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Injective Tauberian Operators on L_1 and Operators with Dense Range on ℓ_∞

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Abstract. There exist injective Tauberian operators on $L_1(0,1)$ that have dense, nonclosed range. This gives injective nonsurjective operators on ℓ_∞ that have dense range. Consequently, there are two quasi-complementary noncomplementary subspaces of ℓ_∞ that are isometric to ℓ_∞.

1 Introduction

A (bounded, linear) operator T from a Banach space X into a Banach space Y is called *Tauberian* provided $T^{**-1}Y = X$. The structure of Tauberian operators when the domain is an L_1 space is well understood and is exposed in Gonzáles and Martínez-Abejón's book [5, Chapter 4]. (For convenience they only consider $L_1(\mu)$ when μ is finite and purely nonatomic, but their proofs for the results we mention work for general L_1 spaces.) In particular, [5, Theorem 4.1.3] implies that when X is an L_1 space, an operator $T\colon X \to Y$ is Tauberian if and only if whenever (x_n) is a sequence of disjoint unit vectors, there is an N such that the restriction of T to $[x_n]_{n=N}^{\infty}$ is an isomorphism (and, moreover, the norm of the inverse of the restricted operator is bounded independently of the disjoint sequence). Here and elsewhere in this paper, by an isomorphism $T\colon E \to F$ we always mean an operator that is an isomorphism from E onto its range, $T(E)$. From this it follows that an injective operator $T\colon X \to Y$ is Tauberian if and only if it isomorphically preserves isometric copies of ℓ_1 in the sense that the restriction of T to any subspace of X that is isometrically isomorphic to ℓ_1 is an isomorphism. (Recall that a subspace of an L_1 space is isometrically isomorphic to ℓ_1 if and only if it is the closed linear span of a sequence of nonzero disjoint vectors [11, Chapter 14.5].) Since Tu is Tauberian if T is Tauberian and u is an isomorphism, one deduces that an injective Tauberian operator from an L_1 space isomorphically preserves isomorphic copies of ℓ_1 in the sense that the restriction of T to any subspace of X that is isomorphic to ℓ_1 is an isomorphism. Thus, injective Tauberian operators from an L_1 space are opposite to ℓ_1-singular operators; *i.e.*, operators whose restriction to every subspace isomorphic to ℓ_1 is *not* an isomorphism.

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The main result in this paper is a negative answer to the following question [5, Problem 1]. Suppose T is a Tauberian operator on an L_1 space. Must T be upper semi-Fredholm; *i.e.*, must the range $\mathcal{R}(T)$ of T be closed and the null space $\mathcal{N}(T)$ of T be finite dimensional? The basic example is a Tauberian operator on $L_1(0,1)$ that has infinite dimensional null space. This is rather striking because the Tauberian condition is equivalent to the statement that there is $c > 0$ such that the restriction of the operator to $L_1(A)$ is an isomorphism whenever the subset A of $[0,1]$ has Lebesgue measure at most c.

In fact, we show that there is an injective, dense range, nonsurjective Tauberian operator on $L_1(0,1)$. Since T is Tauberian, T^{**} is also injective, so $\mathcal{R}(T^*)$ is dense and proper, and T^* is injective because $\mathcal{R}(T)$ is dense.[1]

2 The Examples

We begin with a lemma that is an easy consequence of characterizations of Tauberian operators on L_1 spaces.

Lemma 2.1 *Let X be an L_1 space and let T be an operator from X to a Banach space Y. The operator T is Tauberian if and only if there is $r > 0$ and a natural number N such that if $(x_n)_{n=1}^N$ are disjoint unit vectors in X, then $\max_{1\le n\le N} \|Tx_n\| \ge r$.*

Proof The condition in the lemma clearly implies that if (x_n) is a disjoint sequence of unit vectors in X, then $\liminf_n \|Tx_n\| > 0$, which is one of the equivalent conditions for T to be Tauberian [5, Theorem 4.1.3]. On the other hand, suppose that there are disjoint collections $(x_k^n)_{k=1}^n$, $n = 1, 2, \dots$, with $\max_{1\le k\le n} \|Tx_k^n\| \to 0$ as $n \to \infty$. Then the closed sublattice generated by $\bigcup_{n=1}^\infty (x_k^n)_{k=1}^n$ is a separable abstract L_1 space (meaning that it is a Banach lattice such that $\|x + y\| = \|x\| + \|y\|$ whenever $|x| \vee |y| = 0$) and hence is order isometric to $L_1(\mu)$ for some probability measure μ by Kakutani's theorem (see *e.g.,* [7, Theorem 1.b.2]). Choose $1 \le k(n) \le n$ such that the support of $x_{k(n)}^n$ in $L_1(\mu)$ has measure at most $1/n$. Since T is Tauberian, by [5, Proposition 4.1.8], necessarily $\liminf_n \|Tx_{k(n)}^n\| > 0$, which is a contradiction. ■

The reason that Lemma 2.1 is useful for us is that the condition in the lemma is stable under ultraproducts. Call an operator that satisfies the condition in Lemma 2.1 (r, N)-Tauberian. For background on ultraproducts of Banach spaces and of operators, see [4, Chapter 8]. We use the fact that the ultraproduct of L_1 spaces is an abstract L_1 space and hence is order isometric to $L_1(\mu)$ for some measure μ.

