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CONTINUOUS SELF-MAPS OF THE CIRCLE

J. SCHAER

ABSTRACT. Given a continuous map δ from the circle S to itself we want to find all self-maps $\sigma\colon S \longrightarrow S$ for which $\delta \circ \sigma = \delta$. If the degree r of δ is not zero, the transformations σ form a subgroup of the cyclic group C_r. If $r = 0$, all such invertible transformations form a group isomorphic either to a cyclic group C_n or to a dihedral group D_n depending on whether all such transformations are orientation preserving or not. Applied to the tangent image of planar closed curves, this generalizes a result of Bisztriczky and Rival [1]. The proof rests on the theorem: *Let* $\Delta\colon \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous, nowhere constant, and* $\lim_{x\to-\infty} \Delta(x) = -\infty$, $\lim_{x\to+\infty} \Delta(x) = +\infty$*; then the only continuous map* $\Sigma\colon \mathbb{R} \longrightarrow \mathbb{R}$ *such that* $\Delta \circ \Sigma = \Delta$ *is the identity* $\Sigma = \mathrm{id}_{\mathbb{R}}$.

0. **Introduction.** In [1] Bisztriczky and Rival prove that a simple smooth closed curve admits at most two continuous maps to itself that preserve the direction of the tangent. Actually the authors assume further that the curve have no cusps and be of bounded finite order (*i.e.*, there exists a number n such that the intersection of any line with the curve contains at most n points); besides ruling out accumulation points of inflection points, this also eliminates trivial counterexamples where the curve contains a line segment.

Here we shall show that the conditions that the curve be simple, have no cusps, and be of bounded finite order, are not essential. The problem is rather of an algebraic topological nature. It is concerned with the tangent image of the curve and is addressed in Section 2. To prepare the ground for the proof of the main Theorems (3, 4, and 5) some properties of certain continuous functions on the reals have to be established first; this is done in Section 1. In Section 3, finally, the results are applied to the original geometrical questions, and the result of [1] appears in a more general setting.

1. The Analytical Background.

THEOREM 1. *Let* $\Delta\colon \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous, nowhere constant,* $\lim_{x\to-\infty} \Delta(x) = -\infty$ *and* $\lim_{x\to+\infty} \Delta(x) = +\infty$. *Let* $\Sigma\colon \mathbb{R} \longrightarrow \mathbb{R}$ *be continuous and such that*

$$\Delta \circ \Sigma = \Delta. \tag{1}$$

Then $\Sigma = \mathrm{id}_{\mathbb{R}}$.

PROOF. Let $c \in \mathbb{R}$. Then $\Delta^{-1}(c)$ is compact and contains no intervals. It suffices to show that $\Sigma(x) = x$ for all $x \in \Delta^{-1}(c)$. Replacing $\Delta(z)$ by $\Delta(z) - c$, we may assume $c = 0$.

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Suppose we could prove that

$$\Sigma(x) \leq x \text{ for all } x \in \Delta^{-1}(0). \tag{2}$$

Replacing $\Delta(z)$, $\Sigma(z)$ by $-\Delta(-z)$, $-\Sigma(-z)$, respectively, (2) yields the opposite inequality, and thus $\Sigma(x) = x$ would follow for all $x \in \Delta^{-1}(0)$.

We show that $\Sigma(m) = m$ where $m = \min \Delta^{-1}(0)$. Thus $\Delta(z) < 0$ for $z < m$ and $\Delta(-\infty, m) = (-\infty, 0)$. By (1), $\Delta\big(\Sigma(m)\big) = \Delta(m) = 0$. Hence $\Sigma(m) \in \Delta^{-1}(0)$ and thus $\Sigma(m) \geq m$. On the other hand (1) and our assumptions imply that $\lim_{z\to-\infty} \Sigma(z) = -\infty$. Suppose $\Sigma(z) > m$ for some $z < m$. Then $\Sigma(x) = m$ for some $x \leq z < m$, and hence $\Delta(x) = 0$ by (1), contradicting the definition of m. Hence $\Sigma(z) < m$ for all $z < m$, and therefore $\Sigma(m) = m$.

