PART III

PHYSICAL RANDOMNESS

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On The Correct Definition of Randomness

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The concept of randomness as applied to number sequences is important to the study of the relationship between the foundations of mathematics and physics. A reason is that while randomness is often defined in mathematical-logical terms, the only way one has to generate random number sequences is by means of repetitive physical processes. This paper will examine the question: What definition of randomness is correct in the sense of being the weakest allowable? Why this question is so important will become clear during the course of the discussion.

The main body of this paper is divided into three sections. Section 1. discusses the use of probability theory to describe various statistical processes and some of the alternative definitions of randomness that have been proposed.

In Section 2. a criterion which a definition of randomness should satisfy is proposed. Roughly, the criterion says that with respect to a theory T and the set of processes which are in its domain of explanation, a definition of randomness should be such that no random outcome sequence which can be generated by a process which theory T explains can be used to derive a contradiction within theory T. Following discussion of this criterion it is suggested that the weakest definition which is contradiction-free in the above sense be taken as the correct definition of randomness. Some aspects of this definition of correctness are discussed. In particular its dependence on T is discussed and reasons are given why this T-dependence is not a disadvantage.

Finally, in Section 3. the importance of determining which

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definition is correct in the above sense of being the weakest allowable is shown by considering the use of models of Zermelo-Frankel (ZF) set theory as carriers for the mathematics of quantum mechanics. It is seen that if the correct definition of randomness is sufficiently strong, then there exist models of ZF set theory which do not contain random sequences and thus cannot serve as carriers for the mathematics of quantum mechanics. In this case, contrary to intuition, physics would have something to say about the foundations of quantum mechanics.

1. Probability Theory, Statistical Processes, and Randomness

In order to see why the problem of determining a correct definition of randomness arises, let us consider the use of probability theory to describe various statistical processes. Probability theory is a mathematical theory about measures defined on a σ -field $\mathcal{B}(Z)$ of subsets of some set Z. It also includes the theory of random variables as measureable functions on Z and, as such, touches many aspects of modern analysis.

However, for all its beauty and power, probability theory says nothing whatsoever about any properties which an element of Z may or may not have. The most one can prove in probability theory is that a given Borel subset B of Z is a set of measure 1 with respect to some measure. To conclude from this that an element Ψ of Z lies in B, or equivalently, has the property corresponding to B, requires some extra assumptions. These are usually implied in the use of probability theory to explain stochastic processes, and form part of the interpretative rules between a theory which assigns measures to various stochastic processes and the outcome sequences which the processes generate.

Let us give a specific example. Let $Z = \{0,1\}^N$, the set of all infinite 0-1 sequences and $\mathcal{B}(\{0,1\}^N)$ be the set of all Borel subsets of $\{0,1\}^N$. Let M be a set of stochastic processes each of which generate elements of $\{0,1\}^N$ by emitting 0's or 1's in successive steps. Thus carrying out n steps of a process generates a sequence of n 0's and 1's which, as $n \to \infty$, becomes a sequence in $\{0,1\}^N$.

A theory for M generates a map P from M into the set of all probability measures on $\mathcal{B}(\{0,1\}^N)$. Thus, given any process Q in M, the theory assigns a probability measure P_Q to the process Q. For instance, discrete step quantum stochastic processes which output 0 or 1 can be so described. Examples of this are sequences of successive measurements of projection operators on the same sample where the time interval between the nth and n+1st step, and choice of projection operator for the n+1st step may depend on n and the previous outcomes.

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To each such process and initial state preparing procedure, denoted here by Q, quantum mechanics associates a probability measure P_Q.

Another type of example includes the infinite repetitions of preparing a system in some state ρ, measuring a question valued observable A on the system so prepared, and discarding the system. In this case the measure assigned to each infinite repetition is a product measure $P_A = \otimes p_{A\rho}$ where $p_{A\rho}$ is computed from A and ρ in standard fashion.

So far it is seen that a theory for a collection M of processes assigns probability measure P_Q to each process Q in M. However, the condition that the theory is correct, or agrees with the experiment, is still lacking. To reduce this to essentials and avoid inessential complications, assume the theory is such that for each Q, P_Q is a computable function on the cylinder subsets of $\{0,1\}^N$, i.e., all finite unions of sets of the form $\{a_1\} \times \{a_2\} \times \ldots \times \{a_n\} \times \{0,1\}^N$ where $a_1, a_2, \ldots a_n \in \{0,1\}$.

