

$$\langle a, b, C_1 \le c \le C_2, d\rangle$$

to indicate the set of forms

$$\{\langle a, b, c, d\rangle \mid a, b, c, d \in \mathbb{N}, C_1 \le c \le C_2\}.$$

If $\langle a, b, c\rangle$ fails to represent a prime, we call the smallest prime not represented by $\langle a, b, c\rangle$ its prime truant.

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TABLE 2. Prime truants for certain ternary quadratic forms $\langle a, b, c\rangle$.

a	b	c	Prime truant
1	1	1	7
1	2	2	7
2	2	2	3
2	2	3	17
2	3	3	7
2	3	4	13
2	3	5	43
2	3	6	7
2	3	7	13

PROOF OF THEOREM 1.1. Let $a, b, c \in \mathbb{N}$ be such that $a \le b \le c$ and $\langle a, b, c\rangle$ is prime-universal. As $\langle a, b, c\rangle$ represents 2, by the bounding lemma we have $a \le 2$. As $\langle a, b, c\rangle$ represents 3 but the forms $\langle 1 \le a \le 2\rangle$ do not, by the bounding lemma we have $b \le 3$. It remains to consider the forms

$$\langle 1 \le a \le 2, a \le b \le 3, b \le c\rangle.$$

Observe that $\langle 1, 3, c\rangle$ has prime truant 2 regardless of the choice of c (≥ 3), and hence this form is not prime-universal. We see that

$$\langle 1, 1\rangle \text{ has prime truant } 3, \quad \langle 1, 2\rangle \text{ has prime truant } 5,$$
$$\langle 2, 2\rangle \text{ has prime truant } 3, \quad \langle 2, 3\rangle \text{ has prime truant } 7.$$

Thus, by the bounding lemma, the only possible prime-universal forms are the 14 forms

$$\langle 1, 1, 1 \le c \le 3\rangle, \quad \langle 1, 2, 2 \le c \le 5\rangle, \quad \langle 2, 2, 2 \le c \le 3\rangle, \quad \langle 2, 3, 3 \le c \le 7\rangle.$$

From Dickson [2, page 112],

$$\begin{array}{lll} \langle 1,1,2\rangle & \text{represents all integers not of the form} & 4^k(16\ell + 14), \\ \langle 1,1,3\rangle & \text{represents all integers not of the form} & 4^k(16\ell + 14), \\ \langle 1,2,3\rangle & \text{represents all integers not of the form} & 4^k(16\ell + 10), \\ \langle 1,2,4\rangle & \text{represents all integers not of the form} & 4^k(16\ell + 14), \\ \langle 1,2,5\rangle & \text{represents all integers not of the form} & 25^k(25\ell + \{10 \text{ or } 15\}), \end{array}$$

for any $k, \ell \in \mathbb{N}_0$. The progressions $4^k(16\ell + 10), 4^k(16\ell + 14)$ and $25^k(25\ell + \{10 \text{ or } 15\})$ contain only composite integers, so the five forms in question represent all primes and are prime-universal. All the remaining nine forms are not prime-universal as they have the prime truants listed in Table 2. □

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PROOF OF THEOREM 1.2. We see that if we repeat the proof of Theorem 1.1 under the assumption that $\langle a, b, c\rangle$ represents all the integers in S_{3p}, we arrive at the same conclusion, as $\langle 2, 3, 4\rangle$ and $\langle 2, 3, 7\rangle$ do not represent 17. Since

$$\begin{array}{llll} \langle 1,3,4\rangle & \text{represents} & 3,5,7,17,43 & \text{but not } 2, \\ \langle 1,1,6\rangle & \text{represents} & 2,5,7,17,43 & \text{but not } 3, \\ \langle 1,2,6\rangle & \text{represents} & 2,3,7,17,43 & \text{but not } 5, \\ \langle 1,1,1\rangle & \text{represents} & 2,3,5,17,43 & \text{but not } 7, \\ \langle 2,2,3\rangle & \text{represents} & 2,3,5,7,43 & \text{but not } 17, \\ \langle 2,3,5\rangle & \text{represents} & 2,3,5,7,17 & \text{but not } 43, \end{array}$$

the set S_{3p} is minimal. □

3. Prime-universal forms $\langle a, b, c, d\rangle$

In this section we prove Theorems 1.3, 1.6 and 1.7. We also prove unconditionally that those quadratic forms listed in Table 1 which are not given in (1.1) are prime-universal (see Theorem 3.1).

