BOOLEAN AVERAGES

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1. Introduction. The purpose of this note is to investigate the properties of a mapping α, of a Boolean algebra A into itself, which satisfies the functional equation $\alpha(p \cdot \alpha q) = \alpha p \cdot \alpha q$, where the multiplication is infimum. This is the so-called *averaging identity* of Kampé de Fériét. Garrett Birkhoff has noted **(2)** the occurrence of this identity both in the statistical theory of turbulence and in mathematical logic. That this is more than coincidence is shown by the existence of an explicit connection between the notion of quantification in logic and certain non-linear "extremal operators" in function algebras **(16)**. Certain linear operators are associated in a natural way with these extremal operators. These so-called "generalized means" coincide with the class of linear operators on $C(X)$ which satisfy the averaging identity. The characteristic property of these averaging operators is a decomposition into simpler averaging operators. Roughly speaking, an averaging operator on $C(X)$ is given by an upper semicontinuous decomposition of X and by a "cross-section by probabilities" of the decomposition **(1, 2, 6, 16)**.

Applications of averaging operators are found in the theory of turbulent motion, probability theory, differentiation, martingales, and ergodic theory **(1, 9, 11, 17)**. Birkhoff's survey **(2)** contains an excellent summary and bibliography.

The averaging identity has interesting algebraic implications. Kelley **(6)** has shown that an idempotent isotone projection of $C(X)$ is an averaging operator if and only if the range is a subalgebra. Madame M.-L. Dubreil-Jacotin (Algebra Seminar, Tulane, 1962) has shown that a closure operator in a partially ordered groupoid has a subgroupoid as range if and only if the closure satisfies the averaging identity. Rota **(11)** has remarked that the averaging identity for a linear transformation includes the assertion that the transformation is a module endomorphism over the range.

The averaging identity appears in Boolean algebras in the algebraic formulation of the notion of an existential quantifier in the work of Tarski and his associates **(4, 5, 7)**, and in the work of Halmos **(3)**. In this note, we consider the effect of the averaging identity on different kinds of mappings of a Boolean algebra. The resulting characterizations are again in the nature of cross-sections. Two separate cases occur, depending on whether A is regarded as a lattice or as a ring. In one case, the duality theory of M. H. Stone provides

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the characterization, and the other case is best handled by means of the Pontrjagin duality theorem.

2. Averaging hemimorphisms. In a Boolean algebra A, we let 0 and 1 denote the least and greatest elements, and we write $p \vee q$, $p \wedge q$, and p' to denote the supremum, infimum, and complement in A. We write $p + q$ to denote the symmetric difference $(p' \wedge q) \vee (p \wedge q')$.

A *hemimorphism* of a Boolean algebra A is a mapping α of A into itself such that $\alpha 0 = 0$ and such that $\alpha(p \vee q) = \alpha p \vee \alpha q$ for each p, q in A. This terminology is due to Halmos **(3)**; Jónsson and Tarski **(5)** call these normal and additive mappings. A hemimorphism is always *isotone*, that is $p \leqslant q$ implies $\alpha p \leqslant \alpha q$.

PROPOSITION 1. *For a hemimorphism α the following statements are equivalent:*

(i) $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$, *for each p, q in A;*

(ii) $p \leqslant \alpha q$ *implies* $\alpha p \leqslant \alpha q$, *and* $p \wedge \alpha q = 0$ *implies* $\alpha p \wedge \alpha q = 0$, *for each p, q in A.*

Proof. Suppose (i) is satisfied. If $p \leqslant \alpha q$, then $p = p \wedge \alpha q$, so

$$\alpha p = \alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q.$$

If $p \wedge \alpha q = 0$, then

$$0 = \alpha 0 = \alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q.$$

Conversely, suppose (ii) is satisfied. Since

$$(p + (p \wedge \alpha q)) \wedge \alpha q = 0,$$

then

$$\alpha(p + (p \wedge \alpha q)) \wedge \alpha q = 0,$$

and since $p \wedge \alpha q \leqslant \alpha q$, then $\alpha(p \wedge \alpha q) \leqslant \alpha q$. Write

$$p = (p + (p \wedge \alpha q)) \vee (p \wedge \alpha q),$$

so that

$$\alpha p = \alpha(p + (p \wedge \alpha q)) \vee \alpha(p \wedge \alpha q).$$

Intersecting with αq, we have

$$\begin{aligned}\alpha p \wedge \alpha q = [\alpha(p + (p \wedge \alpha q)) \wedge \alpha q] \vee [\alpha(p \wedge \alpha q) \wedge \alpha q]\\ = 0 \vee \alpha(p \wedge \alpha q) = \alpha(p \wedge \alpha q).\end{aligned}$$

PROPOSITION 2. *If α is a hemimorphism satisfying*

$$\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$$

for each p, q in A, then $\alpha^2 = \alpha$.

Proof. We have $\alpha 1 = \alpha(1 \vee q) = \alpha 1 \vee \alpha q$, so that $\alpha q \leqslant \alpha 1$ for each q in A. Then

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$$\alpha^2 q = \alpha(\alpha q) = \alpha(1 \wedge \alpha q) = \alpha 1 \wedge \alpha q = \alpha q$$

for each q in A.

PROPOSITION 3. *Let α be a hemimorphism of a Boolean algebra A. Then the following are equivalent*:

(i) $\alpha 1 = 1$ *and* $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$, *for each* p, q *in* A;

(ii) $\alpha^2 = \alpha$ *and* $\alpha[(\alpha p)'] = (\alpha p)'$ *for each* p *in* A;

(iii) $\alpha^2 = \alpha$ *and* α *has a subalgebra* B *as range.*

Proof. Suppose (i) to be true. Then $\alpha^2 = \alpha$, by Proposition 2. Moreover, $1 = \alpha 1 = \alpha(\alpha p \vee (\alpha p)') = \alpha^2 p \vee \alpha[(\alpha p)'] = \alpha p \vee \alpha[(\alpha p)']$, from which we have $(\alpha p)' \leqslant \alpha[(\alpha p)']$. We also have

$$0 = \alpha 0 = \alpha((\alpha p)' \wedge \alpha p) = \alpha[(\alpha p)'] \wedge \alpha p,$$

from which $\alpha[(\alpha p)'] \leqslant (\alpha p)'$. Hence $\alpha[(\alpha p)'] = (\alpha p)'$.