Let D be a countably infinite subset of $\mathcal{B}(\{0,1\}^N)$. One says that the theory D-explains the processes in M or D-agrees with experiment ([3],[5],[6]) if, for each Q in M and each B in D, if $P_Q B=1$ then $\Psi_Q \in B$ where Ψ_Q is an outcome sequence associated with the actual carrying out of Q.

Equivalently, one says that the theory, which assigns the map P over M, D-agrees with experiment if for each Q in M, P_Q is "D-correct" for Ψ_Q. That is, Ψ_Q has every property in D which is possessed by P_Q-almost-every sequence ([3],[5],[6]). (One associates a property to each Borel set B where B is the set of sequences which have the property.) From now on, for ease in presentation, the discussion will be restricted to processes Q for which P_Q is a product measure $P_Q = \otimes p_Q$ where p_Q is a probability measure on the four element set of all subsets of {0,1}. Then the theory D-agrees with experiment on M if for each Q so restricted, the outcome sequence Ψ_Q is D-random for P_Q (i.e., for all $B \in D$, $P_Q B=1 \rightarrow \Psi_Q \in B$).

The question of exactly which properties are to be included in D immediately arises. Some properties are such that one has strong intuitive feelings that every random sequence should have them--these properties should clearly be included. For other properties one's intuition is much weaker. Finally, there are many properties for which there is no common agreement whether or not they should be possessed by random sequences. This is quite apparent from the history of many attempts to give a good definition of randomness.

As a review of the development of definitions of randomness for

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infinite outcome sequences shows, many definitions of randomness require that a random sequence be such that the limit mean exists and be invariant under each subsequent selection procedure in a countable set E. Definitions of this type, which differ in how E is defined, have been given by Von Mises [17], Church [7], Wald [18], Loveland ([11],[12]) and Kruse [10].

These definitions were shown by Ville [16] to be deficient in that for any countable set E of subsequent selection procedures, there existed a sequence which passed the tests in E and for which the mean of the first n elements is greater than the limit mean for each n.

Definitions of randomness which avoid this problem are given in terms of set D of properties (or Borel subsets of $\{0,1\}^N$). For definiteness, let $P=\bigotimes p$ where $p\{0\}=p\{1\}=1/2$. Then a sequence Ψ is D-random for P if for each property B in D which is true P almost everywhere is possessed by Ψ. In Martin-Löf's definition [13] D is the set of complements of all Borel sets of P measure zero which are defined in terms of recursively enumerable sequential tests. In other definitions, D is the set of all hyperarithmetic Borel sets [14], the set of all Borel sets with a code in the minimal standard transitive model of ZF set theory [15], the set of Borel sets which are nameable in a set theory [10] or definable in ZF set theory [4].

These definitions can be generalized to apply to any probability measure P by relativizing the definition of D to P ([3],[4],[5],[6]). For example, D can be the set of complements of Borel sets defined from P sequential tests which are recursively enumerable in P, the set of Borel sets which are hyperarithmetic in P, or have codes in $M_0[P]$ (the Cohen extension by P of the minimal model of ZF set theory), or the set of Borel sets which are definable from P in the language of ZF set theory. Other definitions as well as definitions applicable to finite sequences are discussed by Fine [9].

From this brief sketch and further examination of the literature, one sees that many definitions of randomness have been proposed. Each definition is of different strength and for each, one can give rough philosophical arguments why it is to be preferred over the others. However, there do not seem to be any compelling reasons why one definition is to be preferred over another. Intuition does not seem to be of any help either.

It would clearly be desirable to have a precise criterion to use to decide which properties belong to D.

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2. A Possible Criterion for Randomness

It is proposed here that a valid definition of randomness should satisfy the criterion that a sequence accepted as random cannot be used to derive a contradiction within the conjunction of the theory and the interpretative rules which explain any process which can generate the sequence. In more detail, let T be a theory with domain of explanation M. Then the set D of properties (or Borel sets) must be such that for each product probability measure $P = \bigotimes p$, which T assigns to a process in M, it must not be possible to use any outcome sequence Ψ which is D-random for P to derive a contradiction within T.