Lemma 2.2 *Let (X_k) be a sequence of L_1 spaces, and for each k let T_k be a norm one linear operator from X_k into a Banach space Y_k. Assume that there is $r > 0$ and a natural number N such that each operator T_k is (r, N)-Tauberian. Let $\mathcal{U}$ be a free ultrafilter on the natural numbers. Then $(T_k)_\mathcal{U} \colon (X_k)_\mathcal{U} \to (Y_k)_\mathcal{U}$ is (r, N)-Tauberian.*

[1] This solves a problem [10] the second author raised on MathOverFlow.net that led to the collaboration of the authors.

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Here, $(T_k)_{\mathcal{U}}$ is the usual ultraproduct of the sequence (T_k) defined by

$$(T_k)_{\mathcal{U}}(x_k) = (T_k x_k).$$

Proof The vectors (x_k) and (y_k) are disjoint in the abstract L_1 space $(X_k)_{\mathcal{U}}$ if and only if $\lim_{\mathcal{U}} \||x_k| \wedge |y_k|\| = 0$, so it is only a matter of proving that if T is (r, N)-Tauberian from some L_1 space X, then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $x_1, \dots, x_N$ are unit vectors in X and $\||x_n| \wedge |x_m|\| < \delta$ for $1 \le n < m \le N$, then $\max_{1 \le n \le N} \|Tx_n\| > r - \varepsilon$. But if $x_1, \dots, x_N$ are unit vectors that are ε-disjoint as above and $y_1, \dots, y_n$ are defined by

$$y_n := \big[\, |x_n| - \big(|x_n| \wedge (\vee\{|x_m| : m \ne n\})\big)\big] \operatorname{sign}(x_n),$$

then the y_n are disjoint and all have norm at least $1 - N\delta$. Normalize the y_n and apply the (r, N)-Tauberian condition to this normalized disjoint sequence to see that $\max_{1 \le n \le N} \|Tx_n\| > r - \varepsilon$ if $\delta = \delta(\varepsilon, N)$ is sufficiently small. ■

An example that answers [5, Problem 1] is the restriction of an ultraproduct of operators on finite dimensional L_1 spaces constructed in [3].

Theorem 2.3 *There is a Tauberian operator T on $L_1(0,1)$ that has an infinite dimensional null space. Consequently, T is not upper semi-Fredholm.*

Proof An immediate consequence of [3, Proposition 6 & Theorem 1] is that there are $r > 0$ and a natural number N such that for all sufficiently large n there is a norm one (r, N)-Tauberian operator T_n from ℓ_1^n into itself with $\dim \mathcal{N}(T_n) > rn$. The ultraproduct $\widetilde{T} := (T_n)_{\mathcal{U}}$ is then a norm one (r, N)-Tauberian operator on the gigantic L_1 space $X_1 := (\ell_1^n)_{\mathcal{U}}$, and the null space of $\widetilde{T}$ is infinite dimensional. Take any separable infinite dimensional subspace X_0 of $\mathcal{N}(\widetilde{T})$ and let X be the closed sublattice of X_1 generated by X_0. Let Y be the sublattice of X_1 generated by $\widetilde{T}X$ and let T be the restriction of $\widetilde{T}$ to X, considered as an operator into Y. So X and Y are separable L_1 spaces and by Lemmas 2.1 and 2.2 the operator T is Tauberian. Of course, by construction $\mathcal{N}(T)$ is infinite dimensional and reflexive (because T is Tauberian). Thus X is not isomorphic to ℓ_1 and hence is isomorphic to $L_1(0,1)$. So is Y, but that does not matter: Y, being a separable L_1 space, embeds isometrically into $L_1(0,1)$. ■

We want to "soup up" the operator T in Theorem 2.3 to get an injective, non surjective, dense range Tauberian operator on $L_1(0,1)$. We could quote a general result [6, Theorem 3.4] of González and Onieva to shorten the presentation, but we prefer to give a short direct proof.

We recall a simple known lemma.

Lemma 2.4 *Let X and Y be separable infinite dimensional Banach spaces and let $\varepsilon > 0$. Let Y_0 be a countable dimensional dense subspace of Y. Then there is a nuclear operator $u \colon X \to Y$ so that u is injective and $\|u\|_\wedge < \varepsilon$ and $uX \supset Y_0$.*

Proof Recall that an M-basis for a Banach space X is a biorthogonal system $(x_\alpha, x_\alpha^*) \subset X \times X^*$ such that the linear span of (x_α) is dense in X and $\cap_\alpha \mathcal{N}(x_\alpha^*) = \{0\}$. Every separable Banach space X has an M-basis [8]; moreover, the vectors (x_α) in the M-basis can span any given countable dimensional dense subspace of X.