Now suppose $x \in \Delta^{-1}(0)$, $x \neq m$. Let $0 < \epsilon < \max_{m\leq z\leq x} |\Delta(z)|$, and put $x_0 = m$.

Let

$$y_1 = \inf\{z \in (x_0, x) \mid |\Delta(z)| \geq \epsilon\}; \text{ of course } |\Delta(y_1)| = \epsilon.$$
$$x_1 = \inf\{z \in (y_1, x] \mid \Delta(z) = 0\}; \ \Delta(x_1) = 0.$$

Further, as long as the sets in question are not empty,

$$y_i = \inf\{z \in (x_{i-1}, x) \mid |\Delta(z)| \geq \epsilon\}; \ |\Delta(y_i)| = \epsilon.$$
$$x_i = \inf\{z \in (y_i, x] \mid \Delta(z) = 0\}; \ \Delta(x_i) = 0.$$

If there were infinitely many y_i, they would form an increasing bounded sequence and hence have a limit y. Since $|\Delta(y_i)| = \epsilon$, by continuity $|\Delta(y)| = \epsilon$. But $y_i < x_i < y_{i+1}$, and so $\lim_{i\to\infty} x_i = y$ also. Since $\Delta(x_i) = 0$, by continuity$\Delta(y) = 0$, a contradiction. So the sequence $m = x_0 < y_1 < x_1 < y_2 < \cdots$ is finite. Thus, there is a natural number k, such that $y_k < x_k \leq x$ and $|\Delta(z)| < \epsilon$ for $x_k \leq z \leq x$. Let $i \in \{1, \ldots, k\}$, and $z \in (x_{i-1}, y_i)$. Suppose $\Sigma(z) = y_i$; then $|\Delta(z)| = |\Delta\big(\Sigma(z)\big)| = |\Delta(y_i)| = \epsilon$. This contradicts the definitions of x_{i-1} and y_i. Hence

$$\Sigma(z) \neq y_1 \text{ for } x_{i-1} < z < y_i \qquad i = 1, 2, \ldots, k.$$

Similarly

$$\Sigma(z) \neq x_i \text{ for } y_i < z < x_i \qquad i = 1, 2, \ldots, k.$$

As $\Sigma(x_0) = x_0$, these relations yield consecutively

$$\Sigma(y_1) \leq y_1, \Sigma(x_1) \leq x_1, \Sigma(y_2) \leq y_2, \ldots, \Sigma(x_k) \leq x_k.$$

Of course, x_i, y_i depend on ϵ, and so does k. Let us write $x_i(\epsilon)$, $k(\epsilon)$. As Δ is nowhere constant $\lim_{\epsilon\to 0} x_{k(\epsilon)}(\epsilon) = x$. Hence by continuity

$$\Sigma(x) \leq x. \qquad \blacksquare$$

THEOREM 2. *Let $\Delta\colon \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, nowhere constant and periodic. Then all continuous invertible transformations $\Sigma\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that $\Delta \circ \Sigma = \Delta$ form a group*

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$\bar{G}$ which is isomorphic to either the infinite cyclic group C_∞ or the infinite dihedral group $D = C_2 \times C_\infty$, depending on whether $\bar{G}$ does not or does contain decreasing transformations.

PROOF. Without loss of generality, we may assume that the period of Δ is 1:

$$\Delta(x+1) = \Delta(x). \tag{3}$$

By continuity, Δ attains a minimum m and a maximum M in each period. Since Δ is not constant, $m < M$. $\Delta^{-1}\big((m,M)\big)$ is open, and hence is the union of disjoint open intervals (u,v). There are such intervals with $\Delta(u) = m$ and $\Delta(v) = M$ ("ascending"), and such intervals with $\Delta(u) = M$ and $\Delta(v) = m$ ("descending"). If $\Delta(a) = m$ and $\Delta(b') = M$ then by (3) there exists $b > a$ such that $\Delta(b) = M$. Thus $\Delta([a,b]) = [m,M]$. Let $a_0 = \sup\{x \mid x < b, \Delta([x,b]) = [m,M]\}$ and $b_0 = \inf\{x \mid x > a_0, \Delta([a_0,x]) = [m,M]\}$. Then $(a_0,b_0) \subseteq \Delta^{-1}\big((m,M)\big)$, and $\Delta(a_0) = m$, $\Delta(b_0) = M$. The existence of descending intervals is proved similarly.

