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HOW TO CONSTRUCT ALMOST FREE GROUPS

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0. Introduction. Almost free groups were introduced in [**9**] as groups all of whose "small" subgroups are free. Here "small" means generated by fewer elements than the cardinality of the group. This concept is a generalization of locally free. Suppose κ is a cardinal $> \omega$. A group is *κ-free* if every subgroup generated by fewer than κ elements is free. A group of cardinality κ which is κ-free is almost free. There are two related concepts which are closer approximations to freeness.

0.1. *Definition.* If A is κ-free, a subgroup B is *κ-pure* if B is a free factor of $B + D$, where $|D| < \kappa$. ($|D|$ denotes the cardinality of the group or set.) A κ-free group A is *strongly κ-free* if every subset of A of cardinality $<\kappa$ is contained in a κ-pure subgroup generated by fewer than κ elements. A κ-free group, A, is *κ-separable* if every subset of cardinality $<\kappa$ is contained in a free factor generated by fewer than κ elements.

Of these notions that of being strongly κ-free is of particular interest because of the following theorem.

0.2. THEOREM. ([**3**], [**13**]) *A group is $L_{\infty\kappa}$-equivalent to a free group ($L_{\infty\kappa}$-free) if and only if it is strongly κ-free.*

Proof. Since the proof of this result follows that of [**3**] (for abelian groups), we will just indicate how it is proved. The necessity follows from the fact that being strongly κ-free is expressible by a sentence of $L_{\infty\kappa}$ and is true for all free groups. In order to show the sufficiency we can set up as Eklof did partial isomorphisms between free factors of a fixed free group and κ-pure subgroups of the strongly κ-free group.

0.3. *Remark.* The terminology of 0.1 is imported from abelian group theory. In most of this paper we treat groups and abelian groups simultaneously. The reader can substitute direct summand for free factor and direct sum for free product in order to obtain the results for abelian groups. Some of our results will be established only for abelian groups (or non-abelian groups). We will alert the reader to these.

0.4. *Outline of Paper.* This paper has two main parts. The first section is devoted to a description of possible non-free κ-free groups. In the remaining sections, various κ-free groups are constructed. The central idea

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is that every κ-free group of cardinality κ can be expressed as the union of a smooth chain (of length κ) of free groups. For regular κ (this is the only interesting case, c.f. Theorem 1.1), this leads to a function, Γ, which assigns an invariant to κ-free groups. Roughly this invariant tells which elements of the chain are not κ-pure. In Section 1, we study κ-free groups by proving results which say a κ-free group can be expressed as the union of a chain of subgroups satisfying certain properties. For example, Theorem 1.13 restricts the possible range at Γ.

In Section 2, we construct many strongly κ-free groups. This is done by inductively defining smooth chains of free groups. From these inductive constructions there arise two other notions, that of a criterion for freeness and that of an embedding property. A criterion for freeness tells us that the groups constructed at limit stages are free. An embedding property is used to go from one stage to the next. Some of these constructions require consequences of Gödel's Axiom of Constructibility, $(V = L)$. These constructions provide relative consistency results. For example, Theorem 2.15 says if $(V = L)$, then for successor cardinals all strongly κ-free abelian groups possible in light of Lemma 1.9 and Theorem 1.13 actually occur.

In Section 3, we use trees and embeddings to construct κ-free not strongly κ-free groups. In Section 4, we vary the embeddings in order to construct para-free groups. It is shown that we can demand all the non-abelian group constructed in Sections 2 and 3 be para-free as well.

If we restrict ourselves to abelian groups, we can obtain more sophisticated embeddings and criteria for freeness. In Section 5, we prove a number of "real world" results. In particular, we show for all $n > 0$ that Γ applied to strongly ω_n-free abelian groups of cardinality ω_n can take any value.

We end the paper with a number of questions.

0.5. *Definition.* A *cardinal* is an initial ordinal. An *ordinal* is identified with the set of its predecessors $\aleph_\sigma = \omega_\sigma = \{\nu | \nu < \omega_\sigma\}$. A subset, C, of an ordinal, ν, is *closed and unbounded* (a cub) if it is closed under unions and for all $\tau < \nu$ there is an $\alpha \in C$ such that $\alpha > \tau$. A set $E \subseteq \nu$ is *stationary* if it has non-empty intersection with every cub.

0.6. *Acknowledgment.* Some of the results in this paper are from the author's doctoral dissertation. I would like to express my gratitude to my thesis advisor, Paul Eklof, for his help during the preparation of my thesis and the years after.

1. Structure of almost free groups. We wish to analyze some of the structure of non-free almost free groups. The following results tell us we can restrict ourselves to considering κ-free groups where κ is regular and not weakly compact.

1.1. THEOREM. (1) ([**18**]) *If A is κ-free and κ is singular then A is κ^+-free.*

(2) ([**13**]) *If κ is weakly compact (resp. strongly compact) and A is a κ-free group of cardinality κ (resp. of arbitrary cardinality), then A is free.*

Proof. See [**6**] Theorem 2.6 for a proof of (2).

In the following, unless otherwise stated, κ will always be a regular, non-weakly compact cardinal and A a κ-free group of cardinality κ. The most obvious thing we can say of A is that it is a union of a chain of free groups.

1.2. *Definition.* A *κ-filtration* of A is an increasing chain, $\Delta = \{A_\nu | \nu < \kappa\}$ of subgroups of A satisfying for all $\nu < \kappa$:

(i) $|A_\nu| < \kappa$;
(ii) $A = \bigcup_{\nu<\kappa} A_\nu$; and
(iii) if ν is a limit ordinal, $A\nu = \bigcup_{\mu<\nu} A_\mu$.

A chain satisfying (iii) is said to be *smooth*.

The following observation is essentially due to Eklof.

1.3. THEOREM. ([**6**], Theorem 2.5) *Suppose Δ is a κ-filtration of A and $E = \{\nu | A_\nu$ is κ-pure$\}$. A is free if and only if E is not stationary.*

Proof. If E is not stationary, we can find a cub $C \subseteq \kappa$ such that $C \cap E = \emptyset$. Let f enumerate C and let

$$\Delta' = \{B_\nu | B_\nu = A_{f(\nu)}\}.$$

Since we have a κ-filtration of A by κ-pure free groups, we can inductively obtain a set of free generators for A.

