CALCULATING ASTRONOMICAL REFRACTION BY MEANS OF CONTINUED FRACTIONS

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0. Abstract: A continued fraction was derived for the summation of the asymptotic expansion of astronomical refraction. Using simple approximations for the last denominator of the fraction, accurate formulae, useful down to the horizon, were obtained. The method is not restricted to any model of the atmosphere and can thus be used in calculations based on actual aerological measurements.

1. Introduction

The usual way to evaluate the integral of astronomical refraction is to expand it into a series in powers of the tangent or secant of the zenith distance (Newcomb 1906, Oterma 1960, Joshi and Mueller 1974 and enclosed references). This series may, mathematically, have a non-zero radius of convergence, or else it may be a totally divergent asymptotic series, depending upon the model used for the upper atmosphere. However, for the first coefficients of the series there are not, in practice, significant differences in the numerical values, so that it always behaves like an asymptotic expansion. Thus it can be used only up to a certain zenith distance.

According to the theory of continued fractions the formal power series of a continued fraction is usually an asymptotic expansion, and vice versa. In this paper we shall investigate the use of a continued fraction to sum the expansion of astronomical refraction.

2. Development of the refraction integral

2.1. Notations

μ is the index of refraction of the air
r is the distance from the centre of the earth
z is the apparent zenith distance
μ_0, r_o, z_o are the above quantities at the place of observation

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$\psi = (r\mu/r_o\mu_0)^2 - 1$

$S = \sin z_o$

$C = \cos z_o$

Δz is the astronomical refraction

2.2. Expansion into a continued fraction

The usual asymptotic expansion of the integral of the refraction is

$$\Delta z = S\sum_{n=0} (-1)^n \alpha_n C^{-2n-1} \tag{2.1}$$

where the coefficients α_n are the moment integrals

$$\alpha_0 = \int_1^{\mu_0} \frac{d\mu}{\mu} = \log\mu_0$$
$$\alpha_n = \frac{1\times 3\times \cdots \times(2n-1)}{2\times 4\times \cdots \times(2n)}\int_1^{\mu_0} \psi^n \frac{d\mu}{\mu} \tag{2.2}$$

Now, an asymptotic series of this type has what is called an S-fraction expansion (Wall 1948, p. 200), i.e. a continued fraction expansion of the form

$$\Delta z = \cfrac{\alpha_0 S}{C + \cfrac{b_1}{C + \cfrac{b_2}{C + \cfrac{b_3}{C + \cdots}}}} \tag{2.3}$$

A convenient algorithm for computing the partial numerators b_k from the coefficient α_n is the quotient-difference algorithm (Henrici, 1967). The formulae needed in this case are given in section 3.1. It is difficult to investigate the mathematical convergence of this fraction, but as was shown in an earlier paper (Mikkola 1978) this fraction greatly resembles that of the error function and thus is very probably convergent for $C \neq 0$. However, to get a formula that can also be used at the horizon we first write fraction (2.3) in the recursive form

$$\Delta z = \alpha_0 S/g_1(C)$$
$$g_k(C) = C + b_k/g_{k+1}(C) \tag{2.4}$$

and construct for $g_k(C)$ an asymptotic approximation useful for large k (say for k=n). If we put

$$Q_n = g_n(C)/g_{n+1}(C) \tag{2.4}$$

then

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$$g_n(C) = C + b_n Q_n / g_n(C) \tag{2.5}$$

from which follows the formula

$$g_n(C) = \frac{1}{2}(C + \sqrt{(C^2 + 4b_n Q_n)}) \tag{2.6}$$

As shown by Mikkola, 1978, the terms b_n are approximately of the form $b_n = n\beta$, where β is a number of the order $\beta \sim 10^{-3}$. Now it is not difficult to see that for $C>>0$, $Q_n = 1 + \mathcal{O}(\beta)$, and for $C = 0$, $Q_n = 1 + \mathcal{O}(1/n)$ so that a good first approximation is obtained by replacing Q_n by its horizon value, giving

$$g_n(C) \approx \frac{1}{2}(C + \sqrt{(C^2 + q_n)}) \tag{2.7}$$

Here q_n is chosen in order to obtain the correct value for the horizontal refraction. Numerical experiments show that for large n (say $n > 6$) this approximation is, in practice, sufficient. However, to obtain better approximations we may write

$$g_n(C) = C + \frac{b_n}{\frac{1}{2}(C + \sqrt{(C^2 + q_{n+1})})} \tag{2.8}$$

and chose the parameters b_n and q_{n+1} to give suitable values for both the refraction and its derivative at the horizon. On the other hand (2.8) can be written in the form