Let (a_0,b_0) be an ascending interval. Then there are only finitely many ascending intervals in $[a_0,a_0+1]$, and hence in any interval of length 1. Indeed, if there were infinitely many ascending intervals (a_0,b_0), (a_1,b_1), $(a_2,b_2),\ldots$ with $a_0 < a_1 < a_2 \cdots$ then the sequence $a_0,a_1,a_2,\ldots$ would be increasing and bounded above by a_0+1, hence convergent with limit a, say. Since $\Delta(a_i) = m$ and Δ is continuous, $\Delta(a) = m$. But $a_0 < b_0 < a_1 < b_1 < a_2 < b_2 < \cdots$, so the sequence $b_0,b_1,b_2,\ldots$ would also converge to a. Since $\Delta(b_i) = M$ and $\Delta(a) = M$, we have a contradiction.

So let $(a_0,b_0),(a_1,b_1),\ldots,(a_{k-1},b_{k-1})$ be the ascending intervals in $[a_0,a_0+1]$, $k \geq 1$, with $a_0 < a_1 < \cdots < a_{k-1}$. For convenience define a_i, b_i for all $i \in Z$ by

$$a_{i+k} = a_i + 1, b_{i+k} = b_i + 1.$$

In particular, $a_k = a_0 + 1$.

Between any two consecutive ascending intervals (a_i,b_i), (a_{i+1},b_{i+1}) there is a descending interval (c_i,d_i). In fact d_i may be characterized by

$$d_i = \inf\{x \mid x > b_i, \Delta(x) = m\}. \tag{4}$$

Similarly, between any two consecutive descending intervals (c_i,d_i), (c_{i+1},d_{i+1}) there is the ascending interval (a_{i+1},b_{i+1}).

It is clear that all invertible transformations Σ satisfying (1) form a group $\bar{G}$. Since every $\Sigma \in \bar{G}$ is one-to-one and onto, it is either strictly increasing or strictly decreasing. Consider first the set $\bar{H}$ of all increasing transformations Σ in $\bar{G}$. $\bar{H}$ is a subgroup of $\bar{G}$.

Let $\Sigma \in \bar{H}$. Then Σ maps every ascending interval onto an ascending interval. Hence $\Sigma(a_0) = a_e$, and $\Sigma(b_0) = b_e$ for some integer $e \in Z$. d_i is characterized by (4), *i.e.*, $\Delta\big([b_i,d_i)\big) = (m,M]$ and $\Delta([b_i,d_i]) = [m,M]$. $\Delta\Big(\big[b_e,\Sigma(d_0)\big)\Big) = \Delta\Big(\big[\Sigma(b_0),\Sigma(d_0)\big)\Big) = \Delta\circ\Sigma\big([b_0,d_0)\big) = \Delta\big([b_0,d_0)\big) = (m,M]$ and $\Delta\Big(\big[b_e,\Sigma(d_0)\big]\Big) = \Delta\Big(\big[\Sigma(b_0),\Sigma(d_0)\big]\Big) = \Delta\circ\Sigma([b_0,d_0]) = \Delta([b_0,d_0]) = [m,M]$. Hence, by (4), $\Sigma(d_0) = d_e$, and therefore $\Sigma(c_0) = c_e$.

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Similarly we can prove successively $\Sigma(b_1) = b_{e+1}$, $\Sigma(a_1) = a_{e+1}$, and $\Sigma(d_1) = d_{e+1}$, $\Sigma(c_1) = c_{e+1}$, *etc*. In general,

(5) $$\Sigma(a_i) = a_{e+i},\ \Sigma(b_i) = b_{e+i},\ \text{and}\ \Sigma(c_i) = c_{e+i},\ \Sigma(d_i) = d_{e+i}.$$

In particular, $\Sigma(a_0 + 1) = \Sigma(a_k) = a_{e+k} = a_e + 1$, $\Sigma(b_k) = b_e + 1$, $\Sigma(c_k) = c_e + 1$ and $\Sigma(d_k) = d_e + 1$.