Assume Δ and E are as in the hypothesis of the theorem. Suppose $\Delta' = \{B_\nu | \nu \subset \kappa\}$ is some other κ-filtration of A. An easy argument shows $C = \{\nu | B_\nu = A_\nu\}$ is a cub. If A is free there is a κ-filtration Δ' by κ-pure subgroups. Let C be as above. Then $C \cap E = \emptyset$.

1.4. *Description of* Γ. There is a natural equivalence relation on the subsets of κ. Two sets, E_1 and E_2, are equivalent if there is a cub C such that $C \cap E_1 = C \cap E_2$, or equivalently $(E_1 - E_2) \cup (E_2 - E_1)$ is not stationary. For any set $E \subseteq \kappa$, let $\tilde{E}$ indicate the equivalence class determined by E. This gives the Boolean algebra, $P(\kappa)/I$, where I is the ideal of non-stationary sets. We will use $\cup$, $\cap$, 1, 0 for the Boolean algebra operations and constants.

The proof of Theorem 1.3 shows that $\tilde{E}$ is an invariant of A. In [**6**] the function which assigns $\tilde{E}$ to A was called Γ. That is, if Δ is a κ-filtration of A and $E = \{\nu | A_\nu$ is not κ-pure$\}$ then $\Gamma(A) = \tilde{E}$. A major concern of this paper is to discover possible values of Γ. Note that if $\Gamma(A) \neq 1\ (= \tilde{\kappa})$ then A is strongly κ-free (the converse is not true). In [**8**], the model theoretic significance of Γ is explained.

In [**14**], we attempted to give a construction which given an almost free abelian group A such that $\Gamma(A) = \tilde{E}$ would produce for $E_1 \subseteq E$ a B such that $\Gamma(B) = \tilde{E}_1$. Unfortunately, there is a flaw in the construction. We tacitly assumed A was strongly κ-free. The above result is true for strongly κ-free abelian groups but we will see that given certain set theoretic assumptions (namely $V = L$ and there exists a weakly compact cardinal) it fails in general. So Theorem 1.1 of [**14**] should read:

1.5. THEOREM. *If there exists a non-free strongly κ-free abelian group of cardinality κ, then there exists 2^κ strongly κ-free abelian groups of cardinality κ.*

In this paper we will be able to strengthen this result to recover the original result in some cases. Strangely enough Theorem 1.2 of [**14**] and its proof are correct as written.

Although a κ-free group need not be strongly κ-free it is strongly ρ-free for all $\rho < \kappa$. We begin by showing the following.

1.6. THEOREM. *If κ is a regular cardinal and A is κ^+-free then A is strongly κ-free.*

Proof. Suppose first $\kappa > \omega$. Assume the theorem is false and A is a counter example. There is a $B \subseteq A$, $|B| < \kappa$ such that B is not contained in a κ-pure subgroup of cardinality $<\kappa$. We now define for ordinals $\nu < \kappa$ a sequence of subgroups B_ν.

Let $B_0 = B$. If B_ν has been defined, choose $B_{\nu+1} \supseteq B_\nu$ such that $|B_{\nu+1}| < \kappa$ and B_ν is not a free factor of $B_{\nu+1}$. If B_ν has been defined for $\nu < \lambda$ where λ is a limit ordinal, let $B_\lambda = \bigcup_{\nu<\lambda} B_\nu$. Let $B_\kappa = \bigcup_{\nu<\kappa} B_\nu$. By Theorem 1.3, B_κ is not free. However $B_\kappa \subseteq A$ and $|B_\kappa| = \kappa$. So B_κ is free, a contradiction.

Suppose now that A is ω_1-free and B is a finitely generated subgroup which is not contained in an ω-pure subgroup. Choose a sequence of countable subgroups B_n by $B_0 = B$; if $C \subseteq B_n$ is not ω-pure and C is finitely generated then C is not ω-pure as a subgroup of B_{n+1}. Let $B_\omega = \bigcup_{n<\omega} B_n$. By the construction, B is not contained in a finitely generated free factor of B_ω. So B_ω is not free, a contradiction.

For limit cardinals we have a stronger result.

1.7. THEOREM. *If λ is a limit cardinal and A is λ-free then A is strongly λ-free.*

Proof. Since A is λ-free, it is κ^{++}-free for all $\kappa < \lambda$. Suppose $B \subseteq A$ and $|B| = \kappa$. By Theorem 1.6 there is a $C \supseteq B$ such that $|C| = \kappa$ and C is

κ^+-pure. We must show C is λ-pure. Suppose $D \supseteq C$ and $|D| < \lambda$. Since D is free, there is a free factor H of D which contains C and is of cardinality κ. Since C is κ^+-pure, C is a free factor of H and hence of D.

We now turn our attention to abelian groups. An approach to discovering more about κ-free groups is to place stronger constraints on the filtration.

1.8. LEMMA. *Suppose A is a κ^+-free abelian group and κ is regular. If $A \supseteq B$ and $B = \bigcup_{\tau<\mu} B_\tau$ (not necessarily smooth), μ a regular cardinal and each B_τ is μ-pure, then B is μ-pure. In other words the union of a μ-chain of μ-pure subgroups is μ-pure.*

Proof. Suppose $D \subseteq A$ and $|D| < \mu$. Choose B_τ such that $B \cap D \subseteq B_\tau$. Now

$$(B + D)B \simeq D/(B \cap D) \simeq B_\tau + D/B_\tau$$

which is free.

