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On finite soluble groups and the fixed-point groups of automorphisms

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Let p denote a prime, G a finite soluble p'-group and A an elementary abelian p-group of operators on G. Suppose that A has order p^4 and that if $\omega \in A^{\#}$ then $C_G(\omega)$ has nilpotent derived group. Then G has nilpotent derived group.

A number of results have been obtained relating the structure of a finite group to the structure of the fixed-point groups of certain automorphisms of the group. The results so far obtained fall into two classes: those which deduce from the hypotheses that the group is soluble and those which assume solubility and then give more detailed information about the structure of the group. It is to this second group of results that the following theorem belongs.

THEOREM. *Let p denote a prime, G a finite soluble p'-group and A an elementary abelian p-group of operators on G. Suppose that A has order p^4 and that if $\omega \in A^{\#}$ then $C_G(\omega)$ has nilpotent derived group. Then the derived group of G is nilpotent.*

This result differs from earlier results of the same kind in that we have assumed neither the nilpotence of $C_G(\omega)$ for any automorphism ω nor the nilpotence of $C_G(A)$.

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It is also in a sense the best possible theorem in that if the order of A is a proper divisor of p^4, the other assumptions being satisfied, then the conclusion is not necessarily valid. This is shown by means of an example at the end of the paper.

Other results of this kind may be found in [2], [3] and the references given in those papers. We use the notation of [1].

Proof. The theorem is proved by induction on the order of G. Thus we may assume that if H and K are A-subgroups of G, $H \triangleright K$, and either $G \neq H$ or $K \neq 1$ then $(H/K)'$ is nilpotent. We also suppose that G' is not nilpotent. Thus by [2], Lemma 2, the Fitting subgroup $F = F(G)$ is the unique minimal normal A-subgroup of G. F is an elementary abelian r-group for some prime r.

First assume that G/F is nilpotent. By induction, any proper A-subgroup of G/F is abelian. Thus we may assume that G/F is a q-group for some prime q, and $D = (G/F)/\Phi(G/F)$ is the sum of at most two components which are irreducible under A. Since A is abelian, it follows that for any irreducible A-component D_1 of D, $C_{D_1}(\omega) = 1$ or D_1 for each $\omega \in A^\#$. Hence $AD_1/C_A(D_1)$ is a Frobenius group with complement $A/C_A(D_1)$. It follows that $A/C_A(D_1)$ is cyclic. Hence if $B = C_A(D)$ then $|B| \geq p^2$. If $\omega \in B^\#$ then $G = FC_G(\omega)$. Hence as F is abelian, $C_F(\omega)$ is a normal A-subgroup of G for $\omega \in B^\#$. Since F is the unique minimal normal A-subgroup of G, for each $\omega \in B^\#$ we have $C_F(\omega) = 1$ or F. As above we conclude that $|C_B(F)|$ has order at least p. Let $\omega \in C_B(F)^\#$. Then $G = C_G(\omega)$ and hence G' is nilpotent. Since we have assumed G' is not nilpotent, this contradiction shows that G/F is not nilpotent.

By induction $F_2(G)/F$ is abelian and $G/F_2(G)$ is elementary abelian and irreducible under A. Let q and s denote distinct primes such that $|G : F_2(G)|$ is a power of q and s is a divisor of $|F_2(G) : F|$. If S denotes a Sylow s-subgroup of G invariant under A, then $N = N_G(S)$ is an A-subgroup of G complementing F. We can choose a minimal A-invariant q-subgroup Q of N covering $G/F_2(G)$. Since G/F is not nilpotent we can suppose that Q does not centralize S . Now if either Q is not a Sylow q-subgroup of N or S does not cover $F_2(G)/F$ then QS is a proper A-subgroup of N . By induction QS is abelian, a contradiction. Hence $QS = N$. If $S/\Phi(S)$ is not irreducible under the action of AQ then, by Maschke's Theorem, we can find proper AQ-invariant subgroups S_1 and S_2 of S such that $S = S_1S_2$. By induction Q centralizes S_1 and S_2 and hence S - again a contradiction. Thus $S/\Phi(S)$ is irreducible under the action of AQ . Since $[S, Q]$ is normalized by AQ , $S = [S, Q]$.

If $\omega \in A^{\#}$ then $C_Q(\omega)$ is normalized by A and hence, as A acts irreducibly on $G/F_2(G)$, $C_Q(\omega) = 1$ or Q . Now the argument used earlier shows that $B_1 = C_A(Q)$ has order at least p^3 . Similarly $B_2 = C_{B_1}(S)$ has order at least p^2 and $B = C_{B_2}(F)$ has order at least p . Now if $\omega \in B^{\#}$ then $G = C_G(\omega)$. But this is a contradiction as before.

This proves the theorem.

EXAMPLE. Let p, q, r, s be distinct primes such that p is a divisor of $q-1$, $r-1$ and $s-1$. Then we can find solutions α, β , distinct from 1 , of the equations $x^p \equiv 1 \pmod{r}$ and $x^p \equiv 1 \pmod{s}$ respectively.

Let X_1 denote the wreath product of a non abelian group of order pq with a cyclic group of order r . Then X_1 has a normal subgroup Y_1 with a complement Z_1 which is non abelian of order pq . We define an automorphism ϕ of X_1 by letting $y^\phi = y^\alpha$ for $y \in Y_1$ and $z^\phi = z$ for $z \in Z_1$. ϕ has order p . Let U_1 denote the splitting extension of X_1 by $\langle\phi\rangle$.

We now repeat this construction starting with U_1 instead of a non cyclic group of order pq and using s and β instead of r and α respectively. This yields a group U_2 which has a normal subgroup G of index p^3 . A Sylow p-subgroup A of U_2 complements G . It is easy to check that G, A and p satisfy the conditions of the theorem, except that $|A| = p^3$. It is also easily seen that G' is not nilpotent.

NOTE. In this example $G \neq F_2(G)$. It is also possible to construct examples where $G = F_2(G)$ but which are similar to this example in the other relevant details.

References

[1] Daniel Gorenstein, *Finite groups* (Harper and Row, New York, Evanston, London, 1968).

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