LEMMA. *There is at most one $\Sigma \in \bar{H}$ such that $\Sigma(a_0) = a_e$.*

PROOF. Let $\Sigma_1(a_0) = \Sigma_2(a_0)$. Then by (5), $\Sigma_1(a_i) = \Sigma_2(a_i)$ for all i. Let $T = \Sigma_2^{-1} \circ \Sigma_1$. Then $T(a_i) = a_i$ for all i. For any $x_0 \in \mathbb{R}$ there exist a_h, a_j such that $a_h \leq x_0 \leq a_j$. Consider $\bar{\Delta}\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\bar{\Delta}(x) = \begin{cases} m + x - a_h & \text{for } x < a_h \\ \Delta(x) & \text{for } a_h \leq x \leq a_j \\ m + x - a_j & \text{for } x > a_j \end{cases}$$

$\bar{\Delta}$ is continuous, nowhere constant, $\lim_{x\to-\infty} \bar{\Delta}(x) = -\infty$ and $\lim_{x\to+\infty} \bar{\Delta}(x) = +\infty$. Let

$$\bar{T}(x) = \begin{cases} x & \text{for } x < a_h \text{ or } x > a_j \\ T(x) & \text{for } a_h \leq x \leq a_j. \end{cases}$$

Since $T(a_h) = a_h$, $T(a_j) = a_j$ and T is increasing, we have for $x \in [a_h, a_j]$ that $\bar{T}(x) = T(x) \in [a_h, a_j]$. Hence $\bar{\Delta} \circ \bar{T} = \bar{\Delta}$. Also, $\bar{T}$ is continuous. So by Theorem 1, $\bar{T} = \mathrm{id}_{\mathbb{R}}$. In particular $T(x_0) = \bar{T}(x_0) = x_0$. Hence $\Sigma_2^{-1} \circ \Sigma_1(x_0) = T(x_0) = x_0$, $\Sigma_1(x_0) = \Sigma_2(x_0)$ for all $x_0 \in \mathbb{R}$, and the Lemma is proved. ∎

PROOF OF THEOREM 2 (CONTINUED). Let $I = \{e \in Z \mid \exists \Sigma \in \bar{H}$ such that $\Sigma(a_0) = a_e\}$. For $i \in I$ denote by Σ_i, the unique Σ for which $\Sigma(a_0) = a_i$. Since (5) holds for Σ_e, we have $\Sigma_e \circ \Sigma_i(a_0) = \Sigma_e(a_i) = a_{e+i} = \Sigma_{e+i}(a_0)$, and hence, by the Lemma again,

$$\Sigma_e \circ \Sigma_i = \Sigma_{e+i} \text{ for all } e, i \in I.$$

So $\bar{H}$ is a subgroup of the infinite cyclic group of Z under addition, and is therefore cyclic itself. $\bar{H}$ is infinite, since it contains at least the integral shifts $x \longrightarrow x + \ell$ ($\ell \in Z$). If $\bar{G}$ contains only increasing transformations, then $\bar{G} = \bar{H}$ which is cyclic. If $\bar{G}$ contains a decreasing transformation T, then T has a fixed point u. By arguments like the ones used for increasing transformations, in particular (5) and the Lemma, we obtain: T maps every ascending interval into a descending interval and vice-versa. Thus, u is not in an ascending nor in a descending interval. The first ascending interval above u is mapped into the first descending interval below u and vice-versa; the first descending interval above u is mapped into the first ascending interval below u and vice-versa. Successively, the second ascending interval above u is mapped into the second descending interval below u and vice-versa, *etc*. By these images T is uniquely determined. Now T^{-1} has the same properties; thus $T^{-1} = T$, and $T^2 = \mathrm{id}_{\mathbb{R}}$.

Let $T_0\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a fixed decreasing transformation. Then for $\Sigma \in \bar{H}$, $T_0 \circ \Sigma$ is also decreasing, and all decreasing transformations of $\bar{G}$ are of this form. If T is another

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decreasing transformation then $T_0 \circ T$ is increasing; hence $T_0 \circ T = \Sigma$ for some $\Sigma \in \bar{H}$. Thus, $T = T_0^{-1} \circ \Sigma = T_0 \circ \Sigma$.