1.9. LEMMA. *Assume A is as in Lemma 1.8. There is a κ^+-filtration $\Delta = \{A_\nu | \nu < \kappa^+\}$ such that each $A_{\nu+1}$ is κ-pure and if λ is a limit ordinal then A_λ is* $\mathrm{cf}(\lambda)$*-pure.*

Proof. Enumerate A as $\{a_\nu | \nu < \kappa^+\}$. Appealing to Theorem 1.6, choose A_0 to be any κ-pure subgroup of A. Suppose A_ν has been defined. Since $|A_\nu| \leqq \kappa$, we can write A_ν as $\bigcup_{\beta<\kappa} B_\beta$ where $|B_\beta| < \kappa$. Choose κ-pure subgroups C_β such that

$$|C_\beta| < \kappa,\ C_0 \geqq B_0 \cup \{a_\nu\}, \text{ and } C_\beta \supseteq (\bigcup_{\tau<\beta} C_\beta) \cup B_\beta.$$

Let $A_{\nu+1} = \bigcup_{\beta<\kappa} C_\beta$. By Lemma 1.8, $A_{\nu+1}$ is κ-pure. Again by Lemma 1.8, if for limit ordinals λ we define A_λ as $\bigcup_{\nu<\lambda} A_\nu$ then A_λ is $\mathrm{cf}(\lambda)$-pure.

1.10 THEOREM. *If A is a κ^+-free abelian group such that $\Gamma(A) \cap \tilde{E} \neq 0$ where $E = \{\nu | \mathrm{cf}(\nu) = \kappa\}$, then there exists a κ-free non-free abelian group of cardinality κ.*

Proof. Filter A as in Lemma 1.9. For some λ of cofinality κ, there exists A_λ which is κ-pure but not κ^+-pure. Choose $B \supseteq A_\lambda$ such that $|B| = \kappa$ and B/A_λ is not free. Since A_λ is κ-pure, B/A_λ is the group required.

Combining Theorems 1.1(2) and 1.10 yields the following corollary.

1.11. COROLLARY. *If κ is weakly compact and A is κ^+-free abelian then A is strongly κ^+-free. In fact $\Gamma(A) \cap \tilde{W} = 0$, where $W = \{\nu | \mathrm{cf}(\nu) = \kappa\}$.*

This result shows (once appropriate existence results are established) that the existence of a strongly κ-free not κ^+-free abelian group does not necessarily imply the existence of a non-strongly κ-free abelian group.

In [**4**] it is shown that if a κ-free not κ^+-free abelian group exists, then there are 2^{κ^+} strongly κ^+-free not κ^{++}-free abelian groups. Using this we get a strengthening of Theorem 1.5.

1.12. THEOREM. *If κ is either a regular limit cardinal or the successor of a regular cardinal and a κ-free not κ^+-free abelian group exists, then there are 2^κ strongly κ-free not κ^+-free abelian groups.*

Proof. Suppose A is κ-free, not free and $|A| = \kappa$. If A is strongly κ-free then we are done (by Theorem 1.5). If not then $\kappa = \rho^+$ where ρ is a regular cardinal, since for limit cardinals κ every κ-free group is strongly κ-free. By Theorem 1.10, there is a ρ-free not ρ^+-free group. So by the results of [**4**] mentioned above, we are done.

For strongly κ-free abelian groups we can do much better than Corollary 1.11, in our attempt to describe Γ. The next theorem says any strongly κ-free abelian group of cardinality κ has a filtration whose "bad" points include no ordinals with a weakly compact cofinality.

1.13. THEOREM. *Suppose A is a strongly κ-free abelian group and $|A| = \kappa$ then $\Gamma(A) \cap \tilde{W} = 0$ where $W = \{\nu | \mathrm{cf}(\nu)$ is weakly compact$\}$.*

Proof. We can filter A as $\{A_\nu | \nu < \kappa\}$ such that $A_{\nu+1}$ is κ-pure. If $\mathrm{cf}(\lambda)$ is weakly compact then A_λ is the union of a chain of κ-pure subgroups. In order to show A_λ is κ-pure it is enough to prove the following claim.

Claim. Suppose for a weakly compact cardinal, τ, we have free abelian groups $F \supseteq K = \bigcup_{\sigma<\tau} K_\sigma$ and F/K_σ is free. Then F/K is free.

Proof of Claim. Assume not. Let ρ be the least cardinal such that $F \supseteq K = \bigcup_{\sigma<\tau} K_\sigma$ form a counter example to the claim and $|F| = \rho$. First note that F/K is ρ-free. Suppose $G \subseteq F$ and $|G| < \rho$. Since

$$G/(K_\sigma \cap G) \simeq (G + K_\sigma)/K_\sigma \subseteq F/K_\sigma,$$

$G/(K_\sigma \cap G)$ is free. Hence $G, G \cap K = \bigcup_{\sigma<\tau}(G \cap K_\sigma)$ satisfy the hypothesis of the claim. So by the minimality of ρ, $G + K/K \simeq G/K \cap G$ is free. By Theorem 1.1, ρ is regular and $\rho > \tau$.

Choose a ρ-filtration, $\{F_\nu | \nu < \rho\}$, of F such that; F_ν is a direct summand of F and if $(F_\nu + K)/K$ is not ρ-pure in F/K, then $(F_{\nu+1} + K)/(F_\nu + K)$ is not free. Now for each $\sigma < \tau$, define

$$C_\sigma = \{\nu | F_\nu + K_\sigma/K_\sigma \text{ is } \rho\text{-pure in } F/K_\sigma\}.$$

Since F/K_σ is free, C_σ is a cub. Let $C = \bigcap_{\nu<\tau} D_\sigma$. Since $\rho > \tau$ and ρ is regular, C is a cub. Since F/K is not free there is a $\nu \in C$ such that $(F_{\nu+1} + K)/(F_\nu + K)$ is not free. By the definition of C, $(F_{\nu+1} + K_\sigma)/$

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$(F_\nu + K_\sigma)$ is free for all $\sigma < \tau$. Note that

$$(F_{\nu+1} + K)/(F_\nu + K) \simeq F_{\nu+1}/(F_\nu + (K \cap F_{\nu+1})),$$
$$(F_{\nu+1} + K_\sigma)/(F_\nu + K_\sigma) \simeq F_{\nu+1}/(F_\nu + (K_\sigma \cap F_{\nu+1})) \text{ and}$$
$$F_\nu + (K \cap F_{\nu+1}) = \cup_{\sigma<\tau}(F_\nu + (K_\sigma \cap F_{\nu+1})).$$

So

$$F_{\nu+1} \supseteq (F_\nu + (K \cap F_{\nu+1})) = \cup_{\sigma<\tau}(F_\nu + (K_\sigma \cap F_{\nu+1}))$$

satisfy the hypothesis of the claim. Since $|F_{\nu+1}| < \rho$, $(F_{\nu+1} + K)/(F_\nu + K)$ is free. This is a contradiction.