Suppose now that (ii) is true, We have

$$\alpha 1 = \alpha(1 \vee (\alpha 1)') = \alpha 1 \vee \alpha[(\alpha 1)'] = \alpha 1 \vee (\alpha 1)' = 1.$$

If $p \leqslant \alpha q$, then $\alpha p \leqslant \alpha(\alpha q) = \alpha q$. If $p \wedge \alpha q = 0$, then $p \leqslant (\alpha q)'$, so that $\alpha p \leqslant \alpha[(\alpha q)'] = (\alpha q)'$, and thus $\alpha p \wedge \alpha q = 0$. By Proposition 1, (i) is satisfied.

If (i) is satisfied, it is obvious that the range is a subalgebra. Finally, suppose (iii) is satisfied. Then an element p in A is in the range of α if and only if $\alpha p = p$. Since the range is a subalgebra, $(\alpha p)' = \alpha[(\alpha p)']$, and hence (ii) holds.

DEFINITION. *An averaging hemimorphism of a Boolean algebra A is a hemimorphism α such that $\alpha 1 = 1$ and such that $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$ for each p, q in A.*

The following terminology is more or less standard. A *derivative operator* is a hemimorphism α for which $\alpha^2 p \leqslant \alpha p$ for all p in A. An *incompressible* mapping α on A is one for which $\alpha p \leqslant p$ implies $\alpha p = p$. An *extensive* mapping α on A is one in which $p \leqslant \alpha p$ for all p in A. A *self-conjugate* mapping α is one satisfying $\alpha[(\alpha p)'] \leqslant p'$ for all p in A. A *closure operator* is an extensive derivative operator. (An extensive, idempotent, and isotone mapping, whether a hemimorphism or not, is sometimes called a closure; this is the case in the work of Mme. Dubreil-Jacotin mentioned in the Introduction.) The facts about these concepts are easily proved; we shall collect them here without proof **(3, 14, 15, 17)**.

PROPOSITION 4. (i) *A necessary and sufficient condition that a derivative operator be a closure operator is that it be incompressible.* (ii) *A self-conjugate derivative operator α for which $\alpha 1 = 1$ is a closure operator.*

Halmos **(3)** defines an *existential quantifier* on a Boolean algebra A as a mapping $\exists$ of A into itself which satisfies $\exists 0 = 0$, $p \leqslant \exists p$, and

$$\exists(p \wedge \exists q) = \exists p \wedge \exists q.$$

(The last is, of course, the averaging identity.)

PROPOSITION 5 (Halmos). *A mapping α of A into itself is an existential quantifier if and only if it is a closure satisfying $\alpha[(\alpha p)'] = (\alpha p)'$ for all p in A.*

These results give at once the following characterization:

PROPOSITION 6. *Let α be a mapping of A into itself. Then the following are equivalent*:

(i) *α is an existential quantifier*;

(ii) *α is a self-conjugate derivative operator satisfying $\alpha 1 = 1$*;

(iii) *α is an extensive averaging hemimorphism.*

An averaging hemimorphism is almost an existential quantifier; the following shows precisely how it fails to be extensive.

PROPOSITION 7. *If α is an averaging hemimorphism, then $\alpha(p + (p \wedge \alpha p)) = 0$.*

Proof.

$$\alpha(p + (p \wedge \alpha p)) = \alpha(p \wedge (\alpha p)') = \alpha(p \wedge \alpha[(\alpha p)'])$$
$$= \alpha p \wedge \alpha[(\alpha p)'] = \alpha p \wedge (\alpha p)' = 0.$$

A useful characterization of existential quantifiers which exhibits their symmetric behaviour is the following.

PROPOSITION 8. *If α is a hemimorphism, the following are equivalent*:

(i) *α is an existential quantifier*;

(ii) *$\alpha 1 = 1$, $\alpha^2 = \alpha$, and $p \wedge \alpha q = 0$ implies $q \wedge \alpha p = 0$.*

The proof is straightforward, and will be omitted.

PROPOSITION 9. *Let α be an averaging hemimorphism on A, with range B, and let $K = \{p \in A : \alpha p = 0\}$. Then K is an ideal of A, and $K \cap B = (0)$.*

Proof. If $p, q \in K$, then $\alpha(p \vee q) = \alpha p \vee \alpha q = 0$, so that $p \vee q \in K$. If $p \in K$, $q \leqslant p$, then $\alpha q = 0$, since any hemimorphism is isotone. An element p of A belongs to both B and K if and only if both $\alpha p = p$ and $\alpha p = 0$.

We note explicitly that we have shown that the kernel of any hemimorphism is an ideal.

PROPOSITION 10. *Let α be an averaging hemimorphism of A, let B be the range of α, let $K = \{p \in A : \alpha p = 0\}$, let C be the quotient Boolean algebra A/K, and let $\gamma : A \to C$ be the natural projection of A onto C. Then γ maps B isomorphically into C, and if β is defined on C by $\beta\gamma p = \gamma\alpha p$ for all p in A, then β is an existentital quantifier on C having as its range the image γB of B in C.*

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Proof. That γ is one–one on B follows from Proposition 9. We show first that β is well defined on C. If $\gamma p_1 = \gamma p_2$, then $\gamma(p_1 + p_2) = 0$, so that

$$p_1 \vee p_2 = p_1 \vee (p_1 + p_2) = p_2 \vee (p_1 + p_2),$$

and hence

$$\alpha(p_1 \vee p_2) = \alpha p_1 \vee \alpha(p_1 + p_2) = \alpha p_2 \vee \alpha(p_1 + p_2).$$

But $\gamma(p_1 + p_2) = 0$ if and only if $\alpha(p_1 + p_2) = 0$, so that $\alpha p_1 = \alpha p_2$, and thus $\gamma\alpha p_1 = \gamma\alpha p_2$. Hence β is well defined. That β is an averaging hemimorphism is a matter of routine calculation.

It is a little more subtle that the range of β is precisely γB. First, let $p \in B + K$, so that $p = q + k$, $q \in B$, $k \in K$. Since $\alpha k = 0$, then $\alpha(k') = 1$. Then

$$\begin{aligned}\alpha p = \alpha(q + k) &= \alpha((q \wedge k') \vee (q' \wedge k)) = \alpha(q \wedge k') \vee \alpha(q' \wedge k)\\ &= \alpha(\alpha q \wedge k') \vee \alpha(\alpha(q') \wedge k) = (\alpha q \wedge \alpha(k')) \vee (\alpha(q') \wedge \alpha k)\\ &= (\alpha q \wedge 1) \vee (\alpha(q') \wedge 0) = \alpha q = q.\end{aligned}$$

Hence $p = \alpha p + k$ if $p \in B + K$. The converse is obvious. Now $\beta\gamma p = \gamma p$ if and only if $\gamma\alpha p = \gamma p$, if and only if $\gamma(p + \alpha p) = 0$, if and only if

$$0 = \alpha(p + \alpha p),$$

if and only if $p + \alpha p \in K$, if and only if $p \in B + K$. Thus $\gamma p = \beta\gamma p$ implies $\gamma p \in \gamma B$.