We identify the group structure of $\bar{G}$ which as a set equals $\{\mathrm{id}_R, T_0\} \times \bar{H}$. Let Σ be a generator of $\bar{H}$. Then $T_0^{-1}\Sigma T_0$ is increasing and thus equals Σ^k for some $k \in Z$. Since $T_0^{-1}\Sigma T_0$ maps the first ascending interval above u to the first ascending interval below u, it follows that $T_0^{-1}\Sigma T_0 = \Sigma^{-1}$.

Therefore $\bar{G}$ is isomorphic to the infinite dihedral group $\{\mathrm{id}_{\mathbb{R}}, T_0\} \times \bar{H}$. ∎

2. **The topological core.**

THEOREM 3. *Let $\delta: S \longrightarrow S$ be a continuous mapping from the circle to itself, which is nowhere constant and has degree 1. Then the only continuous transformation $\sigma: S \longrightarrow S$ such that*

$$\delta \circ \sigma = \delta \tag{6}$$

is the identity $\sigma = \mathrm{id}_S$.

PROOF. Consider $S = \{z \mid |z| = 1\}$ in the complex plane. Let $\mathbb{R}$ denote the real line and $\pi: \mathbb{R} \longrightarrow S$ be given by $\pi(x) = e^{2\pi i x}$. Then $(\mathbb{R}, \pi)$ is a universal covering of S. For any continuous map $\varphi: S \longrightarrow S$ we define $\bar{\varphi}: \mathbb{R} \longrightarrow S$ by $\bar{\varphi} = \varphi \circ \pi$. $\bar{\varphi}$ is periodic, $\bar{\varphi}(x+1) = \bar{\varphi}(x)$, and $\bar{\varphi}$ can be lifted to a continuous map $\Phi: \mathbb{R} \longrightarrow \mathbb{R}$ such that $\pi \circ \Phi = \bar{\varphi} = \varphi \circ \pi$ (see [2], p. 342). The degree of φ is the integral constant $\Phi(x+1) - \Phi(x)$ $(x \in \mathbb{R})$. If Φ_0 is such a lifting, then all other liftings are of the form $\Phi(x) = \Phi_0(x) + n$, where n is a fixed arbitrary integer. We shall say Φ represents φ and write

$$\rho(\Phi) = \varphi. \tag{7}$$

ρ is a mapping from the set of all representing functions on $\mathbb{R}$ onto the set of continuous functions on S.

Lifting δ and σ yields maps $\Delta \in \rho^{-1}(\delta)$ and $\Sigma \in \rho^{-1}(\sigma)$ such that

$$\pi \circ \Delta = \delta \circ \pi \text{ and } \pi \circ \Sigma = \sigma \circ \pi.$$

The degree of δ is 1, so

$$\Delta(x+1) - \Delta(x) = 1 \tag{8}$$

Condition (6) is equivalent to $\Delta\big(\Sigma(x)\big) = \Delta(x) + n$ for some integral constant n. Replacing $\Sigma(x)$ by $\Sigma(x) + n$, we have that $n = 0$ by condition (8). Thus $\Delta \circ \Sigma = \Delta$. So Δ and Σ satisfy the conditions of Theorem 1. Hence $\Sigma = \mathrm{id}_{\mathbb{R}}$ and $\sigma = \rho(\Sigma) = \mathrm{id}_S$. ∎

THEOREM 4. *Let $\delta: S \longrightarrow S$ be a continuous mapping of degree $r \geq 1$ that is nowhere constant. Then there are at most r continuous mappings $\sigma: S \longrightarrow S$ such that*

$$\delta \circ \sigma = \delta. \tag{9}$$

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Each such σ satisfied $\sigma^r = \mathrm{id}_S$. The collection of all such σ form a cyclic group G whose order is a divisor of r.