Although the above is the best possible result for successor cardinals (cf. Theorem 2.13), for some regular limit cardinals we can obtain more information.

1.14. *Definition*. A cardinal, κ, is a *Mahlo* cardinal if the set of regular cardinals less than κ is stationary in κ. (The first Mahlo cardinal (if any) is less than the first weakly compact cardinal.)

1.15. THEOREM. *Suppose κ is a cardinal and $E \subseteq \kappa$ is such that $F = \{\nu < \kappa | E \cap \nu$ is stationary in ν and ν is a regular cardinal$\}$ is stationary in κ. For no A does $\Gamma(A) = \tilde{E}$.*

Note the hypothesis forces κ to be a Mahlo cardinal.

Proof. Assume that A is a counterexample. Choose a κ-filtration $\{A_\nu | \nu < \kappa\}$ of A such that: If A_ν is not κ-pure, then A_ν is not a free factor of $A_{\nu+1}$; and $|A_\nu| = |\nu|$. By the assumption on F and E we can find a cardinal $\tau \in F$ such that $\{\nu < \tau | A_\nu$ is not a free factor of $A_{\nu+1}\}$ is stationary in τ. Since $|A_\nu| = |\nu| < \tau$, A_τ is not free. This is a contradiction.

This not only eliminates 1 as a possible value for Γ but also such sets as $\{\nu | \mathrm{cf}(\nu) = \omega\}$.

2. Construction of strongly κ-free groups. In this section we begin our description of the construction of almost free groups. We will use Gödel's axiom of constructability, $(V = L)$, for some of our constructions. The constructible universe, L, is a pleasant place where many otherwise independent set theoretic principles, including GCH, hold. We will explain these principles as we need them. Aside from any intrinsic interest constructions in L may have, they provide consistency results for the existence of certain groups. These show the results of Section 1 are the best general results possible.

In order to motivate the constructions to come, we will sketch the construction in [**9**] of an ω_1-free group.

2.1. *Higman's Construction.* We start with a countably generated free group A_0. If we have defined A_ν, we choose a set of free generators $\{a_n | n \in \omega\}$ for A_ν. We then take the $A_{\nu+1}$ a group freely generated by $\{b_n | n \in \omega\}$. We embed A_ν in $A_{\nu+1}$ by identifying a_n with $b_n(b_{n+1})^{-2}$.

Remark. This embedding has two properties we will see again:

(i) Any finitely generated free factor of A_ν is a free factor of $A_{\nu+1}$ (a complementary factor of $\langle a_0, \ldots, a_n \rangle$ is $\langle b_m | m \rangle n \rangle$); and
(ii) A_ν is not a free factor of $A_{\nu+1}$.

If λ is a limit ordinal we let $A_\lambda = \bigcup_{\nu<\lambda} A_\nu$. We let $A = \bigcup_{\nu<\omega_1} A_\nu$. To show this construction works we must show A_ν is free and that A is not. Since $\mathrm{cf}(\lambda) = \omega$, using part (i) of the remark we can write A_λ as the union of countable chain of free groups each of which is a free factor of the succeeding elements of the chain. The group A is ω_1-free since every countable subset is contained in some A_ν. Since for all ν A_ν is not ω_1-pure, A is not free (Theorem 1.3).

In fact A is not strongly ω_1-free. Let B be the smallest normal subgroup containing a^2 for each $a \in A$. Then $|A| > \omega$ and $|A/B| = \omega$. Since B is a definable subgroup, this is expressible in $L_{\infty\omega_1}$. As this is not true for any free group, A is not strongly ω_1-free.

One way of viewing the above construction is that two ingrediants are involved; an embedding at one free group in another, and a criterion for freeness. The embedding is used to pass from one stage to the next. The criterion for freeness is used to guarantee that the groups constructed at limit stages are free. Eklof, by using moderately sophisticated set theory, was able to use a simple criterion, namely the union of smooth chain of free groups each of which is a free factor of the next group in the chain is free (c.f. [**4**], Theorem 2.2).

We now define the relevant set theoretic idea.

2.2. *Definition.* Suppose κ is a regular cardinal. A set $E \subseteq \kappa$ is *sparse* if $E \cap \nu$ is not stationary in ν.

Jensen [**11**] has established the following principle which has since been called $E(\kappa)$.

2.3. THEOREM. (Jensen) *Assume* $(V = L)$. $E(\kappa)$ *holds for each regular non-weakly compact cardinal* κ *where* $E(\kappa)$ *asserts that there is a sparse stationary subset of* κ *consisting of ordinals of cofinality* ω.

This allows us to construct κ-separable groups. This construction is closely related to that of [**5**].

2.4. THEOREM. *Suppose* $E(\kappa)$ *holds. If* κ *is a regular cardinal then there are* 2^κ κ*-separable groups of cardinality* κ.

Proof. Choose E to witness $E(\kappa)$. We will define a κ-filtration inductively. As well as the groups A_ν, we will need certain auxiliary groups. For each $\nu < \kappa$, we will construct a group A_ν satisfying:

(i) $A_\nu \subseteq A_\tau$, if $\tau \geqq \nu$;

(ii) $A_\lambda = \bigcup_{\nu<\lambda} A_\lambda$, if λ is a limit ordinal; and

(iii) A_ν is a free factor of $A_{\nu+1}$ if and only if $\nu \notin E$.