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Take M-bases (x_n, x_n^*) and (y_n, y_n^*) for X and Y, respectively, normalized so that $\|x_n^*\| = 1 = \|y_n\|$ and such that the linear span of (y_n) is Y_0. Choose $\lambda_n > 0$ such that $\sum_n \lambda_n < \varepsilon$ and set $u(x) = \sum_n \lambda_n \langle x_n^*, x\rangle y_n$. ■

Theorem 2.5 *There is an injective, nonsurjective, dense range Tauberian operator on $L_1(0,1)$.*

Proof By Theorem 2.3 there is a Tauberian operator T on $L_1(0,1)$ that has an infinite dimensional null space. By Lemma 2.4 there is a nuclear operator $\widetilde{v}\colon \mathcal{N}(T) \to L_1(0,1)$ that is injective and has dense range, and we can extend $\widetilde{v}$ to a nuclear operator v on $L_1(0,1)$. We can choose $\widetilde{v}$ such that $\widetilde{v}(\mathcal{N}(T)) \cap TL_1(0,1)$ is infinite dimensional by the last statement in Lemma 2.4. This guarantees that the Tauberian operator $T_1 := T + v$ has an infinite dimensional null space (this allows us to avoid breaking the following argument into cases).

Now $\mathcal{N}(T_1) \cap \mathcal{N}(T) = \{0\}$, so again by Lemma 2.4 and the extension property of nuclear operators, there is a nuclear operator $u\colon L_1(0,1)/\mathcal{N}(T) \to \ell_1$ such that the restriction of u to $Q_{\mathcal{N}(T)}\mathcal{N}(T_1)$ is injective and has dense range (here for a subspace E of X, the operator Q_E is the quotient mapping from X onto X/E). Finally, define $T_2\colon L_1(0,1) \to L_1(0,1) \oplus_1 \ell_1$ by $T_2 x := T_1 x \oplus uQ_{\mathcal{N}(T)}x$. Then T_2 is an injective Tauberian operator with dense range. T_2 is not surjective because $P_{\ell_1}T_2$ is nuclear by construction, where P_{ℓ_1} is the projection of $L_1(0,1) \oplus_1 \ell_1$ onto $\{0\} \oplus_1 \ell_1$. Since $L_1(0,1) \oplus_1 \ell_1$ is isomorphic to $L_1(0,1)$, this completes the proof. ■

Corollary 2.6 *There is an injective, dense range, nonsurjective operator on ℓ_∞. Consequently, there is a quasi-complementary, noncomplementary decomposition of ℓ_∞ into two subspaces each of which is isometrically isomorphic to ℓ_∞.*

Proof Let T be an injective, dense range, nonsurjective Tauberian operator on $L_1(0,1)$ (Theorem 2.5). Since T is Tauberian, T^{**} is also injective, so T^* has dense range but is not surjective because its range is not closed, and T^* is injective because T has dense range. The operator T^* translates to an operator on ℓ_∞ that has the same properties because L_∞ is isomorphic to ℓ_∞ by an old result due to Pełczyński (see, *e.g.*, [1, Theorem 4.3.10]) (notice however that, unlike T^*, the operator on ℓ_∞ cannot be weak* continuous).

For the "consequently" statement, let S be any norm one injective, dense range, nonsurjective operator on ℓ_∞. In the space $\ell_\infty \oplus_\infty \ell_\infty$, which is isometric to ℓ_∞, define $X := \ell_\infty \oplus \{0\}$ and $Y := \{(x, Sx) : x \in \ell_\infty\}$. Obviously X and Y are isometric to ℓ_∞ and $X + Y = \ell_\infty \oplus S\ell_\infty$, which is a dense proper subspace of $\ell_\infty \oplus_\infty \ell_\infty$. Finally, $X \cap Y = \{0\}$, since S is injective, so X and Y are quasi-complementary, non complementary subspaces of $\ell_\infty \oplus_\infty \ell_\infty$. ■

Theorem 2.5 and the MathOverFlow question [10] suggest the following problem. Suppose X is a separable Banach space (so that X^* is isometric to a weak* closed subspace of ℓ_∞) and X^* is nonseparable. Is there a dense range operator on X^* that is not surjective? The answer is no. Argyros, Arvanitakis, and Tolias [2] constructed a separable space X such that X^* is nonseparable, hereditarily indecomposable (HI), and every strictly singular operator on X^* is weakly compact. Since X^* is HI, every

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operator on X^* is of the form $\lambda I + S$ with S strictly singular. If $\lambda \neq 0$, then $\lambda I + S$ is Fredholm of index zero by Kato's classical perturbation theory. On the other hand, since every weakly compact subset of the dual to a separable space is norm separable, every strictly singular operator on X^* has separable range.[2]

Any operator T on ℓ_∞ that has dense range but is not surjective has the property that 0 is an interior point of $\sigma(T)$. This follows from [9, Theorem 2.6], where it is shown that $\partial\sigma(T) \subset \sigma_p(T^*)$ for any operator T acting on a $C(K)$ space that has the Grothendieck property.

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[2] Thanks to Spiros Argyros for bringing this example to our attention.

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