This result gives an adequate algebraic characterization of averaging hemimorphisms as extensions of existential quantifiers. The nature of this extension is greatly clarified by means of the topological representation of these operators.

3. Representation of averaging hemimorphisms. A compact and totally disconnected Hausdorff space X is called a Boolean space. If 2 denotes the two-element Boolean algebra, endowed with the discrete topology, let $2(X)$ denote the algebra of all continuous functions from X to 2. The Stone duality theorem **(12)** asserts that every Boolean algebra A is isomorphic to an algebra $2(X)$, where X is a Boolean space uniquely defined within homeomorphisms by A. We call X the *dual space* of A, and we shall identify A with $2(X)$. If B is another Boolean algebra, with dual space Y, and if γ is a homomorphism of A into B, then there is a continuous mapping g of Y into X such that $\gamma p(y) = p(gy)$ for each $p \in A$ and each $y \in Y$; conversely each continuous mapping $g : Y \to X$ defines in this way a homomorphism $\gamma : A \to B$. The homomorphism γ and the continuous mapping g determine each other uniquely, and each is called the other's dual. Either is one–one if and only if the other is onto its range.

If B is a subalgebra of a Boolean algebra A, there is an ambiguity concerning the representation of B as an algebra of functions: the above convention requires us to regard B both as an algebra of functions on the dual space X of A and as the algebra of continuous functions on its own dual space Y. We resolve this in favour of the former interpretation. Thus, if B is a subalgebra of $A = 2(X)$, and if Y is the dual space of B, let $j : B \to 2(Y)$ be the isomorphism given by Stone's theorem. If i is the identity injection of B into A, then the mapping ij^{-1} is an isomorphism of $2(Y)$ into $2(X)$. Its dual $\pi : X \to Y$ is called the projection of X associated with, or determined by, the subalgebra B of A. The mapping π is related to j by the equation $jp(\pi x) = p(x)$ for any p in B and any x in X. Every continuous mapping π of a Boolean space X onto a Boolean space Y determines a subalgebra B of $2(X)$, where B has dual space Y. The subalgebra B consists of those functions in $2(X)$ which are constant on each of the sets $\pi^{-1}\,\pi x$, $x \in X$.

If $A = 2(X)$ and if $F \subset X$ is a closed set, let $O(F)$ be the set of all these elements in A which vanish on the closed set F. The set $O(F)$ is an ideal of A; each ideal of A is of this form, and the ideals of A and the closed sets are thereby in one–one correspondence. The quotient algebra $2(X)/O(F)$ is a Boolean algebra isomorphic with $2(F)$.

This is an account of the needed portions of the Stone duality theory **(7)**. The extension to hemimorphisms is also needed **(3, 14)**. If α is a hemimorphism of $2(X)$ into $2(Y)$, then its dual α^* is a Boolean relation from Y to X, and $\alpha p(y) = \sup\{p(x) : y\,\alpha^* x\}$. Conversely, every Boolean relation from Y to X determines a hemimorphism in this way. The usual rules of duality hold. The relation dual to a hemimorphism α is defined by setting $y\,\alpha^*\,x$ if and only if $p(x) \leqslant \alpha p(y)$ for all p in A.

One more concept is needed. Let $\mathfrak{F}(X)$ denote the collection of all non-empty closed subsets of X. We shall endow $\mathfrak{F}(X)$ with the so-called Hausdorff–Vietoris–Frink topology. This is defined as follows. For each finite collection $U_1, \ldots, U_n$ of open subsets of X, let $\langle U_1, \ldots, U_n\rangle$ denote the collection of elements E in $\mathfrak{F}(X)$ such that $E \cap U_j$ is not empty, for each $j = 1, \ldots, n$, and such that $E \subset U_1 \cup \ldots \cup U_n$. The smallest topology for $\mathfrak{F}(X)$ which contains all such sets is the desired topology. The sets $\langle U_1, \ldots, U_n\rangle$ form a basis for the open sets of this topology, in fact, and, since X is a compact Hausdorff space, the space $\mathfrak{F}(X)$ is also a compact Hausdorff space. The space X is homeomorphic with a closed subset of $\mathfrak{F}(X)$, in the obvious fashion.

Properties of this topology are given in considerable detail in the paper of Michael **(8)**, which is the only reference needed on this topic in the sequel. We shall require principally the following facts about $\mathfrak{F}(X)$. (i) The union of a compact family of closed sets in X is a closed set in X, where compactness of the family is in terms of the topology in $\mathfrak{F}(X)$. (ii) If π_0 maps Z onto Y, let $\mathfrak{D}(\pi_0) = \{\pi_0^{-1}y : y \in Y\}$ be regarded as a subset of $\mathfrak{F}(Z)$, where Y and Z are compact Hausdorff spaces. Then π_0 is an open mapping if and only if $\mathfrak{D}(\pi_0)$ is a closed subset of $\mathfrak{F}(Z)$. (iii) If π_0, Y, Z are as in (ii), and if we define a mapping τ of Y into $\mathfrak{F}(Z)$ by $\tau y = \pi_0^{-1}\,y$, then τ is continuous if and only if $\mathfrak{D}(\pi_0)$ is closed in $\mathfrak{F}(Z)$. These results are, respectively, Theorem 2.5.1, Corollary 5.12, and Proposition 5.11.2 of Michael **(8)**.

The space $\mathfrak{F}(X)$ is sometimes called the *hyperspace* of X, which suggests the following terminology.

DEFINITION. *Let X and Y be Boolean spaces, and let π be a continuous mapping of X onto Y. By a hypersection of π is meant a continuous mappping $\tau : Y \to \mathfrak{F}(X)$ such that $\tau y \subset \pi^{-1}y$ for each y in Y.*

Recall that a *cross-section* of π is a continuous mapping $\gamma : Y \to X$ such that the composite $\pi\gamma$ is the identity mapping on Y. Since X may be regarded as a subspace of $\mathfrak{F}(X)$, a cross-section of π may be regarded as a special case of a hypersection.