For each $\nu \notin E$ and $\tau \geqq \nu$ we will construct a free group $B_{\nu\tau}$ satisfying:

(iv) $A_\nu * B_{\nu\tau} = A_\tau$;

(v) $B_{\nu\tau} \subseteq B_{\nu\rho}$, if $\rho \geqq \tau$; and

(vi) $B_{\nu\tau} * B_{\tau\rho} = B_{\nu\rho}$, if $\nu \leqq \tau \leqq \rho$ and $\nu, \tau \notin E$.

If we have these groups and set $A = \bigcup_{\nu<\kappa} A_\nu$, A is not free (Theorem 1.3). Further if $\nu \notin E$, it is clear that $A = A_\nu * (\bigcup_{\nu<\tau} B_{\nu\tau})$. So A is one of the groups required to prove the theorem. Note that $\Gamma(A) = \tilde{E}$.

We will now define the necessary subgroups by induction. For notational convenience a set of generators will always be free. There are several cases.

Case 0. ($\nu = 0$) Let $A_0 = \langle 1 \rangle$, $B_{00} = \langle 1 \rangle$.

Case 1. ($\nu = \rho + 1, \rho \notin E$). Let $A_\nu = A_\rho * \langle a_\nu \rangle$, and $B_{\nu\nu} = \langle 1 \rangle$. For $\tau \leqq \rho$ and $\tau \notin E$, let $B_{\tau\nu} = B_{\tau\rho} * \langle a_\nu \rangle$. It is clear that if the groups defined prior to stage ν satisfy (i)–(vi), so do A_ν and the $B_{\tau\nu}$.

Case 2. (ν is a limit ordinal) Let $A_\nu = \bigcup_{\rho<\nu} A_\rho$. To see that A_ν is free, let $\{\tau_\xi | \xi < \mathrm{cf}(\nu)\}$ be a closed unbounded sequence in ν such that for all ξ, $\tau_\xi \notin E$. (This is possible by the choice of E.) Also choose $\tau_0 = 0$. So

$$A_\nu = * (B_{\tau_\xi \tau_{\xi+1}})_{\xi<\mathrm{cf}(\nu)}.$$

If $\tau < \nu$ and $\tau \notin E$, let $B_{\tau\nu} = \bigcup_{\tau<\xi<\nu} B_{\tau\xi}$. (This definition is forced by (i)–(vi).) If $\nu \notin E$, let $B_{\nu\nu} = \langle 1 \rangle$; otherwise leave it undefined. For much the same reason that A_ν is free, the $B_{\tau\nu}$ satisfy (iv)–(vi).

Case 3. ($\nu = \rho + 1, \rho \in E$) This case is the crux of the construction. Since $\rho \in E$, $\rho = \bigcup_{n<\omega} \tau_n$ where τ_n is a successor ordinal for each n. Let

$$A' = A_{\tau_0} * (B_{\tau_{n+1}\tau_n+1})_{n<\omega}.$$

Then

$$A_\nu = A' * \langle a_{\tau_n+1} \rangle_{n<\omega}.$$

Let $A_\nu = A' * \langle a_{n\nu} \rangle_{n<\omega}$. Identify a_{τ_n+1} with $(a_{n\nu})(a_{n+1\nu})^{-2}$ and extend this to the embedding of A_ρ into A_ν which is the identity on A'. Note that A_ρ is not a free factor of A_ν. To see this, let C be the smallest normal subgroup of A_ν containing A_ρ. Then A_ν/C is an abelian group which contains a non-zero element which can be divided by arbitrary powers of 2.

We now must define the $B_{\tau\nu}$. As usual let $B_{\nu\nu} = (1)$. Before we define the other $B_{\tau\nu}$ note the following. For all $m < \omega$, $\langle a_{n\nu} \rangle_{n<\omega}$ is freely generated by

$$\{a_{0\nu}(a_{1\nu})^{-2}, \ldots, a_{m\mu}(a_{m+1\nu})^{-2}\} \cup \{a_{m+1\nu}, \ldots\}.$$

This follows since $\{a_{0\nu}(a_{1\nu})^{-2}, \ldots, a_{m\nu}(a_{m+1\nu})^{-2}, a_{m+1\nu}\}$ is a set of $m + 2$ generators of a free group of rank $m + 2$, namely $\langle a_{0\nu}, \ldots, a_{m+1\nu}\rangle$. So these elements must freely generate this group (cf. [**12**], Corollary 2.13.1, p. 110).

For each of the τ_n, let

$$B_{\tau_n\nu} = *\ (B_{\tau_{m+1}\tau_{m+1}})_{n\leqq m} * \langle a_{m\nu}\rangle_{n\leqq m}.$$

If $\tau \notin E$ and $\tau < \nu$, choose n such that τ_n is the least of the $\tau_n \geqq \tau$. Define $B_{\tau\nu} = B_{\tau\tau_n} * B_{\tau_n\nu}$. By the observation in the preceding paragraph it is clear that if $\tau < \nu$ and $\tau \notin E$, then $A_\tau * B_{\tau\nu} = A_\nu$. Since

$$a_{\tau_m+1} = a_{m\nu}(a_{m+1\nu})^{-2} \text{ and } a_{\tau_m+1} \in B_{\tau\xi} \text{ for } \tau < \tau_m + 1 \leqq \xi,$$

$B_{\tau\xi} * B_{\xi\nu} = B_{\tau\nu}$ where $\tau \leqq \xi \leqq \nu$ and $\tau, \xi \notin E$. Also if $\tau \leqq \xi \leqq \nu$, then $B_{\tau\xi} \subseteq B_{\tau\nu}$ $(\tau \in E)$. So properties (i)–(vi) hold.

This construction produces a κ-separable group A such that $\Gamma(A) = \tilde{E}$ where E is a set witnessing $E(\kappa)$. We can complete this proof if we know there are 2^κ non-equivalent sets witnessing $E(\kappa)$ (This is an idea of Shelah). The existence of these sets follows from the existence of one by the following celebrated results of [**19**].

THEOREM. *Assume κ is a regular cardinal. Every stationary subset of κ can be partitioned into κ disjoint stationary subsets.*

As previously noted $E(\kappa)$ holds for all regular non-weakly compact cardinals in L. In the real world $E(\omega_1)$ holds. (Let $E = \{\nu | \mathrm{cf}(\nu) = \omega\}$.) So with no set theoretic hypothesis, we have a construction of an ω_1-separable group.