PROOF OF THEOREM 1.3. Let $a, b, c, d \in \mathbb{N}$ be such that $a \le b \le c \le d$ and suppose that $\langle a, b, c, d\rangle$ is prime-universal. As $\langle a, b, c, d\rangle$ represents 2, by the bounding lemma we have $a \le 2$. As $\langle a, b, c, d\rangle$ represents 3 but $\langle 1\rangle$ and $\langle 2\rangle$ do not, we have $b \le 3$ by the bounding lemma. As $\langle a, b, c, d\rangle$ represents 7 but $\langle 1, 1\rangle$, $\langle 1, 2\rangle$, $\langle 2, 2\rangle$ and $\langle 2, 3\rangle$ do not (and as $\langle 1, 3\rangle$ does not represent 5) we have $c \le 7$ for all choices of a and b. This leaves us to consider the set of forms $\langle 1 \le a \le 2, a \le b \le 3, b \le c \le 7, c \le d\rangle$. We see that $\langle 1, 1, 4, d\rangle, \ldots, \langle 1, 1, 7, d\rangle$ cannot represent 3; $\langle 1, 2, 6, d\rangle, \ldots, \langle 1, 2, 7, d\rangle$ cannot represent 5; $\langle 1, 3, 3, d\rangle, \ldots, \langle 1, 3, 7, d\rangle$ cannot represent 2; $\langle 2, 2, 4, d\rangle, \ldots, \langle 2, 2, 7, d\rangle$ cannot represent 3. This leaves the forms

$$\begin{array}{lllll} \langle 1,1,1,d\rangle, & \langle 1,1,2,d\rangle, & \langle 1,1,3,d\rangle, & & \\ \langle 1,2,2,d\rangle, & \langle 1,2,3,d\rangle, & \langle 1,2,4,d\rangle, & \langle 1,2,5,d\rangle, & \\ \langle 2,2,2,d\rangle, & \langle 2,2,3,d\rangle, & & & \\ \langle 2,3,3,d\rangle, & \langle 2,3,4,d\rangle, & \langle 2,3,5,d\rangle, & \langle 2,3,6,d\rangle, & \langle 2,3,7,d\rangle. \end{array}$$

The prime truant of $\langle 1, 1, 1\rangle$ is 7 and hence we have $d \le 7$ for $\langle 1, 1, 1, d\rangle$. The forms $\langle 1, 1, 1, 1 \le d \le 7\rangle$ are universal [10] and hence are not prime-universal. Similarly, the prime truant of $\langle 1, 2, 2\rangle$ is 7 and the forms $\langle 1, 2, 2, 2 \le d \le 7\rangle$ are also all universal [10]. The remaining forms with $a = 1$, namely,

$$\langle 1,1,2,d\rangle, \quad \langle 1,1,3,d\rangle, \quad \langle 1,2,3,d\rangle, \quad \langle 1,2,4,d\rangle, \quad \langle 1,2,5,d\rangle$$

represent all prime numbers by Theorem 1.1. The conditions on d given in Table 1 for these forms are so that the corresponding forms are not universal [10].

The remaining forms to consider all have $a = 2$ and so they are not universal as they do not represent 1. Observe that if $\langle 2, 2, 2, d\rangle$ is prime-universal then it represents 3

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TABLE 3. Prime truants for certain quaternary quadratic forms $\langle a, b, c, d\rangle$.

a	b	c	d	Prime truant
2	2	3	8	17
2	2	3	11	17
2	2	3	16	17
2	2	3	17	41
2	3	3	3	7
2	3	3	4	13
2	3	3	6	7
2	3	4	4	17
2	3	4	6	23
2	3	4	7	17
2	3	4	10	23
2	3	4	12	13
2	3	4	13	23
2	3	5	7	43
2	3	5	19	43
2	3	5	28	43
2	3	5	34	43
2	3	5	37	43
2	3	5	39	43
2	3	5	42	43
2	3	6	6	7
2	3	6	7	23
2	3	7	7	13
2	3	7	9	13
2	3	7	10	23
2	3	7	11	17
2	3	7	12	13
2	3	7	13	17

and so we must have $d = 3$. But $\langle 2, 2, 2, 3\rangle$ does not represent 17 and thus $\langle 2, 2, 2, d\rangle$ is not prime-universal for any positive integer $d \geq 2$.