THEOREM I. *Let B be a subalgebra of a Boolean algebra A, let π be the projection of the dual space X of A onto the dual space Y of B, and let α be an averaging hemimorphism of A having B as its range. Then there exists a hypersection $\tau : Y \to \mathfrak{F}(X)$ of π such that $\alpha p(x) = \sup\{p(x_1) : x_1 \in \tau\pi x\}$. Conversely, if $\tau : Y \to \mathfrak{F}(X)$ is a hypersection of π, and if αp is defined by*

$$\alpha p(x) = \sup\{p(x_1) : x_1 \in \tau\pi x\},$$

then $\alpha p \in B$, and α is an averaging hemimorphism on A having B as its range.

Proof. Suppose first that τ is a hypersection of π. Since $\mathfrak{F}(X)$ is a compact Hausdorff space, the collection $\{\tau y : y \in Y\}$ is a closed and compact set in $\mathfrak{F}(X)$. If we set $Z = \bigcup\{\tau y : y \in Y\}$, then Z is a closed and compact set in X, by (i) above. Since $\tau y \subset \pi^{-1} y$, it follows easily that $\tau y = Z \cap \pi^{-1} y$. Let π_0 denote the restriction of π to Z; then $\tau y = \pi_0{}^{-1} y$ in Z. From this follows

$$\tau(Y) = \{\tau y : y \in Y\} = \{\pi_0{}^{-1} y : y \in Y\} = \mathfrak{D}(\pi_0),$$

where we use the notation of (ii) above. Since $\mathfrak{F}(Z)$ is a closed subspace of $\mathfrak{F}(X)$, then $\tau : Y \to \mathfrak{F}(Z)$ is continuous, by remark (iii) above. Then the mapping π_0 is an open mapping on Z.

Now define a relation ρ on X by setting $x \,\rho\, x_1$ if and only if $x_1 \in \tau\pi x$. For a fixed element x in X, the set of all x_1 in X such that $x \,\rho\, x_1$ is the set $\tau\pi x$, which is closed in X. This asserts that ρ is a point-closed relation **(14)**. For any subset $U \subset X$, the set $\rho^{-1} U$ is the set of all those x in X such that $U \cap \tau\pi x$ is not empty. We show that this set is $\pi^{-1} \pi_0(U \cap Z)$. Suppose $u \in U \cap \tau\pi x$; then $u \in U \cap Z \cap \tau\pi x$, since $\tau\pi x \subset Z$, and hence

$$\pi_0 u = \pi_0\tau\pi x = \pi x.$$

Thus $x \in \pi^{-1} \pi_0 u$, and therefore $x \in \pi^{-1} \pi_0(U \cap Z)$. Conversely, if $x \in \pi^{-1} \pi_0(U \cap Z)$, then $\pi x = \pi_0 u$ for some u in $U \cap Z$. Since $\tau\pi x = \pi_0{}^{-1}\pi x$, then $u \in \pi_0{}^{-1} \pi x = \tau\pi x$, and therefore $U \cap \tau\pi x$ is not empty. Thus $\rho^{-1} U = \pi^{-1} \pi_0(U \cap Z)$. If U is a closed-open set in X, then $U \cap Z$ is closed-open in Z. Since π_0 is an open mapping on Z, and since π_0 is continuous from the compact Hausdorff space Z to the compact Hausdorff space Y, the set

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$\pi_0(U \cap Z)$ is closed-open in Y. Hence $\pi^{-1}\pi_0(U \cap Z)$ is closed-open in X. This means that ρ is a Boolean relation in X. If we define

$$\alpha p(x) = \sup\{p(x_1) : x \rho x_1\} = \sup\{p(x_1) : x_1 \in \pi x\},$$

then αp is continuous on X, and α is a hemimorphism of A into itself **(14)**.

We now show that α is an averaging hemimorphism on A with range B. If $\pi x_0 = \pi x$, then $\alpha p(x_0) = \alpha p(x)$, for if $x_1 \in \tau\pi x$, then $\pi_0 x = \pi x = \pi x_1$, and thus $x_1 \in \tau\pi x_0$. Consequently αp is constant on each $\pi^{-1}\pi x$, $x \in X$, and therefore αp is in B. On the other hand, if p is in B, then $\alpha p = p$, for if $x_1 \in \tau\pi x$, then $\pi_0 x_1 = \pi x$, $\pi x_1 = \pi x$, so that $p(x_1) = p(x)$. This shows that $\alpha^2 p = \alpha p$ and that the range of α is B. By Proposition 3, α is an averaging hemimorphism. This establishes the second half of the theorem.

To prove the first part, we shall begin by constructing from an averaging hemimorphism α a hypersection τ. We shall then let α_1 be the averaging hemimorphism defined by the hypersection τ, as in the second half of the theorem as just proved. The theorem will clearly be proved if we show that $\alpha = \alpha_1$.

Thus, suppose α is an averaging hemimorphism of A onto B. Let

$$K = \{p \in A : \alpha p = 0\},$$

and let Z be the closed subset of X such that $K = O(Z)$. Let π_0 be the restriction of π to Z, so that $\pi_0 : Z \to Y$. By the Stone duality, the dual of π_0 is the restriction of the homomorphism $\gamma : A \to C = A/K$ to the subalgebra B. Since this homomorphism γ is one–one on B, then π_0 maps Z onto Y. (See Proposition 10.) Now, in the notation of (ii) above, let $\mathfrak{D}(\pi_0) = \{\pi_0^{-1}y : y \in Y\}$, regarded as a subset of $\mathfrak{F}(Z)$. The sets $\pi_0^{-1}y$ are closed, by the continuity of π_0. Since the averaging hemimorphism β defined in Proposition 10 is an existential quantifier for the algebra C, and since π_0 is the projection of Z onto Y associated with the subalgebra γB of C, it follows from a result of Halmos **(3)** that π_0 is an open mapping of Z onto Y. Then $\mathfrak{D}(\pi_0)$ is a closed subset of $\mathfrak{F}(Z)$, and if $\tau : Y \to \mathfrak{F}(Z)$ is defined by $\tau y = \pi_0^{-1}y$, then τ is continuous. (See (ii) and (iii) above.) Since $\mathfrak{F}(Z)$ is (homeomorphic to) a closed subset of $\mathfrak{F}(X)$, the mapping τ may be regarded as a continuous mapping of Y into $\mathfrak{F}(X)$. Since π_0 is the restriction of π to Z, then $\pi_0^{-1}y \subset \pi^{-1}y$, and hence $\tau y \subset \pi^{-1}y$. Thus τ is a hypersection of π.