In [**7**] groups of cardinality κ which are strongly κ-free and indecomposable were constructed in L. So in L if we want κ-separable groups we must do more than construct strongly κ-free groups. Curiously we have the following.

2.5. THEOREM. *Assume Martin's Axiom and $2^\omega \geqq \omega_2$. If A is a strongly ω_1-free abelian group and $|A| = \omega_1$, then A is ω_1-separable.*

Proof. We will sketch the modifications to Shelah's proof that a group of type II or III is a Whitehead group (cf. [**17**] or [**5**]). Suppose B is an ω_1-pure countable subgroup of A. We have the exact sequence $0 \to B \to A \to B/A \to 0$. It is easy to show B/A is again strongly ω_1-free and hence not of type I. Suppose D is ω_1-free and not of type I. In Shelah's proof all that is needed to show that $0 \to Z \to C \to D \to 0$ splits is that Z is free and countable. So $0 \to B \to A \to A/B \to 0$ splits. Hence B is a direct summand of A.

If in Theorem 2.4 we had only been interested in constructing strongly κ-free groups, we could have omitted the $B_{\tau\nu}$, as they are only used to

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show the group is κ-separable. We can now describe the construction of strongly κ-free groups more abstractly.

2.6. *Definition*. Suppose κ is a regular cardinal. We say κ has the embedding property $F(\kappa)$ if there exists a free group F, and a smooth chain of subgroups $K_\nu (\nu < \kappa)$ such that:

(i) $|F| = \kappa$;

(ii) K_ν is generated by fewer than κ elements and K_ν is a free factor of F; and

(iii) $\bigcup_{\nu<\kappa} K_\nu = K$ and K is not a free factor of F.

2.7. *Remark*. $F(\omega)$ is true. For abelian groups $F(\kappa)$ is implied by the existence of a κ-free non-free abelian group of cardinality κ. This comes from the following lemma.

2.8. Lemma. ([**4**]) *Suppose F is a free abelian group of cardinality κ. If F/K is κ-free and $K_1 \subseteq K$ is a direct summand of K such that $|K_1| < \kappa$, then F/K_1 is free.*

Proof. Choose a direct summand D of F such that $D \supseteq K_1$ and $|D| < \kappa$. Since $(D + K)/K$ is free, D/K_1 is free. So F/K_1 is free.

The situation for (non-abelian) groups seems more complicated. We can however show the following.

2.9. Lemma. *Suppose ρ is a regular cardinal and there is a sparse stationary subset $E \subseteq \rho$ such that for all $\nu \in E$, $\mathrm{cf}(\nu) = \kappa$. If $F(\kappa)$ is true then $F(\rho)$ is true. In particular for regular ρ, if $E(\rho)$ is true then so is $F(\rho)$.*

We will omit the proof of this lemma. It is a rather horrible inducive construction, somewhat similar to that of Theorem 2.4. This lemma is proved in [**13**]. The general construction can now be given.

2.10. Theorem. *Suppose κ is regular and $E \subseteq \kappa$ is a sparse stationary subset consisting of limit ordinals such that for all $\nu \in E$ if $\rho = \mathrm{cf}(\nu)$, then $F(\rho)$ is true. There is a strongly κ-free group A of cardinality κ such that $\Gamma(A) = \tilde{E}$.*

Proof. The proof is very similar to that of Theorem 2.4. So we will just sketch the construction. Again we construct a κ-filtration $\{A_\nu | \nu < \kappa\}$ of free groups such that A_ν is a free factor of each succeeding group if $\nu \notin E$. If $\nu \in E$ then A_ν is not a free factor of $A_{\nu+1}$. We have the same cases as before.

Case 0. Let $A_0 = \langle 1 \rangle$.

Case 1. ($\nu = \rho + 1$, $\rho \notin E$) Given A_ρ, define A_ν to be the free product of A_ρ and a free group on $|\nu|$ generators.

Case 2. (ν is a limit ordinal) Let $A_\nu = \bigcup_{\tau<\nu} A_\tau$. Since E is a sparse, A_ν can be written as the union of a smooth chain of free groups each of which is a free factor of the succeeding group.

Case 3. ($\nu = \rho + 1$, $\rho \in E$) Suppose $\tau = \mathrm{cf}(\rho)$ and $F \supset K = \bigcup_{\xi<\tau} K_\xi$ witness that $F(\tau)$ holds. Again since E is sparse, we can write A_ρ as the union of a smooth τ-chain $\{A_{f(\xi)}|\xi < \tau\}$. We impose the conditions that $f(\xi) \notin E$ and that if $K_{\xi+1}$ is freely generated by μ elements over K_ξ, then $A_{f(\xi+1)}$ is freely generated by at least μ elements over $A_{f(\xi)}$. Using the above setup we can choose a free group for A_ν and embed A_ρ into A_ν so that A_ρ is not a free factor of A_ν but each $A_{f(\xi)}$ is. Finally if $\beta < \nu$ and $\beta \notin E$, then there is a ξ such that $f(\xi) > \beta$. So A_β is a free factor of $A_{f(\xi)}$ and hence of A_ν.

In order to push this method we need examples of sparse stationary sets.

2.11. *Example.* If κ is regular then $\{\nu < \kappa^+|\mathrm{cf}(\nu) = \kappa\}$ is a sparse stationary subset of κ^+. An induction as in [**4**] shows that there exists a 2^{ω_n} strongly ω_n-free groups of cardinality ω_n.

In L we can do much better. Jensen [**11**] showed the principle $\square_\kappa$ holds in L. We need the following consequence of $\square_\kappa$, which is due essentially to Magidor and independently to Solovay.

2.12. THEOREM. ([**16**], Lemma 5.4) *Assume $\square_\kappa$. If $E \subseteq \kappa^+$, then E is the union of at most κ sparse sets.*

Combining several results we have a complete description in L of the range of Γ on abelian groups for successor cardinals.