Such a criterion for D has its support in the observation that it must be some criterion such as this which gives one an intuitive feeling about which properties a random sequence must possess. If one feels that every random sequence should have property B where PB=1 for a product measure P, it must be the case that for some theory T and domain M, which are within his realm of experience, there is a sequence Ψ, which does not have B and which is random for P with respect to some other properties, which can be used to derive a contradiction in T. If a contradiction is not derivable within T for any such sequence Ψ, then as far as T is concerned, one has no basis for his feeling that a random sequence must have property B. (This aspect will be referred to later on.)

Note that the criterion is defined relative to a theory T whereas one's intuitive feeling is based on common experience with many theories. Some properties, such as invariance under subsequence selection procedures, are common to many theories. That is, by the above criterion it should be possible to derive contradictions in many theories from sequences which do not have these properties. For these properties one has strong intuitive feelings that they should be included in D. Other properties may be more theory specific. That is, they are contradiction free for some theories and not for other theories. In this case one's intuition is much weaker, or even nonexistent.

The proposed criterion for randomness derives its use from the requirement that properties which are not contradiction free for a theory T must be included in D, as far as T is concerned. Thus the criterion is not very useful with respect to any theory which has a small domain of explanation. In particular, this criterion would not be of much use for a theory which explains only one process. The value of the criterion should become evident for comprehensive theories which explain large classes of processes. One reason is that such theories give many relationships between various processes. As a result, there are many more possibilities for contradictions to appear. Another

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reason, which may be quite important, is that a really comprehensive theory may have processes within its domain which, under suitable interpretation, talk about other processes within its domain. But this is a subject for future work.

Here is a simple example which illustrates how the criterion works. In quantum mechanics, let α denote an observation procedure for a question observable $Q\alpha$ (a projection operator) and let s denote a preparation procedure which prepares a system in a pure state ρ_s. Each infinite repetition of doing s followed by α is characterized by a map X from N, the set of natural numbers into the set of time regions where for each j X(j) is the (calendar) time region occupied by the jth repetition of doing s followed by α. Let $\Psi^X_{s\alpha}$ denote the outcome sequence in $\{0,1\}^N$ associated with the infinite repetition (X,s,α). Since (X,s,α) is a process in the domain of explanation of quantum mechanics, one requires that $\Psi^X_{s\alpha}$ be such that $\bar{M}\Psi^X_{s\alpha}$ exist, and

$\bar{M}\Psi^X_{s\alpha} = P^X_{s\alpha}(\{1\})$ where $P^X_{s\alpha}(\{1\}) = Tr\rho_s Q\alpha$ is a probability measure on the four element set of subsets of $\{0,1\}$. $\bar{M}\Psi$ denotes the limit mean of Ψ.

Let $G:\{0,1\}^N \to \{0,1\}^N$ be given by $(G\Psi)(j)=\Psi(g(j))$ for each j in N where $g:N\to N$ denotes a (sequence independent) subsequence selection procedure which an observer can actually carry out. $G\Psi$ is the subsequence of Ψ selected by g. It is now asserted that $\bar{M}G\Psi^X_{s\alpha} = \bar{M}\Psi^X_{s\alpha}$.

To prove this, assume the contrary, i.e., that $\bar{M}G\Psi^X_{s\alpha} \neq \bar{M}\Psi^X_{s\alpha}$. Consider the subsequence X_G of repetitions where $X_G(j)=X(g(j))$ for each j. Now (X_G,s,α) is also a sequence of repetitions of doing s followed by α and is in the domain of explanation of quantum mechanics. Thus the outcome sequence $\Psi^{X_G}_{s\alpha}$ associated with (X_G,s,α) is also such that $M\Psi^{X_G}_{s\alpha}$ exists and equals $P^{X_G}_{s\alpha}(\{1\})$ where $P^{X_G}_{s\alpha} = \otimes\, p^{X_G}_{s\alpha}$ is the measure assigned to (X_G,s,α). Since the range set of X_G is a subset of X one has that $G\Psi^X_{s\alpha}$ is also an outcome sequence associated with (X_G,s,α), or[2] $G\Psi^X_{s\alpha} = \Psi^{X_G}_{s\alpha}$. By assumption, one has then that $\bar{M}\Psi^{X_G}_{s\alpha} \neq \bar{M}\Psi^X_{s\alpha}$ which yields $p^X_{s\alpha} \neq p^{X_G}_{s\alpha}$. But this implies that the repetitions in (X_G,s,α) either prepare a system in a different state or measure a different question observable than do the repetitions in $(X-X_G,s,\alpha)$. In either case this contradicts the original assumption that each repetition of s and α in X prepares a system in a fixed pure state[3] ρ_s and measures a fixed observable $Q\alpha$. So one has

$$\bar{M}G\Psi^X_{s\alpha} = \bar{M}\Psi^X_{s\alpha}.$$

This example shows that a correct definition of randomness must be sufficiently strong to include all sequence-independent subsequence