Now let α_1 be the averaging hemimorphism given by

$$\alpha_1 p(x) = \sup\{p(x_1) : x_1 \in \tau\pi x\} = \sup\{p(x_1) : x \rho x_1\},$$

where ρ is the relation defined in the first part of the proof. We show that $\alpha = \alpha_1$. Let α^* be the dual relation of α, so that $x \alpha^* x_1$ if and only if $p(x_1) \leqslant \alpha p(x)$ for all p in A. If x_1 is not in Z, then there is an element p_0 in A such that $p_0(x_1) = 1$ and such that p_0 is in $O(Z)$. Then $\alpha p_0 = 0$, so that $p_0(x_1) \leqslant \alpha p_0(x)$ is false. Thus $x \alpha^* x_1$ implies that x_1 is in Z. Moreover, this also implies that $q(x_1) \leqslant q(x)$ for all q in B. Since q' is in B whenever q is

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in B, it follows that $x\,\alpha^*\,x_1$ implies $q(x_1) = q(x)$ for all q in B. We have therefore shown that $\alpha^* \subset \rho$. Thus $\alpha = \alpha^{**} \leqslant \rho^* = \alpha_1$ **(14)**, so that $\alpha p \leqslant \alpha_1 p$ for each p in A. Thus $\alpha_1 p = 0$ implies $\alpha p = 0$. On the other hand, if $\alpha p = 0$, then p is in $K = O(Z)$. By the definition of α_1, it follows that $\alpha_1 p = 0$. Hence $\alpha p = 0$ if and only if $\alpha_1 p = 0$. Since $\alpha(p \wedge (\alpha p)') = 0$, then

$$\alpha_1(p \wedge (\alpha p)') = 0.$$

Since αp is in B, then $(\alpha p)'$ is in B, and hence $\alpha_1((\alpha p)') = (\alpha p)'$. Since α_1 is an averaging hemimorphism,

$$0 = \alpha_1(p \wedge (\alpha p')') = \alpha_1 p \wedge (\alpha p)',$$

so that $\alpha_1 p \leqslant \alpha p$. Thus $\alpha_1 p = \alpha p$ for each p in A, and the proof is complete.

This result occupies an intermediate position between the characterization of retractions as given by the Stone duality theory and the characterization of existential quantifiers as given by the Halmos duality theory. A *retraction* is an idempotent homomorphism γ of A onto the subalgebra B. A hemimorphism is a homomorphism if and only if its dual relation is a function whose domain is the dual space of B, in which case the dual relation is the function given by the Stone duality theory. In terms of the above theorem, an averaging hemimorphism is a retraction if and only if the hypersection maps Y into X (recall that X is homeomorphically embedded in its hyperspace). Thus an averaging hemimorphism is a retraction if and only if its hypersection is in fact a cross-section of the projection on the dual space. At the other extreme, an averaging hemimorphism is an existential quantifier if and only if the only element which it annihilates is 0. This is equivalent to having $Z = X$ in the notation of the theorem, which in turn is exactly the requirement that $\tau y = \pi^{-1} y$ for each y in Y. Under this condition, the continuity of τ and the openness of π are the same thing **(8**; Corollary 5.12**)**.

4. Averaging transformations. A Boolean algebra is a ring under the addition $p + q$ and the multiplication $p \wedge q$. It is, moreover, a linear algebra over the field 2 in a natural way. This section is devoted to simple properties of mappings of a Boolean algebra which are linear transformations. We record the following obvious fact.

PROPOSITION 11. *Let α be a mapping of a Boolean algebra A into a Boolean algebra B. The following are equivalent*:

(i) $\alpha(p + q) = \alpha p + \alpha q$ *for all* p, q *in* A;

(ii) $\alpha(p \vee q) + \alpha(p \wedge q) = \alpha p + \alpha q$ *for all* p, q *in* A;

(iii) *if* $p \wedge q = 0$, *then* $\alpha(p \vee q) = \alpha p + \alpha q$ *for all* p, q *in* A.

We shall call a mapping satisfying any one of the above conditions a *linear transformation*. We might equally call such a mapping a B-valued measure on A.

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PROPOSITION 12. *If α is a linear transformation of A into B, then $\alpha(p') = (\alpha p)'$ for each p in A if and only if $\alpha 1 = 1$.*

Proof. $\alpha(p') = \alpha(1 + p) = \alpha 1 + \alpha p$ and $(\alpha p)' = 1 + \alpha p$.

PROPOSITION 13. *Let α be a linear transformation of A into itself. The following are equivalent:*

(i) $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$ *for all* p, q *in* A;

(ii) $p \leqslant \alpha q$ *implies* $\alpha p \leqslant \alpha q$, *and* $p \wedge \alpha q = 0$ *implies* $\alpha p \wedge \alpha q = 0$ *for all* p, q *in* A.

Proof. This proof is identical with the proof of Proposition 1, and this similarity is a reflection of the fact that the averaging identity is defined solely in terms of the multiplicative semigroup of A.

PROPOSITION 14. *Let α be a linear transformation of A into itself and satisfying $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$ for all p, q in A. Then the range of α is a subring of B, and the following are true for all p in A:*

(i) $\alpha(p \wedge \alpha p) = \alpha p$;

(ii) $\alpha(p \vee \alpha p) = \alpha^2 p$;

(iii) $\alpha p \geqslant \alpha^2 p = \alpha^3 p$.

Proof. That the range is closed under addition is a consequence of linearity, and that it is closed under multiplication is a consequence of the averaging identity. (The range therefore fails to be a subalgebra if and only if it fails to contain 1.) Setting $q = p$ in the averaging identity yields (i). This and Proposition 11 give (ii) at once. Finally,

$$\alpha(\alpha p) = \alpha(\alpha p \wedge \alpha p) = \alpha^2 p \wedge \alpha p,$$

so that $\alpha^2 p \leqslant \alpha p$; moreover,

$$\alpha^3 p = \alpha(\alpha^2 p) = \alpha(\alpha p \wedge \alpha^2 p) = \alpha^2 p \wedge \alpha^2 p = \alpha^2 p.$$

This result cannot be improved without further restrictions; see the example in the next section.

PROPOSITION 15. *If α is a linear transformation of A into itself satisfying $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$ for all p, q in A, then the following are equivalent:*

(i) $\alpha^2 = \alpha$;

(ii) $\alpha 1$ *is an identity element for the range of* α.

PROPOSITION 16. *If α is a linear transformation of A into itself satisfying $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$ for each p, q in A, then the following are equivalent:*

(i) $\alpha^2 = \alpha$, *and the range is a subalgebra*;

(ii) $\alpha 1 = 1$.