PROOF. The case $r = 1$ is given by Theorem 3. So let $r > 1$. As in the proof of Theorem 3, we shall represent the mappings δ, $\sigma\colon S \longrightarrow S$ by the corresponding liftings Δ, $\Sigma\colon \mathbb{R} \longrightarrow \mathbb{R}$. The condition (9) translates into

$$\Delta\big(\Sigma(x)\big) = \Delta(x) + k \tag{10}$$

for some integral constant k. We define a map κ from the set G of all solutions of (9) to Z_r which associates with every solution σ of (9) the value of $k = \kappa(\sigma)$ modulo r.

This mapping κ is well-defined. If Δ_1 is another representative of δ, then $\Delta_1(x) = \Delta(x) + n$ and $\Delta_1\big(\Sigma(x)\big) - \Delta_1(x) = \Delta\big(\Sigma(x)\big) - \Delta(x)$. If Σ_1 is another representative of σ, then $\Sigma_1(x) = \Sigma(x) + n$. Now (8) is replaced by

$$\Delta(x+1) - \Delta(x) = r.$$

Therefore, $\Delta\big(\Sigma_1(x)\big) - \Delta(x) = \Delta\big(\Sigma(x) + n\big) - \Delta(x) = \Delta\big(\Sigma(x)\big) + nr - \Delta(x) = k + nr \equiv k \pmod r$.

κ is a group homomorphism: if σ_1, σ_2 are solutions of (9) then

$$\kappa(\sigma_1 \circ \sigma_2) = \kappa(\sigma_1) + \kappa(\sigma_2). \tag{11}$$

Indeed, if Σ_1, Σ_2 represents σ_1, σ_2, respectively, then $\Delta \circ (\Sigma_1 \circ \Sigma_2)(x) - \Delta(x) = \Delta \circ \Sigma_1\big(\Sigma_2(x)\big) - \Delta(x) = \Delta\big(\Sigma_2(x)\big) + \kappa(\sigma_1) - \Delta(x) = \kappa(\sigma_2) + \kappa(\sigma_1)$.

We show that κ is one-to-one: If for two solutions σ_1, σ_2 of (9), $(\Delta \circ \Sigma_1)(x) \equiv (\Delta \circ \Sigma_2)(x) \equiv k \pmod r$, consider $\tau = \sigma^{-1} \circ \sigma_2$. If $\bar{\Sigma}_1$ represents σ_1^{-1}, then $T = \bar{\Sigma}_1 \circ \Sigma_2$ represents τ, and $(\Delta \circ T)(x) = (\Delta \circ \bar{\Sigma}_1)\big(\Sigma_2(x)\big) \equiv \Delta\big(\Sigma_2(x)\big) - k \equiv \Delta(x)$. Say $(\Delta \circ \bar{\Sigma}_1 \circ \Sigma_2)(x) = \Delta(x) + nr$. Then $\bar{T}(x) = T(x) - n$ also represents τ and $(\Delta \circ \bar{T})(x) = \Delta(x)$. Hence by Theorem 1, $\bar{T} = \mathrm{id}_{\mathbb{R}}$, $\tau = \mathrm{id}_S$, and $\sigma_1 = \sigma_2$. So G is isomorphic to a subgroup of $\mathbb{Z}_r$. ■

THEOREM 5. *Let $\delta\colon S \longrightarrow S$ be a continuous mapping which is nowhere constant and has degree 0. Then all invertible continuous mappings $\sigma\colon S \longrightarrow S$ that satisfy*

$$\delta \circ \sigma = \delta \tag{12}$$

form a group G which is isomorphic to either the cyclic group $\mathbb{Z}_n$ or to the dihedral group D_n, for some integer n, depending on whether G does not or does contain orientation reversing transformations.

PROOF. Instead of (8) we have

$$\Delta(x+1) - \Delta(x) = 0, \tag{13}$$

i.e., Δ is periodic with a period 1. The condition (12) is equivalent to $\Delta\big(\Sigma(x)\big) = \Delta(x) + \ell$ for some integer ℓ. By continuity and (13), the range of Δ is bounded. Hence $\ell = 0$. Thus,