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selection procedures which an observer can actually carry out. If this notion corresponds to (sequence-independent) effectively calculable subsequence selection procedures then, by Church's thesis, one has some of the recursive subsequence selection procedure. These are included in Church's definition [7] of randomness as well as in all stronger definitions.

This example also shows the usefulness of the criterion for a comprehensive theory. Suppose for example a theory T explained measurement repetitions for only one sequence X of time regions and all other repetitions were outside the domain of explanation of T. Then the repetition (X_G, s, α) would not be in the domain of T so the theory would say nothing about the value of $\overline{M}\Psi^{X}_{s\alpha}G$.

So far, the criterion has been applied here to random sequences and processes to which product measures $P = \otimes p$ are assigned. However, the discussion of Section 1. shows that the criterion can be easily extended to cover all stochastic processes in the domain of a theory, not just those which are independent and identically distributed. In this case the notion defined in Section 1., of a measure P being D-correct for an outcome sequence Ψ, replaces the concept of Ψ being D-random for P. Details of the extension will be left to the reader.

This criterion of nonderivability of a contradiction as a theory dependent condition on the set D has the property that it places no limits on how strong a definition of randomness can be. Suppose D is contradiction free for some theory T in the sense that for every product measure P which is assignable by T to some process in T's domain of explanation, no sequence which is D-random for P can be used to derive a contradiction within T. Then any set D^1 where D is a proper subset of D^1 is also contradiction free for T.

However, the criterion does place a limit on how weak a definition of randomness can be. If D is too small then, with respect to a given theory T plus interpretative rules, for some product measure P which is assignable by T, there may exist sequences which are D-random for P and which can be used to derive a contradiction within T.

This suggests that it may be appropriate to consider the correct definition of randomness to be the weakest definition which is contradiction-free in the above sense. Such a consideration finds support in the fact that, with respect to some theory T, D should not contain superfluous sets. As was noted before, on an intuitive basis, if for some property B, a contradiction is not derivable within T for any sequence which does not have B, but is D-{B} random for some product measure P, then as far as T is concerned, one really has no basis for his feeling that a random sequence must have property B.

One can give a more precise definition of the notion of a correct definition as follows: Let T be a theory with domain of explanation M (For simplicity it is assumed that each Q in M generates a 0-1 outcome sequence). Let $P = \otimes p$ be a fixed product probability measure such

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that T assigns P to some process in M. Let D_P be a countably infinite set of Borel subsets of $\{0,1\}^N$ which is contradiction free for T in the sense described. (That is, for no sequence Ψ which is D_P random for P can one derive from ψ a contradiction in the conjunction of T and the interpretative rules which explain the processes in M.)

For each Borel set B in D_P for which PB=1, let $D_P^B = D_P - \{B\}$ the set of all sets in D_P except B. B is P-nontrivial if $\cap\{E:E\ D_P^B$ and PB=1}-B is not empty. One says that D_P-randomness for D is the weakest allowable definition for T if for each P-nontrivial B in D_P, there exists a sequence ψ not in B which is D_P^B random for P, and which can be used to obtain a contradiction in the theory T plus interpretative rules which explain the processes in M.

The above definition is incomplete in that the explicit dependence on the measure P must be removed. Taking a cue from the fact that for usual definitions of D-randomness, D contains sets of the form $\{\Psi:M\ \Psi = r\}$ for many real numbers r between 0 and 1, D is defined by $D=\cup\{D_P$:P is a product measure and T assigns P to some process in M}.

It is suggested that D-randomness with D as defined above is the correct definition of randomness for T. It is correct in the sense described above of being the weakest allowable which is contradiction free.