These both follow from the observation

$$\alpha^2 p = \alpha(\alpha p) = \alpha(1 \wedge \alpha p) = \alpha 1 \wedge \alpha p.$$

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DEFINITION. *An averaging transformation of a Boolean algebra A is a linear transformation α of A into itself satisfying $\alpha 1 = 1$ and $\alpha(p \wedge \alpha q) = \alpha p \wedge \alpha q$ for all p, q in A.*

We pause for a moment to record the following well-known and easily proved fact clarifying the relation between linear transformations and hemimorphisms.

PROPOSITION 17. *Let α be a mapping of a Boolean algebra A into a Boolean algebra B. Then the following are equivalent:*

(i) *α is a homomorphism;*

(ii) *α is an isotone linear transformation;*

(iii) *α is a linear hemimorphism.*

PROPOSITION 18. *Let B be a subalgebra of A, let β be an averaging hemimorphism of A having range B, and let α be a linear transformation of A having range B. Then the following are equivalent:*

(i) *α is an averaging transformation, and $\beta p = 0$ implies $\alpha p = 0$ for all p in A;*

(ii) *$(\beta(p'))' \leqslant \alpha p \leqslant \beta p$ for all p in A.*

Proof. Suppose (i) is true. Since $\beta(p \wedge (\beta p)') = 0$, by Proposition 7, then $\alpha(p \wedge (\beta p)') = 0$. Since $\alpha((\beta p)') = (\beta p)'$, because α is the identity on the range B, and since α is averaging, we have $\alpha p \wedge (\beta p)' = 0$. Hence $\alpha p \leqslant \beta p$. This gives $(\alpha p)' = \alpha(p') \leqslant \beta(p')$, so that $(\beta(p')') \leqslant \alpha p$.

Conversely, suppose (ii) is true. Then, clearly, $\beta p = 0$ implies $\alpha p = 0$. For any q in B, $q = \beta q = (\beta(q'))'$, so that $\alpha q = q$, and hence α is the identity on B. This gives $\alpha^2 = \alpha$ at once. Since, for any p in A, αp is in B, then $\beta\alpha p = \alpha p$, and $(\beta((\alpha p)'))' = \alpha p$. If, for p, q in A, we have $p \leqslant \alpha q$, then $\beta p \leqslant \beta\alpha q = \alpha q$. If $p \wedge \alpha q = 0$, then $\beta(p \wedge \alpha q) = 0$. Hence $p \wedge \alpha q = 0$ implies

$$0 = \beta(p \wedge \alpha q) = \beta(p \wedge \beta(\alpha q)) = \beta p \wedge \beta\alpha q = \beta p \wedge \alpha q.$$

Thus $\alpha p \wedge \alpha q = \alpha p \wedge \beta p \wedge \alpha q = 0$. This shows that α satisfies the average identity, and completes the proof of the proposition.

PROPOSITION 19. *Let $\exists$ be an existential quantifier on A, with range B, and let $\forall$ be the universal quantifier defined by $\forall p = (\exists(p'))'$. Let α be a linear transformation of A onto B. Then the following are equivalent:*

(i) *$\forall p \leqslant \alpha p \leqslant \exists p$ for all p in A;*

(ii) *α is an averaging transformation on A.*

This is a special case of the previous proposition. The topological methods of the next section enable us to give a sharper version (Proposition 20).

5. Representation of averaging transformations. The additive group of a Boolean algebra is an abelian group in which every element has order two. Since an averaging transformation is a special endomorphism of this group, the appropriate tool for describing it is the Pontrjagin duality theory for abelian groups **(10, 13)**.

For the moment let A and B denote simply two discrete abelian groups in which every element has order two. We write the group operation additively, By a character of A is usually meant a (necessarily) continuous homomorphism of A into the multiplicative group of complex numbers of unit modulus. Since every element of A has order two, it is evident that the group of all characters is isomorphic to the additive group of all linear transformations of A into 2. If ξ is such a linear transformation, we let $\langle p, \xi\rangle$ denote the value of ξ at the element p of A. Let $\tilde{A}$ denote the set of all linear transformations of A into 2, endowed with the pointwise operation of addition, and given the compact-open topology. The group $\tilde{A}$ so topologized is called the *character group* of A, and its elements are called *characters* of A. Since A is discrete, the group $\tilde{A}$ is compact. (A character is thus a linear functional of A regarded as a vector space over 2, or is a 2-valued measure on A.)

A homomorphism $\alpha : A \to B$ induces and is induced by a continuous homomorphism $\tilde{\alpha} : \tilde{B} \to \tilde{A}$, so that $\langle \alpha p, \eta\rangle = \langle p, \tilde{\alpha}\eta\rangle$ for each p in A and each η in $\tilde{B}$. The mappings α and $\tilde{\alpha}$ are each other's *adjoint*, and as usual "one-one" and "onto" are properties adjoint the one to the other.

If B is a subgroup of A, let $B^{\perp}$ be the subset of $\tilde{A}$ given by

$$B^{\perp} = \{\xi \in \tilde{A} : p \in B \text{ implies } \langle p, \xi\rangle = 0\}.$$

It is a closed subgroup of $\tilde{A}$. The mapping $\tilde{\pi} : \tilde{A} \to \tilde{B}$ which assigns to each ξ in $\tilde{A}$ its restriction to B is a homomorphism whose kernel is $B^{\perp}$. It is the adjoint of the identity mapping $i : B \to A$; consequently $\tilde{A}/B^{\perp} \cong \tilde{B}$. If B is a subgroup of A, we call this mapping $\tilde{\pi} : \tilde{A} \to \tilde{B}$ the *projection* associated with B.

The Pontrjagin duality theorem asserts that A is the character group of $\tilde{A}$. This, together with the above results, implies that if K is a subgroup of A, then $K^{\perp}$ is isomorphic to the character group of A/K.

Suppose that α is a homomorphism of A into itself, such that $\alpha^2 = \alpha$. Let B be the subgroup of A which is the range of α, and let $\tilde{\alpha}$ be the adjoint mapping. For any p in B and any η in $\tilde{B}$, we have

$$\langle p, \tilde{\pi}\tilde{\alpha}\eta\rangle = \langle ip, \tilde{\alpha}\eta\rangle = \langle \alpha p, \eta\rangle.$$

Since $\alpha p = p$ for any p in B, we have $\langle p, \tilde{\pi}\tilde{\alpha}\eta\rangle = \langle p, \eta\rangle$ for all p in B, so that $\tilde{\pi}\tilde{\alpha}\eta = \eta$ for any η in $\tilde{B}$. Thus $\tilde{\alpha}$ is a cross-section of the projection $\tilde{\pi}$. Conversely, if $\tilde{\alpha}$ is a cross-section of $\tilde{\pi}$, then the adjoint α is a homomorphism of A onto B which is the identity on B, so that, in particular, $\alpha^2 = \alpha$.