$$\Delta \circ \Sigma = \Delta \tag{14}$$

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So Theorem 2 applies, and all solutions Σ to (14) form a group $\bar{G}$ which is either infinite cyclic or dihedral. Let ρ be defined as in the proof of Theorem 3. The subgroup $H \subseteq G$ of orientation preserving transformations $\sigma: S \longrightarrow S$ is represented by $\rho^{-1}(H) = \bar{H}$, the subgroup of $\bar{G}$ of all increasing transformations $\Sigma: \mathbb{R} \longrightarrow \mathbb{R}$. By the proof of Theorem 2, $\bar{H}$ is infinite cyclic, say $\bar{H} = \langle A \rangle$. $\bar{H}$ contains an element A^n which is the unit shift: $A^n(x) = x + 1$. Then $\alpha = \rho(A)$ satisfies $\alpha^n = \mathrm{id}_S$. (In fact, n is a divisor of k, the number of increasing intervals of Δ in a unit interval). ρ is clearly a group homomorphism. Its kernel consists of all integral shifts $A^{n\ell}(\ell \in Z)$, and so the image $H = \rho(\bar{H})$ is the finite cyclic group spanned by α.

If G has no orientation reversing transformation, the $G = H \cong \mathbb{Z}_n$ as claimed. If G contains an orientation reversing transformation τ then $\bar{G}$ contains a representing decreasing transformation T. By Theorem 2, $\bar{G} = \{\mathrm{id}_{\mathbb{R}}, T\} \times \bar{H}$ is infinite dihedral. Its image $G = \rho(\bar{G}) = \{\mathrm{id}_S, \tau\} \times H \cong D_n$. ∎

3. **The geometrical application.** Let a simple closed curve Γ with continuous tangent be parameterized by the map γ with domain a circle S:

$$\gamma: S \longrightarrow \Gamma.$$

Assigning, in a continuous way, an orientation to the tangent, *i.e.*, a unit tangent vector, defines a map

$$\delta': \Gamma \longrightarrow S.$$

If Γ has no cusps, then the total change of the direction of the unit tangent vector, as the point traverses Γ once, is one full rotation: its rotation index is 1 or -1. (The rotation index of a simple closed curve with cusps can be any integer; it may even be any half integer, if the concept is generalized to undirected tangents). A direction preserving map is a map $\sigma': \Gamma \longrightarrow \Gamma$ such that $\delta' \circ \sigma' = \delta'$.

If we allow the curve Γ to be not simple, γ may not be one-to-one, and δ' is not defined at double points. So we have to assign directions of tangents to Γ to points of the parameter circle S rather than to points of the curve Γ, *i.e.*, we define a mapping

$$\delta: S \longrightarrow S$$

that assigns to every point t of the parameter circle in a continuous way a unit vector $\delta(t)$, which is a tangent vector of Γ at $\gamma(t)$. Thus, the search for direction preserving maps $\sigma': \Gamma \longrightarrow \Gamma$ is replaced by the search for continuous maps σ on the parameter circle S,

$$\sigma: S \longrightarrow S \text{ such that } \delta \circ \sigma = \delta.$$

In order to force uniqueness of σ, we have to retain some of the essential properties of simple closed curves of finite order. Instead of finite order we require only that Γ contains no line segment, *i.e.*, δ is nowhere constant. Instead of simpleness, we require the following properties.

1. The rotation index of Γ is 1, *i.e.*, the degree of δ is 1.

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We shall investigate later what happens if that condition is dropped.

2. The product $\gamma \times \delta$ is one-to-one.

This ensures continuity of the parameter transformation σ. Otherwise the different loops of Γ starting and ending at a point of self-osculation may be traversed in any order. This condition is certainly satisfied if the curve does not touch itself.

Now Theorem 3 applies and proves the result of Bisztriczky and Rival under relaxed conditions. The mapping γ and the particular shape and features of the curve Γ turn out to be rather unimportant.

3.1 *Other rotation indices.* Note that the rotation index of a curve Γ is the degree r of its tangent image δ. If $r > 1$, then Theorem 4 applies. It is easy to construct curves Γ for which G is any desired subgroup of Z_r. Examples are given in Figure 1 for $r = 6$.