Several comments about this definition are in order. First, the restriction of B to the P-nontrivial sets in D_P is necessary because if B is not P-nontrivial then there are no Ψ not in B which are D_P^B random for P. Also, the above restriction to product measures can easily be relaxed. As was noted earlier, the notion of a measure P being D-correct for an outcome sequence is not limited to product measures. Thus in the definition of D one can extend the union to cover all measures assigned by T to processes in M and thus include all sequences generated by all processes in M.

This definition for correctness for D-randomness has the property that it depends strongly on the theory T under consideration. Clearly a definition of randomness which is correct in the sense given for one theory may not be correct for another theory. This may seem to be a defect of the definition, but, in the author's opinion, it is not.

First of all, some peculiar theory plus interpretative rules might require a very strange definition of randomness in order that the theory plus interpretative rules be contradiction free over its domain of explanation. The definition given here of a correct definition of randomness will satisfy his requirement whereas a definition which is theory independent may not.

Also, it is quite possible that a definition like the one given which is theory dependent may appear to be theory independent. To see this, consider again, how one arrives at an intuitive feeling that random sequences must have a property B. Suppose B is a property which a

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sequence Ψ, generated by a process in the domain of explanation of some theory T, must have in order that T be contradiction free. Furthermore, suppose our accumulated experience is such that this is true for B for many sequences, processes, and theories. Then one feels strongly that B is a property that random sequences should have. This suggests that one might remove the theory dependence by defining D-randomness by quantifying over all theories which explain any process which yields the sequence in question. However, this is a dubious procedure since it is not clear what is meant by all theories plus rules of interpretation which explain any process which generates Ψ. Also, there may exist properties B which are required by sequences generated by processes in the domain of explanation of only some, but not all, theories.

A better procedure is to consider a succession of theories whose domains of explanation are successively larger or more comprehensive. As a theory becomes more comprehensive the possibilities for deriving a contradiction from a sequence generated by a process in the domain of the theory become greater. So the definition of D randomness which is correct in the sense defined becomes stronger as the relevant theory becomes more comprehensive.

In particular, if a theory has a domain of explanation which includes many smaller domains with each explained by some other weaker theory, a definition of randomness which is correct in the sense defined for the comprehensive theory will include all those properties which are common to the correct definitions of randomness for the weaker theories which explain subdomains of the domain of the comprehensive theory. It is in this sense that the definition of randomness which is correct for a comprehensive theory, may appear to be theory independent with respect to those properties which are common to the correct definitions for many weak and seemingly unrelated theories. Furthermore, it is just those properties about which one has strong intuitive feelings that every random sequence should possess them.

It is to be emphasized that most of the discussion of this section, including that of the criterion that a definition must satisfy as well as that of the correct definition, is heuristic and imprecise. To make it precise one has to define exactly what is meant by saying that "one can derive from a sequence Ψ which can be generated by a process in M, a contradiction in a theory which explains the processes in M." Also it is necessary to assign a well ordering to the properties (or Borel subsets) of sequences in $\{0,1\}^N$. Such a well ordering may be necessary to prove that a unique set D exists which is correct in the sense given.

The further detailed formulation and working out of these definitions will require much more work. In the rest of the paper, the importance of finding out which definition of randomness is correct in the above sense will be given further support. This will be done by considering the use of different models of ZF set theory as carriers for the mathematics of physics.

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4. ZF Models as Carriers for the Mathematics of Physics

In order to understand what it means to use a model of Zermelo Frankel set theory as a carrier for the mathematics of physics, one first notes that all the mathematics used so far by physics and most of mathematics itself can be described in intuitive set theory. Open most any comprehensive treatise on mathematics and you will find an initial chapter on the rudiments of set theory. Furthermore, such treatises are full of statements such as "A Hilbert space is a set such that...", "a C* algebra is a set such that...", "The set of complex numbers...", etc.

There are many ways to axiomatize the intuitive concept of set. By far the most extensively developed and studied is ZF set theory. This theory which axiomatizes much of the intuitive concept of set is powerful enough to include most of the mathematics done to date and to include all the mathematics used so far by physics. What this means is that all the mathematical theorems and results used so far by physics, can be recast as theorems and results about sets. For example, any n-ary operation with certain properties can be described as a set of ordered n+1-tuples with certain properties. For example, in the theory of Hilbert spaces, the operation "•" of multiplication of vectors by scalars as a map from C x H to H, becomes a part of ordered triples of elements of C x H x H whose properties are given by the Hilbert space axioms transcribed to corresponding properties about sets. By the use of such a defining axiom for "•" as well as similar defining axioms for the other Hilbert space operations, and the axioms of ZF set theory, one proves theorems about Hilbert spaces as corresponding theorems about sets.