This much is standard group theory, and applies with trivial changes to any locally compact abelian group. Suppose henceforth that A and B are Boolean algebras, and that B is a subalgebra of A. Let X be the dual space of A as given in the Stone representation theory. Each point x in X determines a ring-theoretic homomorphism of A onto 2, and each such homomorphism determines a point x in X. Hence we may regard X as a subset of $\tilde{A}$. Since A is given the discrete topology, the topologies of X and $\tilde{A}$ in the Stone and Pontrjagin theories rsepectively are simply the topologies induced in these spaces as subsets of the product space 2^A in the Tychonoff product topology. In brief, X is a compact subset of $\tilde{A}$. Since the identity $i : B \to A$ is a ring-isomorphism, the adjoint $\tilde{\pi} : \tilde{A} \to \tilde{B}$ must map X onto the dual space Y of B, regarded as a subset of $\tilde{B}$. The restriction of $\tilde{\pi}$ to X is the projection of X onto Y associated with the subalgebra B.

If $A = 2(X)$, an element of the character group $\tilde{A}$ is said to be *supported* by a closed subset F of X if the element belongs to $O(F)^{\perp}$. Since $O(F)^{\perp}$ is isomorphic to the character group of $A/O(F)$, and since $A/O(F)$ is isomorphic to $2(F)$, it follows that an element of the character group of A which is supported by F may be regarded as an element of the character group of $2(F)$.

DEFINITION. *Let π be a continuous mapping of a Boolean space X onto a Boolean space Y. By a character-section of π is meant a continuous mapping σ of Y into the character group of $A = 2(X)$ such that, for each y in Y, the character σy is supported by $\pi^{-1} y$, and such that $\langle 1, \sigma y\rangle = 1$ for each y in Y.*

This is the analogue of the notion of a probability section, as defined in **(16)**.

THEOREM II. *Let B be a subalgebra of a Boolean algebra A, let π be the projection of the dual space X of A onto the dual space Y of B, and let α be an averaging transformation of A having B as its range. Then there exists a character-section σ of π such that $\alpha p(x) = \langle p, \sigma\pi x\rangle$ for each p in A and each x in X. Conversely, if σ is a character-section of π, and if $\alpha p(x) = \langle p, \sigma\pi x\rangle$, then αp is in B, and the mapping α is an averaging transformation of A having B as its range.*

Proof. For any p in A we define a function p^+ on Y by setting

$$p^+(y) = \sup\{p(x) : \pi x = y\}.$$

Since p is in A, it is a continuous function on X, so that the set $\{x \in X : p(x) = 1\}$ is closed and compact. Since π is continuous, the set

$$\{y \in Y : p^+(y) = 1\} = \pi\{x \in X : p(x) = 1\}$$

is compact and closed in Y. Let P^+ be the complement of this set, so that P^+ is open. Let y_0 be an element of P^+, and let P be an open-closed set containing y_0 and contained in P^+. Let Q be the complement of P and let q^+ be the characteristic function of Q. Then q^+ is continuous and vanishes at y_0. Moreover, if $q^+(y) = 0$, then y is in P, so that $p^+(y) = 0$. Hence $p^+(y) \leqslant q^+(y)$ for all y in Y, and $q^+(y_0) = 0$.

Suppose first that α is an averaging transformation on A with range B. Let $\tilde{\alpha}$ be the adjoint of α. Then $\tilde{\pi}\tilde{\alpha}y = y$ for each y in $Y \subset \tilde{B}$. We show that $\tilde{\alpha}y$

is in $O(\pi^{-1}y)^{\perp}$. Let p belong to $O(\pi^{-1}y)$, so that $p(x) = 0$ for all x satisfying $\pi x = y_0$. Let p^+ be defined as above, and let q^+ be constructed as before. Since q^+ is continuous on Y, there is an element q in B such that $jq = q^+$. Since $p(x) \leqslant p^+(\pi x) \leqslant q^+(\pi x)$ for all x in X, then $p \leqslant q$ in A. Since $\alpha q = q$, we have $p \leqslant \alpha q = q$, and since α satisfies the averaging identity, we have $\alpha p \leqslant \alpha q = q$. On the other hand,

$$0 = q^+(y_0) = jq(y_0) = j\alpha q(y_0).$$

Thus

$$j\alpha p(y_0) = 0 = \langle \alpha p, y_0\rangle = \langle p, \tilde{\alpha} y_0\rangle.$$

This shows that $\tilde{\alpha}y_0$ is in $O(\pi^{-1}y_0)^{\perp}$, as desired. Finally we note that $\langle 1, \tilde{\alpha}y\rangle = 1$ for each y in Y. If we let σ denote the restriction of $\tilde{\alpha}$ to the subset Y of $\tilde{B}$, then σ is the desired character-section of π.

Conversely, let σ be a character-section of π, and define $\alpha p(x) = \langle p, \sigma\pi x\rangle$. It is at once clear that αp is continuous, that α is linear, that αp is in B, and that $\alpha 1 = 1$. Suppose $p \leqslant \alpha q$; we show that $\alpha p \leqslant \alpha q$. We need only show that $\alpha q(x) = 0$ implies that $\alpha p(x) = 0$. If $\alpha q(x) = 0$, and if $\pi x_1 = \pi x$, then $\alpha q(x_1) = 0$. Hence αq is in $O(\pi^{-1}\pi x)$, so that also p is in $O(\pi^{-1}\pi x)$. Since $\sigma\pi x$ is in $O(\pi^{-1}\pi x)^{\perp}$, then $\langle p, \sigma\pi x\rangle = 0$; in other words, $\alpha(x)p = 0$, as desired. Since α is linear and since $\alpha 1 = 1$, then $(\alpha q)' = \alpha(q')$ for any q in A. Hence $p \wedge \alpha q = 0$ if and only if $p \leqslant \alpha(q')$. As we have just seen, the latter implies that $\alpha p \leqslant \alpha(q') = (\alpha q')$, and hence that $\alpha p \wedge \alpha q = 0$. Thus α is an averaging transformation, and the proof is complete.

A very slight modification of the above argument yields the following result, sharpening Proposition 19; we omit the details.

PROPOSITION 20. *Let B be a subalgebra of the Boolean algebra A, and let π be the projection of X onto Y. A necessary and sufficient condition that a linear transformation α of A into B be an averaging transformation is that*

$$\inf\{p(x_1) : \pi x_1 = \pi x\} \leqslant \alpha p(x) \leqslant \sup\{p(x_1) : \pi x_1 = \pi x\}$$

for each p in A and each x in X.