$$g_n(C) = C(1 - \frac{2b_n}{q_{n+1}}) + \frac{2b_n}{q_{n+1}}\sqrt{(C^2 + q_{n+1})} \tag{2.9}$$

thus, due to the above fitting conditions we in fact have formula

$$g_n(C) = g_n'(0)C + \sqrt{(C^2(1 - g_n'(0))^2 + g_n^{\,2}(0))} \tag{2.10}$$

Here the prime indicates a derivative with respect to C. If approximation of this type is also used for g_{n+1}, then, due to the similarity of the formulae, their errors are quite similar and a good value for the ratio g_n/g_{n+1} is obtained. Thus we have for the quantity Q_n in formula (2.6) the very good approximation

$$Q_n \approx \frac{g_n'(0)C + \sqrt{(C^2(1 - g_n'(0))^2 + g_n^2(0))}}{g_{n+1}'(0)C + \sqrt{(C^2(1-g_{n+1}'(0))^2 + g_{n+1}^2(0))}} \tag{2.11}$$

From formulae (2.4) it is easy to obtain recursion formulae for the quantities $g_n(0)$ and $g_n'(0)$ (given in section 3.1). However, to start the recursions the horizon refraction

$$\Delta z_\perp = \int_1^{\mu_0} \psi^{-1/2} \frac{d\mu}{\mu} \tag{2.11}$$

and the derivative

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$$\Delta z_{\perp}' = \left\{\frac{d}{dC}\int_{1}^{\mu_0} (C^2 + \psi)^{-1/2}\,\frac{d\mu}{\mu}\right\}_{C=0} \tag{2.11}$$

are needed. Making the formal substitution $C^2 + \psi = \Theta^2$ we obtain

$$\Delta z_{\perp}' = 2\left\{\frac{d}{dC}\int_{\infty}^{C} \frac{d\log\mu(\psi=\Theta^2-C^2)}{d\psi}\,d\Theta\right\}_{C=0} = 2\left(\frac{d\log\mu}{d\psi}\right)$$

or (2.12)

$$\Delta z_{\perp}' = \frac{\mu_0'/\mu_0}{1 + \mu_0'/\mu_0}$$

Here μ_0' is the derivative of the refractive index with respect to the height (at ground and r_o = unit of distance).

3. Results and discussion

3.1. Collection of formulae

The quotient-difference algorithm for computing the partial numerators b_k of the continued fraction:

Using the expansion coefficients α_l defined in (2.2) we start with

$$\left.\begin{aligned} B_{1,l} &= \alpha\ /\alpha_{l-1} \\ B_{2,l} &= B_{1,l} - B_{1,l-1} \end{aligned}\right\} \quad l = 1,2,\ldots,n \tag{3.1}$$

and continued by means of

$$B_{k,l} = \begin{cases} \dfrac{B_{k-1,l}}{B_{k-1,l-1}}\; B_{k-2,l-1}, & \text{if } k \text{ is odd} \\[2ex] B_{k-1,l} - B_{k-1,l-1} + B_{k-2,l-1}, & \text{if } k \text{ is even} \\ k = 3,4,\ldots,n \end{cases} \tag{3.2}$$

which gives b_k values of:

$$b_k = B_{k,k} \tag{3.3}$$

The recursion formulae for the horizontal values of g_n and g_n'. We start with

$$\Delta z_{\perp} = \int_{1}^{\mu_0} \psi^{-1/2}\frac{d\mu}{\mu}$$

(3.4)

$$\Delta z_{\perp}' = \frac{\mu_0'/\mu_0}{1+\mu_0'/\mu_0}$$

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and use the recursions

$$g_1(0) = \alpha_0/\Delta z_\perp \;;\quad g_1'(0) = -\alpha_0^{-1} g_1^{\,2}(0)\Delta z_\perp'$$
$$g_{k+1}(0) = b_k/g_k(0) \;;\quad g_{k+1}'(0) = \frac{g_{k+1}(0)}{g_k(0)}(1-g_k'(0)) \tag{3.5}$$
$$k = 1,2,3,\ \ldots,n$$

The approximation for g_n is

$$Q_n = \frac{g_n'(0)C + \sqrt{C^2(1-g_n'(0))^2 + g_n^{\,2}(0)}}{g_{n+1}'(0)C + \sqrt{C^2(1-g_{n+1}'(0))^2 + g_{n+1}^{\,2}(0)}}$$
$$g_n(C) = \frac{1}{2}(C + \sqrt{(C^2 + 4b_nQ_n)}) \tag{3.6}$$