For the remaining forms $\langle 2, 2, 3, d\rangle$ and $\langle 2, 3, 3 \leq c \leq 7, d\rangle$ we use Table 2 to determine the prime truant of the given ternary subform, and hence, by the bounding lemma, the possible remaining prime-universal forms are

$$\langle 2, 2, 3, 3 \leq d \leq 17\rangle, \quad \langle 2, 3, 3, 3 \leq d \leq 7\rangle, \quad \langle 2, 3, 4, 4 \leq d \leq 13\rangle,$$
$$\langle 2, 3, 5, 5 \leq d \leq 43\rangle, \quad \langle 2, 3, 6, 6 \leq d \leq 7\rangle, \quad \langle 2, 3, 7, 7 \leq d \leq 13\rangle.$$

From Table 3 we eliminate those forms in the sets above which contain a prime truant. The remaining forms agree with those given in Table 1. □

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TABLE 4. All positive integers not represented by certain $ax^2 + by^2 + cz^2$.

a	b	c	Integers not represented
2	2	3	$8n + 1, 9^k(9n + 6)$
2	3	3	$9^k(3n + 1)$
2	3	6	$3n + 1, 4^k(8n + 7)$
2	3	8	$8n \pm 1, 32n + 4, 9^k(9n + 6)$
2	3	9	$3n + 1, 9n + 6, 4^k(16n + 10)$
2	3	12	$16n + 6, 9^k(3n + 1)$
2	3	18	$3n + 1, 8n + 1, 9^k(9n + 6)$
2	5	6	$9^k(9n + 3), 25^k(25n \pm 10), 4^k(8n + 1)$
2	5	10	$8n + 3, 25^k(5n \pm 1)$
2	5	15	$9^k(9n + 3), 4^k(16n + 10), 25^k(5n \pm 1)$

PROOF OF THEOREM 1.6. The forms with $a = 1$ in Table 1 are not universal [10] and represent all primes by Theorem 1.1 as they have a prime-universal ternary subform. There are 50 forms with $a = 2$. None of these represent 1 so they are not universal. Of these, 23 have $\langle 2, 2, 3\rangle$, $\langle 2, 3, 3\rangle$, $\langle 2, 3, 6\rangle$, $\langle 2, 3, 8\rangle$, $\langle 2, 3, 9\rangle$, $\langle 2, 3, 12\rangle$, $\langle 2, 3, 18\rangle$, $\langle 2, 5, 6\rangle$, $\langle 2, 5, 10\rangle$ or $\langle 2, 5, 15\rangle$ as a ternary subform. These 10 ternaries are all regular and so the positive integers that they do not represent belong to a finite number of progressions of positive integers of the type $A^k(B\ell + C)$. These progressions, which are taken from Dickson's table [2, pages 112–113], are given in Table 4.

Using this in conjunction with a technique of Ramanujan [10] allows us to prove that these 23 quaternaries are prime-universal (see Theorem 3.1). The remaining 27 quaternaries have either the nonregular ternary $\langle 2, 3, 4\rangle$ or the nonregular ternary $\langle 2, 3, 5\rangle$ as a ternary section. Assuming the validity of Conjectures 1.4 and 1.5, we deduce that the remaining quaternaries are prime-universal. This proves Theorem 1.6. □

THEOREM 3.1. *The following 23 quadratic forms are prime-universal:*

(i) $\langle 2, 2, 3, h\rangle$, $h \in \{3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15\}$;
(ii) $\langle 2, 3, 3, h\rangle$, $h \in \{5, 7\}$;
(iii) $\langle 2, 3, h, 8\rangle$, $h \in \{4, 5, 7\}$;
(iv) $\langle 2, 3, h, 9\rangle$, $h \in \{4, 5\}$;
(v) $\langle 2, 3, 5, 6\rangle$;
(vi) $\langle 2, 3, 5, 10\rangle$;
(vii) $\langle 2, 3, 5, 12\rangle$;
(viii) $\langle 2, 3, 5, 15\rangle$;
(ix) $\langle 2, 3, 5, 18\rangle$.