We have mentioned before the result of Kelley **(6)** that an idempotent isotone operator on $C(X)$ satisfies the averaging identity if and only if its range is a subalgebra. Proposition 17 shows that, for averaging transformations in a Boolean algebra, the assumption of isotonicity is quite restrictive. Proposition 16 asserts that any averaging transformation of a Boolean algebra is idempotent, and has a subalgebra as range. It may well be asked if an idempotent linear transformation having a subalgebra as range necessarily satisfies the averaging identity. In view of the fact that idempotent linear transformations of A onto B correspond to cross-sections of the mapping $\tilde{\pi} : \tilde{A} \to \tilde{B}$, this can be settled by determining the comparative size of $\tilde{\pi}^{-1}y$ and $O(\pi^{-1}y)^{\perp} \cap \tilde{\pi}^{-1}y$, for y in Y, and thereby seeing how restrictive is the requirement that a cross-section of $\tilde{\pi}$ be a character-section. (In particular, if $\tilde{\pi}^{-1} y$ is large compared with $O(\pi^{-1} y)^{\perp} \cap \pi^{-1} y$ for each y in Y, the condition that an idempotent linear transformation of A onto B be an averaging transformation becomes a rather severe restriction. We remark that the existence of idempotent linear transformations is a trivial matter, and follows from the fact that A is a vector space over the field 2.)

For any ξ in $\tilde{A}$, $\tilde{\pi}\xi$ is the restriction of ξ to B, so that $\tilde{\pi}\xi = y$ if and only if $\langle p, \xi\rangle = \langle p, y\rangle$ for all p in B. Then for p in $O(\pi^{-1} y) \cap B$ we have $\langle p, y\rangle = 0$, and if ξ is in $\tilde{\pi}^{-1} y$, then $\langle p, \xi\rangle = 0$. Hence $\tilde{\pi}^{-1} y \subset (O(\pi^{-1} y) \cap B)^{\perp}$. On the other hand, $\tilde{\pi}^{-1} y$ and $B^{\perp}$ have no element in common. For if ξ is in $\tilde{\pi}^{-1} y$, then $\langle p, \xi\rangle = \langle p, y\rangle$ for all p in B, and since 1 is in B, then $1 = \langle 1, y\rangle = \langle 1, \xi\rangle$, so that ξ is not in $B^{\perp}$.

Now let ξ belong to $O(\pi^{-1} y)^{\perp}$, but not to $B^{\perp}$; we show that ξ belongs to $\tilde{\pi}^{-1} y$. If p is in $O(\pi^{-1} y) \cap B$, then $\langle p, \xi\rangle = \langle p, y\rangle = 0$. This shows that the kernel N of $\tilde{\pi}\xi$ contains $O(\pi^{-1} y) \cap B$. Since ξ is not in $B^{\perp}$, then N is not all of B. On the other hand $O(\pi^{-1} y) \cap B$ is a maximal ideal of the algebra B, so that the quotient of the algebra by this ideal is isomorphic to 2. Hence we also have $B/N \cong 2$, and therefore $O(\pi^{-1} y) \cap B = N$. This shows that $\tilde{\pi}\xi = y$.

The above paragraph shows that $B^{\perp} \cup O(\pi^{-1} y)^{\perp} \subset B^{\perp} \cup \tilde{\pi}^{-1} y$. Since $B^{\perp} \cup \tilde{\pi}^{-1} y = \tilde{\pi}^{-1}\{0, y\}$, it follows that $B^{\perp} \cup \tilde{\pi}^{-1} y$ is a group, and that $\tilde{\pi}^{-1} y$ is a coset of $B^{\perp}$. Thus $B^{\perp} + O(\pi^{-1} y) \subset B^{\perp} \cup \tilde{\pi}^{-1} y$. Since

$$B^{\perp} + O(\pi^{-1} y)^{\perp} = (B \cap O(\pi^{-1} y))^{\perp},$$

we have therefore shown that $B^{\perp} \cup \tilde{\pi}^{-1} y = (B \cap O(\pi^{-1} y))^{\perp}$. Then

$$O(\pi^{-1} y)^{\perp} = (O(\pi^{-1} y)^{\perp} \cap B^{\perp}) \cup (O(\pi^{-1} y)^{\perp} \cap \tilde{\pi}^{-1} y).$$

Consequently it suffices to determine the size of the group

$$O(\pi^{-1} y)^{\perp} \cap B^{\perp} = (B + O(\pi^{-1} y))^{\perp}.$$

Since $(B + O(\pi^{-1} y))^{\perp}$ is isomorphic to the character group of

$$A/(B + O(\pi^{-1} y)),$$

then it is a large group if and only if this latter quotient group is a large group. But

$$(A/(O(\pi^{-1} y) + B)) \cong (A/(O(\pi^{-1} y)))/((B + O(\pi^{-1} y))/O(\pi^{-1} y))$$

as abelian groups. Now $A/O(\pi^{-1} y) \cong 2(\pi^{-1} y)$, even as Boolean algebras, while $(B + O(\pi^{-1} y))/O(\pi^{-1} y) \cong B/B \cap O(\pi^{-1} y)$. Since $B \cap O(\pi^{-1} y)$ is a maximal ideal in B, then $B/B \cap O(\pi^{-1} y) \cong 2$. Hence

$$A/(O(\pi^{-1} y) + B) \cong 2(\pi^{-1} y)/2,$$

which is a large group whenever $\pi^{-1} y$ is large.

Hence idempotent linear transformations with algebras as range are not automatically averaging transformations. We give here a very simple example

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of an averaging transformation which can serve as prototype for many examples. Let A be the algebra of all subsets of a finite set X. Define α on A by defining $\alpha p = 0$ if p contains an even number of elements, and by setting $\alpha p = 1$ if p contains an odd number of elements. It is trivial (Proposition 11) that α is linear, and equally trivial (Proposition 13) that it satisfies the averaging identity. If X itself has an even number of elements, we have a linear transformation showing the limit of applicability of Propositions 14–16. If X is a set with an odd number of elements, we have an example of an averaging transformation which is not a homomorphism.

This example also furnishes a linear transformation α whose dual relation α^* is empty. In general, if α is a linear transformation of A into itself and if $\alpha 1 = 1$, it is easily seen that α^* is a function defined on a (possibly empty) closed subset D of X. The above example can be modified so that D is a single non-isolated point, or so that D is an infinite point set having void interior.

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