2.13. THEOREM. *Assume ($V = L$). Suppose κ is a successor cardinal and $E \subseteq \kappa$. There exists a strongly κ-free group A of cardinality κ such that $\Gamma(A) = \tilde{E}$ if $\{\nu \in E|\mathrm{cf}(v)$ is weakly compact$\}$ is not stationary.*

Proof. We can assume for all $\nu \in E$ that $\mathrm{cf}(\nu)$ is not weakly compact. Suppose $\kappa = \rho^+$. Choose $\{E_\nu|\nu < \rho\}$ such that each E_ν is sparse and $E = \bigcup_{\nu<\rho} E_\nu$. For each $\nu < \rho$, construct A_ν such that $\Gamma(A_\nu) = \tilde{E}_\nu$. This is possible since $F(\mathrm{cf}(\tau))$ holds for each $\tau \in E_\nu$. Let $A = *(A_\nu)_{\nu<\rho}$.

2.14. THEOREM. *Assume ($V = L$). Suppose κ is a successor cardinal and $E \subseteq \kappa$. There is a strongly κ-free abelian group A with $\Gamma(A) = \tilde{E}$ if and only if $\{\nu \notin E|\mathrm{cf}(\nu)$ is weakly compact$\}$ is not stationary.*

These constructions give more information. For abelian groups let $\approx$ be the least equivalence relation closed under $\simeq$ and direct sums with free abelian groups. This allows us to describe the groups better.

2.15. THEOREM. *Assume ($V = L$). Suppose κ is a successor cardinal and $\theta:\kappa \to$ abelian groups is a function such that: $\theta(\nu)$ is free if ν is a successor cardinal or $\mathrm{cf}(\nu)$ is weakly compact; and $\theta(\nu) = F/K = \bigcup_{\alpha<\mathrm{cf}(\nu)} k_\alpha$, F and F/K_α are free, and $|F| < \kappa$. There exists a strongly κ-free abelian group A of cardinality κ with a κ-filtration $\Delta = \{A_\nu|\nu < \kappa\}$ such that $A_{\nu+1}/A_\nu \approx \theta(\nu)$.*

To prove Theorem 2.15 it is necessary to show that if $F \supseteq k = \bigcup_{\alpha<\rho} K_\alpha$, F and F/K_α are free, then $K = \bigcup_{\alpha<\rho} H_\alpha$ (smooth) where F/H_α is free. To see this fix X a set of free generators for K. Let $H_{\alpha+1}$ be the group generated by $X \cap K_{\alpha+1}$.

2.16. *Remark.* Again the logical significance of this result is explained in [**8**]. The relevant algebraic notion is quotient-equivalence. If A and B are strongly κ-free abelian groups of cardinality κ, they are *quotient equivalent* if there are κ-filtrations $A = \bigcup_{\nu<\kappa} A_\nu$, $B = \bigcup_{\nu<\kappa} B_\nu$ such that for every $\nu < \kappa$, $A_{\nu+1}/A_\nu \simeq B_{\nu+1}/B_\nu$. Quotient-equivalence does not imply isomorphism (cf. [**8**], Section 5). By Lemma 1.9 and Theorem 1.13 every quotient-equivalence class of strongly κ-free abelian groups is determined by a function $\theta : \kappa \rightarrow$ abelian groups satisfying the hypothesis of Theorem 2.15. So Theorem 2.15 characterizes, in L, all quotient-equivalence classes of strongly κ-free abelian groups.

3. Construction of κ-free groups which are not strongly κ-free. We now turn to a rather mysterious class of groups, those which are κ-free but not strongly κ-free. Again set theory enters essentially into our construction.

3.1. *Definition.* A partially ordered set $(T, <)$ is a *tree* if for all $t \in T$, $\{x|x < t\}$ is well ordered. If $t \in T$, the *height* of t, $\mathrm{ht}(t)$, is the ordinal ν such that $(\{x|x < t\}, <) \simeq (\nu, >)$. The *height* of T is $\sup\{\mathrm{ht}(t)|t \in T\}$. A *branch*, B, is a map from an ordinal ν such that for all $\tau < \alpha < \nu$, $\mathrm{ht}(B(\alpha)) = \alpha$ and $B(\tau) < B(\alpha)$. If $B : \nu \rightarrow T$ is a branch, ν is the *length* of B.

3.2. *Example.* The *full binary tree of height* κ is the set of sequences of 0's and 1's of length $<\kappa$, ordered by inclusion.

3.3. *Definition.* Assume κ is regular. A tree of height κ is a *Canadian* tree if $|T| = \kappa$ and T has κ^+ branches of length κ.

3.4. *Remarks.* This somewhat unfortunate name originated with Frank Tall and has been used by Baumgartner [**1**]. The concept itself is older (cf. [**15**]).

The full binary tree of height ω is a Canadian tree. If $2^\kappa = \kappa$ then there is a Canadian tree of height κ, namely the full binary tree of height κ. In particular if $(V = L)$, there is such a tree for each regular κ. The existence of a Canadian tree of height κ does not depend on $2^k = \kappa$. For example in the usual forcing extension of L, which make 2^κ whatever is desired, there are Canadian trees of all regular heights. However the existence of a Canadian tree does not seem to follow from ZFC. The situation is explained by the following theorem of [**15**].

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3.5. THEOREM (Mitchell). (1) *If* ZFC *and there exists an inaccessible cardinal is consistent then it is consistent with* ZFC *that there is no Canadian tree of height* ω_1 *and* $2^{\omega_1} = \omega_2$.

(2) *If it is consistent with* ZFC *that for some regular cardinal* κ *there is no Canadian tree of height* κ, *then* ZFC *and there exists an inaccessible is consistent.*

The following method of defining a group is a modification of a construction of Shelah.