The axioms and formulas of ZF set theory, and of any other mathematical theory for that matter, are meaningless symbol strings. They are given meaning by interpreting them in a model. A model for a (first order) axiom system is a collection, or universe, of mathematical objects together with relations and functions and constants as interpretations of the relation, function, and constant symbols of the formal theory, and with variables ranging over the collection. Under this interpretation, formulas of the formal theory are either true or false; in particular, all axioms and theorems are true.

In mathematics generally, and specifically in the mathematics used by physics, many different groups, algebras, Hilbert spaces, etc., are considered and used. This is possible because the corresponding first order axiom systems, have many different models.

On the other hand, there is only one (standard) whole number set as a model of Peano arithmetic (a first order theory), and only one (standard) complex number set C which is a model of the axioms of an algebraically complete field of characteristic 0 (a first order theory) and which has $2^{\aleph_0}$ elements. (By "one" is meant one up to isomorphism). It is this set C which, in intuitive set theory, can also be built up from the natural numbers in any one of several ways,

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and which is referred to as <u>the</u> scalar field of complex numbers.

Now it is only in the last fifteen years that, as a result of the work of Paul Cohen [8], the existence of many different nonisomorphic models of ZF set theory has been proved. Nowadays, much of the modern work in set theory is directed towards the construction of models with different properties. These models differ by "esoteric" mathematical properties, such as whether the continuum hypothesis is true or false, or the model universe is Gödel constructible or not, etc.. However, all these models are equivalent as far as the "conventional" mathematics is concerned. The reason is that, as noted before, every theorem of the system under study becomes under transcription a theorem of ZF set theory, and all ZF theorems are true in all ZF models.

In particular, each model M contains a profusion of the mathematical systems used so far by physics. All the usual groups, Hilbert spaces, and C* algebras, and the sets of natural numbers, real numbers, and complex numbers, etc., which are used in physics exists as sets in M. Furthermore, their properties are defined relative to M. That is, the universe of M is considered to be the entire mathematical universe for all these systems. This is what is meant by a ZF model M being a carrier for the mathematics of physics.

It follows from the above arguments that, because all models are completely equivalent as far as the mathematics used so far by physics is concerned, one would expect that all ZF models would be equivalent as carriers for the mathematics of physics. Thus, it should make no difference which model one used for the mathematics of physics.

This has been investigated in detail ([1],[2]) for the standard transitive model V and any standard transitive model M (with $M \subseteq V$) as carriers for the mathematics of quantum mechanics. V is that part of the intuitive mathematical universe which is a model of ZF set theory. In brief, if $\mathcal{H}_M$ and $B_M(\mathcal{H}_M)$ are the respective Hilbert space and algebra of bounded linear operators on $\mathcal{H}_M$ in M, which describe the physical states and observables of a system, then there exists a Hilbert space $\mathcal{H}$ and algebra $B(\mathcal{H})$ of bounded linear operators in V along with monomorphisms $U:\mathcal{H}_M \to \mathcal{H}$ and $W:B_M(\mathcal{H}_M) \to B(\mathcal{H})$ which preserve the norms. This means that if M is used as the carrier for the mathematics of quantum mechanics, use of $\mathcal{H}_M$ and $B_M(\mathcal{H}_M)$ in M to describe the physical states and observables of the physical system gives the same statistics as does use of $\mathcal{H}$ and $B(\mathcal{H})$ in V. (Of course, this holds for only that part of $\mathcal{H}$ and $B(\mathcal{H})$ in the respective ranges of U and W). This result is clearly necessary if the two models are to be equivalent as carriers for the mathematics of quantum mechanics.

However, the point to be made here is that, contrary to the above, it may be the case that all ZF models are not equivalent as carriers for the mathematics of quantum mechanics. In brief the argument is as follows ([1],[2]): In quantum mechanics, connection between theory and experiment is made by the requirement that to each infinite repetition of measurements of some observable on an ensemble of systems, each prepared in the same state, there is associated an infinite, random

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sequence of outcomes whose limit mean equals the theoretical expectation value. (We assume that for many types of measurements the system is not prepared in an eigenstate of the observable.)