Now the refraction can be evaluated by means of the formula

$$\Delta z = \cfrac{\alpha_0 S}{C + \cfrac{b_1}{C + \cfrac{b_2}{C + \cdots \cfrac{\cdots}{C + \cfrac{b_{n-1}}{g_n(C)}}}}} \tag{3.7}$$

3.2. Results of some numerical tests

To test the reliability of the approximations for g_n the results obtained using the continued fraction formula were compared with a direct numerical integration using a polytropic model atmosphere. Table 1 gives the errors for different z and n values when approximations of the type (2.7) were used for g_n in (3.7). Table 2 shows the errors when formulae (3.6) are used. As we can see, formulae (3.6) are surprisingly accurate as they yields an negligible error even for n = 1.

More details about the method using a polytropic model atmosphere are given in an earlier paper of the author (Mikkola 1978).

4. Acknowledgements

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Table 1.

Errors when using the formula (2.7) for g_n.

z \ n	1	2	3	4	5	6	7	8	9
80°.00	3".30	-0".10	0".01	-0".00	0".00	0".00	0".00	0".00	0".00
81.00	4.34	-0.16	0.01	-0.00	0.00	0.00	0.00	0.00	0.00
82.00	5.85	-0.25	0.03	-0.00	0.00	0.00	0.00	0.00	0.00
83.00	8.10	-0.42	0.06	-0.01	0.00	-0.00	0.00	0.00	0.00
84.00	11.54	-0.73	0.12	-0.01	0.00	-0.00	0.00	0.00	0.00
85.00	16.95	-1.32	0.27	-0.04	0.01	-0.00	0.00	-0.00	0.00
86.00	25.64	-2.48	0.65	-0.12	0.02	-0.01	0.00	-0.00	0.00
87.00	39.39	-4.75	1.62	-0.37	0.08	-0.07	0.01	-0.01	0.01
87.50	48.51	-6.54	2.59	-0.69	0.16	-0.16	0.03	-0.02	0.02
88.00	58.56	-8.85	4.09	-1.26	0.30	-0.35	0.08	-0.06	0.05
88.50	67.56	-11.51	6.25	-2.28	0.55	-0.78	0.21	-0.16	0.17
89.00	70.37	-13.66	8.75	-3.84	0.93	-1.59	0.52	-0.36	0.49
89.25	66.25	-13.81	9.63	-4.65	1.13	-2.13	0.76	-0.50	0.79
89.50	55.53	-12.51	9.49	-5.07	1.25	-2.54	1.01	-0.63	1.13
89.75	34.95	-8.57	7.06	-4.19	1.05	-2.30	1.03	-0.59	1.23
90.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Table 2.

Errors when using the formulae (3.6) for g_n.

z \ n	1	2	3	4	5	6	7	8	9
80°.00	-0".02	0".00	0".00	0".00	0".00	0".00	0".00	0".00	0".00
81.00	-0.03	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
82.00	-0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
83.00	-0.07	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
84.00	-0.10	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00
85.00	-0.15	0.00	0.01	0.00	0.00	0.00	-0.00	-0.00	0.00
86.00	-0.19	-0.00	0.04	0.01	0.00	-0.00	-0.00	-0.00	0.00
87.00	-0.19	-0.01	0.00	0.02	0.00	-0.01	-0.00	-0.00	-0.00
87.50	-0.13	-0.02	0.18	0.04	0.00	-0.01	-0.01	-0.01	-0.00
88.00	-0.01	-0.03	0.29	0.06	0.00	-0.03	-0.02	-0.01	-0.00
88.50	0.16	-0.04	0.41	0.08	-0.01	-0.07	-0.05	-0.03	-0.01
89.00	0.30	-0.03	0.47	0.08	-0.03	-0.12	-0.09	-0.06	-0.01
89.25	0.31	-0.03	0.42	0.06	-0.05	-0.13	-0.10	-0.07	-0.01
89.50	0.25	-0.02	0.31	0.04	-0.05	-0.12	-0.09	-0.06	-0.01
89.75	0.10	-0.01	0.12	0.01	-0.03	-0.06	-0.05	-0.03	-0.00
90.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

5. References

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DISCUSSION

B. Garfinkel: For what zenith distances is it possible to apply Mikkola's formulae?

J. Kakkuri: answered that he has no complete information on this subject.

B. Garfinkel, J.A. Hughes, K. Poder and J. Saastamoinen: discussed the possibilities of refraction calculation near to zenith distances of 90°.