For the various quaternary quadratic forms in Theorem 3.1, the method of proof is the same, only the values change. We describe the method in general. Suppose

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$\langle a, b, c, d\rangle$ is a quaternary quadratic form we wish to show is prime-universal and which also has a regular ternary subform, say $\langle a, b, c\rangle$. We use Table 4 to determine all positive integers represented by $\langle a, b, c\rangle$. This is given as a list of progressions such that $\langle a, b, c\rangle$ can represent all positive integers which do not belong to the corresponding progressions. For all sufficiently large primes p not represented by $\langle a, b, c\rangle$, we can determine a positive integer m using the properties of the given progressions, such that

$$p - dm^2 \text{ is represented by } \langle a, b, c\rangle,$$

and hence it follows that p is represented by $\langle a, b, c, d\rangle$. We choose p sufficiently large so that $p - dm^2$ is a positive integer, and for the remaining small primes not represented by $\langle a, b, c\rangle$ it can be shown that they are represented by $\langle a, b, c, d\rangle$.

We give complete details for part (i) of Theorem 3.1. For the remaining parts, we state only the conditions on the prime p, the lower bound for p, the relevant ternary subform and the corresponding positive integer m.

Proof of Theorem 3.1.

Part (i). Fix $h \in \{3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15\}$. It can be shown that $\langle 2, 2, 3, h\rangle$ represents every prime $p \le 9h$. Let p be a prime with $p > 9h$. From Table 4 we see that $\langle 2, 2, 3\rangle$ represents all positive integers not of the forms $8\ell + 1$ and $3^{2k}(9\ell + 6)$ $(k, \ell \in \mathbb{N}_0)$. Thus $\langle 2, 2, 3\rangle$ represents all primes not of the form $8\ell + 1$, and hence so does $\langle 2, 2, 3, h\rangle$. Now suppose $p \equiv 1 \pmod 8$. As $h \not\equiv 0 \pmod 8$ it follows that the positive integer $p - 9h$ satisfies $p - 9h \equiv 1 - h \not\equiv 1 \pmod 8$. Further, as $p \not\equiv 0 \pmod 3$ it follows that $p - 9h \ne 3^{2k}(9\ell + 6)$ for any choice of $k, \ell \in \mathbb{N}_0$. Thus there exist integers x, y and z such that

$$p - 9h = 2x^2 + 2y^2 + 3z^2, \quad \text{and so } p = 2x^2 + 2y^2 + 3z^2 + h3^2,$$

proving that $\langle 2, 2, 3, h\rangle$ is prime-universal for every h in the given set.

Part (ii). Let $h \in \{5, 7\}$. We use the general method with the ternary subform $\langle 2, 3, 3\rangle$, $p \equiv 1 \pmod 3$, $p > 28$ and

$$m = \begin{cases} 1 & \text{if } h = 5, \\ 1 & \text{if } h = 7, p \equiv 1, 4 \pmod 9, \\ 2 & \text{if } h = 7, p \equiv 7 \pmod 9. \end{cases}$$

Part (iii). Let $h \in \{4, 5, 7\}$. We use the general method with the ternary subform $\langle 2, 3, 8\rangle$, $p \equiv \pm 1 \pmod 8$, $p > 180$ and

$$m = \begin{cases} 3 & \text{if } h = 4, 7, \\ 6 & \text{if } h = 5. \end{cases}$$

Part (iv). Let $h \in \{4, 5\}$. We use the general method with the ternary subform $\langle 2, 3, 9\rangle$, $p \equiv 1 \pmod 3$, $p > 20$ and

$$m = \begin{cases} 2 & \text{if } h = 4, p \equiv 1 \pmod 9, \\ 1 & \text{if } h = 4, p \equiv 4, 7 \pmod 9, \\ 2 & \text{if } h = 5. \end{cases}$$

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Part (v). We use the general method with the ternary subform $\langle 2,5,6\rangle$, $p \equiv 1 \pmod 8$, $p > 300$ and $m = 10$.

Part (vi). We use the general method with the ternary subform $\langle 2,5,10\rangle$, at least one of $p \equiv 3 \pmod 8$ or $p \equiv \pm 1 \pmod 5$, $p > 192$ and

$$m = \begin{cases} 1 & \text{if } p \equiv 3 \pmod 8, p \equiv 1 \pmod 5, \\ 2 & \text{if } p \equiv 3 \pmod 8, p \equiv -1 \pmod 5, \\ 5 & \text{if } p \equiv 3 \pmod 8, p \not\equiv \pm 1 \pmod 5, \\ 4 & \text{if } p \not\equiv 3 \pmod 8, p \equiv 1 \pmod 5, \\ 8 & \text{if } p \not\equiv 3 \pmod 8, p \equiv -1 \pmod 5. \end{cases}$$

Part (vii). We use the general method with the ternary subform $\langle 2,3,12\rangle$, $p \equiv 1 \pmod 3$, $p > 80$ and $m = 4$.