Note carefully, we do not require that one be able to carry out, by any finite time, an infinite number of measurements - something which is impossible. However, it is clear that no matter how many repetitions we have carried out, we can always do one more. Since the number of possible repetitions is arbitrarily large, the theory-experiment connection must be given in terms of infinite repetitions and outcome sequences.

It follows from the above that if a ZF model is to be a carrier for the mathematics of quantum mechanics, it must contain random number sequences. In particular, it must contain infinite 0,1 sequences which are random (i.e., those which are associated with the measurement of question observables). However, it can be shown ([15],[1],[2]) that for sufficiently strong definitions of randomness, there exist ZF models which do not contain any random outcome sequences. In particular, for randomness defined in terms of Borel sets which have codes in the *minimal standard transitive model* M_o, *or in terms of Borel sets* which are ZF definable, M_o does not contain any such random sequences.

One sees then that if a sufficiently strong definition of randomness is correct, M_o cannot be a carrier for the mathematics of quantum mechanics. However, if weaker definitions, for example, those given in terms of recursively enumerable or hyperarithmetic Borel sets are correct, then the above argument fails as all standard ZF models contain sequences which are random according to these definitions.

It should be noted that the above argument is much stronger for ZF models as carriers for the mathematics of quantum mechanics than it is for classical mechanics. The reason is that in classical mechanics randomness plays only a minor role, as statistical fluctuations are a consequence of an observer's lack of knowledge rather than something more fundamental. In particular, for pure classical states, which are the generators of all states, one single measurement of any observable is sufficient to give a number to be compared with theory, and repetitions need not be discussed. In quantum mechanics, randomness is much more important. The reason is that even for pure states, for many observables, statistical fluctuations of outcomes of repeated measurements are present <u>in principle</u> and must be taken into account.

It is thus seen that it is important to prove rigorously which definition of randomness is correct in the sense of being the weakest allowable. If such a definition is sufficiently strong, some models of ZF set theory can be excluded as carriers for the mathematics of quantum mechanics. In this case physics would have something to say about the foundations of mathematics at a deep level. However, if such a definition is weaker, then one cannot exclude ZF models on the basis of randomness and the above argument cannot be used to support a deep relationship between the foundations of physics and mathematics.

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The above argument by which strong definitions of randomness exclude the minimal model M_o can also be extended ([1],[2]) to the concept of statistical independence of a sequence Ψ from a sequence Φ. In particular, corresponding to each definition of randomness one can give a definition of statistical independence of one sequence from another. One can then show that for strong definitions of statistical independence, the Cohen extensions $M_o[\Psi]$ of M_o by any random outcome sequence Ψ can also be excluded as carriers for the mathematics of quantum mechanics.

In conclusion, it should be stressed that the preceding argument is an if-then statement. If the correct definition of randomness is sufficiently strong, then physics has something to say about the foundations of mathematics. Nothing has been said about the more important and difficult problems of showing how strong the definition, which is correct in the sense defined in Section 2., must be.

As the discussion in Section 2. on the dependence of the strength of the correct definition of randomness on the theory T has noted, it is for very comprehensive theories that one has the best chance of showing that the correct definition is sufficiently strong. It is felt that, of the theories being used at present, only quantum mechanics, or some extension of it, will be sufficiently comprehensive so that the corresponding correct definition of randomness will be sufficiently strong. However, one may have to await further development of the theory of macroscopic systems in quantum mechanics. Then the description of interacting systems may be sufficiently advanced to describe systems which can be interpreted as one subsystem carrying out repeated measurements on another type of subsystem. It may be possible to prove that for such an extension of quantum mechanics, the corresponding correct definition of randomness is sufficiently strong to enable one to conclude that physics has something to say about the foundations of mathematics. But this will have to await further developments.

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Notes

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[2]The equality $G\Psi^{X}_{s\alpha} = \Psi^{X}_{s\alpha}G$ is not needed for the proof.

It is sufficient to require that $\overline{M}G\Psi^{X}_{s\alpha} = \overline{M}\Psi^{X}_{s\alpha}G$.

[3]The requirement that ρ_s be pure is not necessary. However, it avoids potential problems which would result from considering a sequence of preparations of a mixed state as a mixed sequence of preparation of different pure states.

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