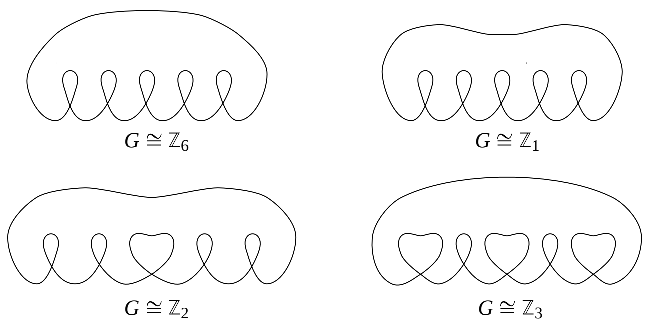

Figure 1

Curves with rotation index $r = 0$ may or may not admit direction preserving mappings which are neither one-to-one nor onto. For example, the simple closed curves with cusps in [1] all have rotation index 0. In Figure 2(a) the only direction preserving map, besides the identity, maps the curve on the more heavily outlined arc. In Figure 2(b), however, the two direction preserving maps are invertible and form the group $G \cong Z_2$.

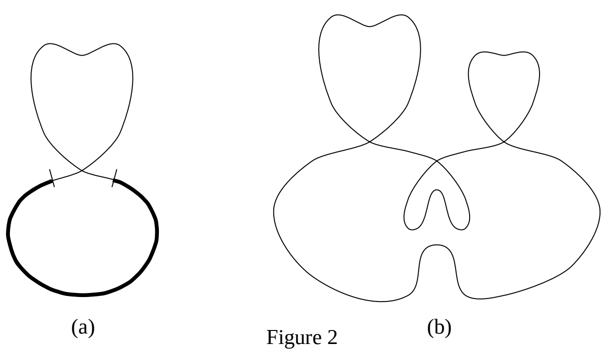

Figure 2

According to Theorem 5 all invertible continuous transformations $\Gamma \longrightarrow \Gamma$ form a group G which is isomorphic to either Z_n or D_n for some integer n. For every positive integer n there exist curves for which $G \cong Z_n$, and curves for which $G \cong D_n$. See Figure 3.

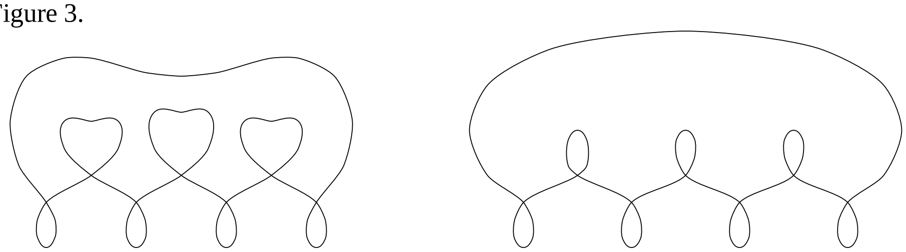

Figure 3

3.2 *Undirected tangents.* So far we have considered oriented tangents only, *i.e.*, parameter transformations σ for which $\delta \circ \sigma = \delta$. Dropping the orientation means to look also for transformations σ such that $\delta \circ \sigma = -\delta$.

The two directions along a tangent are identified by a further projection $\pi_0 \colon S \longrightarrow S$ given by the squaring map

$$\pi_0(e^{2\pi iz}) = e^{2\pi i \cdot 2z}.$$

The mapping $\delta \colon S \longrightarrow S$ is replaced by

$$\delta_0 = \pi_0 \circ \delta = \pi_0 \circ (-\delta).$$

If δ has degree r, then δ_0 has degree $2r$. Thus we have the following consequence of Theorem 4.

COROLLARY. The set of continuous mappings σ that preserve undirected tangents to a curve with tangent map δ of degree $r \geq 1$ is identical to the set of continuous mappings σ that preserve directed tangents with tangent map $\delta_0 = \pi_0 \circ \delta$ of degree $2r$. Thus the solutions form a subgroup of $\mathbb{Z}_{2r}$.

Intuitively, the undirected tangent to a curve with rotation index r turns $2r$ times around (it is true, only by an angle π each time) as the point traverses the curve once. This is true even for a half-integer rotation index. In this case there exists, of course, no continuous unit vector field along Γ.

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Department of Mathematics and Statistics
The University of Calgary
2500 University Drive N.W.
Calgary, Alberta T2N 1N4

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