Part (viii). We use the general method with the ternary subform $\langle 2,5,15\rangle$, $p \equiv 1 \pmod 5$, $p > 108$ and

$$m = \begin{cases} 6 & \text{if } p \equiv 1 \pmod 5, \\ 2 & \text{if } p \equiv -1 \pmod 5. \end{cases}$$

Part (ix). We use the general method with the ternary subform $\langle 2,3,18\rangle$, at least one of $p \equiv 1 \pmod 3$ or $p \equiv 1 \pmod 8$, $p > 180$ and

$$m = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 3, p \equiv 1 \pmod 8, \\ 3 & \text{if } p \equiv 2 \pmod 3, p \equiv 1 \pmod 8, \\ 4 & \text{if } p \equiv 1 \pmod 3, p \not\equiv 1 \pmod 8. \end{cases} \qquad \square$$

Proof of Theorem 1.7. Suppose that $\langle a,b,c,d\rangle$ represents every prime in S_{4p} and not every integer in S_{4u}. Then $\langle a,b,c,d\rangle$ does not represent every integer in $\{1,2,3,5,6,7,10,14,15\}$ and so is not universal by the fifteen theorem (see [1]). Proceeding as in the proof of Theorem 1.6, universal forms are eliminated and the representability of every prime in S_{4p} by $\langle a,b,c,d\rangle$ leads to the set of forms in Table 1. Moreover, this set is minimal as

$$\begin{array}{llll}
\langle 1,3,3,4\rangle & \text{represents} & 3,5,7,13,17,23,41,43 & \text{but not } 2, \\
\langle 2,4,5,6\rangle & \text{represents} & 2,5,7,13,17,23,41,43 & \text{but not } 3, \\
\langle 1,2,6,6\rangle & \text{represents} & 2,3,7,13,17,23,41,43 & \text{but not } 5, \\
\langle 2,3,3,8\rangle & \text{represents} & 2,3,5,13,17,23,41,43 & \text{but not } 7, \\
\langle 2,3,3,4\rangle & \text{represents} & 2,3,5,7,17,23,41,43 & \text{but not } 13, \\
\langle 2,3,4,4\rangle & \text{represents} & 2,3,5,7,13,23,41,43 & \text{but not } 17, \\
\langle 2,3,4,6\rangle & \text{represents} & 2,3,5,7,13,17,41,43 & \text{but not } 23, \\
\langle 2,2,3,17\rangle & \text{represents} & 2,3,5,7,13,17,23,43 & \text{but not } 41, \\
\langle 2,3,5,42\rangle & \text{represents} & 2,3,5,7,13,17,23,41 & \text{but not } 43.
\end{array} \qquad \square$$

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4. Final remarks

Let $q(x_1, x_2, x_3)$ be a positive-definite ternary quadratic form with integral coefficients and set

$$D = \det\left(\frac{\partial^2 q}{\partial x_i \partial x_j}\right).$$

Duke [3] has shown that if n is a positive squarefree integer for which the congruence $q(x_1, x_2, x_3) \equiv n \pmod{8D^3}$ has a solution, then $q(x_1, x_2, x_3) = n$ is solvable in integers x_1, x_2, x_3 if $n > cD^{337}$, for some positive constant c. This theorem enables us to deduce that $2x^2 + 3y^2 + 4z^2$ represents all sufficiently large odd integers n and that $2x^2 + 3y^2 + 5z^2$ represents all sufficiently large primes p. Unfortunately, the size of 'sufficiently large' is out of reach!

Bhargava announced that he had proved that for any infinite set of positive integers T, there is a finite subset $S \subseteq T$ such that if an integral positive-definite quadratic form represents all integers in S, then this form represents all integers in T (see [8]). This was proved in more generality by Kim *et al.* [7]. Bhargava asserted (see [5, Theorem C, page 674]) that if T is the set of prime numbers, then the corresponding smaller finite set S is given by

$$S = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 67, 73\},$$

but no proof of this has appeared in the literature to the authors' knowledge. Should this set S be proven to be correct, then it is straightforward to verify that the unproven forms in Table 1 represent all primes in S, and Theorems 1.6 and 1.7 would be unconditional.

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