3.6. THEOREM. *Suppose there exists a Canadian tree of height* κ *and that* $F(\kappa)$ *holds. There is a* κ^+*-free group which is not strongly* κ^+*-free.*

Proof. Let $(T, <)$ be a Canadian tree of height κ and $\{B_\nu | \nu < \kappa^+\}$ be a set of pairwise different branches of length κ. Let $F \supseteq K = \bigcup_{\tau<\kappa} K_\tau$ witness $F(\kappa)$. Choose groups H_τ such that $K_\sigma = *(H_\tau)_{\tau<\sigma}$. Note if λ is a limit ordinal, then $H_\lambda = \langle 1 \rangle$. For each $t \in T$, we take a free group H_t isomorphic to H_τ where $\tau = \mathrm{ht}(t)$. Take $G_0 = *(H_t)_{t\in T}$. For each $\nu < \kappa^+$ choose free groups $A_\nu \supseteq D_\nu = *(C_{\nu\tau})_{\tau<\kappa}$ an isomorphic copy of $F \supseteq K = *(H_\tau)_{\tau<\kappa}$. Let G be the amalgamated product of G_0 and the A_ν's where we identify $C_{\nu\tau}$ with $H_{B_\nu(\tau)}$. In other words we attach A_ν to G_0 by identifying D_ν along branch B_ν.

We must now show G is K^+-free. This follows easily once we have established the following claim.

Claim. Suppose P is a presentation of G and for every $Q \subseteq P$ such that $|Q| = \kappa$ there is $P \supseteq Q' \supseteq Q$ such that Q' is the presentation of a free group. Then G is κ^+-free.

Proof of Claim. Suppose for some $G' \subseteq G$, G' is not free and $|G| = \kappa$. We can choose $Q \subseteq P$ with $|Q| = \kappa$ such that each element of G' is a word in the generator of Q and if two words in Q are equal (as elements of G) then their equality is deducible from the relations of Q. So if $P \supseteq Q' \supseteq Q$ then Q' is the presentation of a group which contains an isomorphic copy of G'. So Q' is not the presentation of a free group.

Since we have described a presentation for G it will suffice to show appropriate subsets of this presentation are presentations of free groups. Suppose we have any κ branches of T. For notational convenience we can assume the branches are B_σ, $\sigma < \kappa$. We can also assume each $t \in T$ is $B_\sigma(\rho)$, for some σ, $\rho < \kappa$. Take the presentation, Q, whose generators are G_0, A_σ $(\sigma < \kappa)$ and whose relations are the identifications described above. To see the group presented by Q is free, we construct the group as a free product of free groups.

Take $F = *(F_\sigma)_{\sigma<\kappa}$, the free product of κ free groups of rank κ. Choose an isomorphism from A_0 to F_0. In general at stage σ, we let ρ be the least ordinal such that $B_\sigma(\rho) \neq B_\beta(\rho)$, for all $\beta < \sigma$. Such a ρ exists, since κ is regular. Let $\bar{A}_\sigma$ be a complementary factor of $*(C_{\sigma\tau})_{\tau<\rho}$ in A_σ. We then choose an isomorphism from $\bar{A}_\sigma$ to F_σ. This process shows Q is the presentation of a free group.

To see that G is not strongly κ^+-free, note that G_0 cannot be contained in any κ^+-pure subgroup and $|G_0| = \kappa$. In fact let $G_0{}^N$ be the smallest normal subgroup of G containing G_0 and let K^N be the smallest normal subgroup of F containing K. Then G/G^N is isomorphic to a free product of κ^+ copies of F/K^N.

Although the existence of a Canadian tree of height ω_1 implies the existence of an ω_2-free group which is not strongly ω_2-free, the converse is false (for abelian groups) (cf. Theorem 5.2).

One reason why κ-free not strongly κ-free groups are mysterious from a logical point of view is that they need not be $L_{\infty\kappa}$-equivalent.

3.7. THEOREM. (1) *Assume* ($V = L$). *If* κ *is a regular non-weakly compact cardinal then there exist* 2^κ ($=\kappa^+$) κ^+*-free groups which are pairwise not* $L_{\infty\kappa^+}$*-equivalent.*

(2) *There exist* 2^ω ω_1*-free groups which are pairwise not* $L_{\infty\omega_1}$*-equivalent.*

(3) *If there exists a Canadian tree of height* ω_n *there are* 2^{ω_n} ω_{n+1}*-free groups which are pairwise not* $L_{\infty\omega_{n+1}}$*-equivalent.*

(4) *Suppose that there is a strongly* κ*-free abelian group which is not* κ^+*-free and that there is a Canadian tree of height* κ. *There exist* 2^κ κ^+*-free abelian groups which are pairwise not* $L_{\infty\kappa^+}$*-equivalent.*

Proof. We will only prove (4). The proofs of the others are similar. Suppose A_1 and A_2 are κ-free abelian groups of cardinality κ such that $\Gamma(A_1) \neq \Gamma(A_2)$. Using the construction of the last theorem we can construct κ^+-free abelian groups G_1 and G_2. These groups have the property that G_i satisfies the sentence, "any set of size κ is contained in a subgroup B such that G_i/B satisfies the sentence 'every set of size κ is contained in a κ^+-pure subgroup which is the direct sum of copies of A_1' ". This is a sentence of $L_{\infty\kappa^+}$. So G_1 is not $L_{\infty\kappa^+}$-equivalent to G_2. Since we can find 2^κ κ-free groups whose images under Γ are pairwise different, we are done.

4. Para-free groups. The groups constructed in this section have no clear abelian analogues: Baumslag [**2**] opened the study of para-free groups in an attempt to find a non-free group of cohomological dimension one. We now know this is an impossible task. In any case it is interesting that groups can be constructed which share with free groups not only the local property of being (strongly) κ-free but also the global property of being para-free.

4.1. *Definition*. Let G be a group. If $a, b \in G$, $[a, b]$ denotes the *comutator* $a^{-1}b^{-1}ab$ of a and b. The subgroups G_n of the *lower central series* are defined by $G_1 = G$ and $g \in G_{n+1}$ if $g = [g_1, g_2]$ where $g_1 \in G_n$ and $g_2 \in G$. The sequence $G/G_2, G/G_3, \ldots$ is the *lower central sequence* of G. Two groups G and H have the same lower central sequence if there are isomorphisms $\theta_n : H/H_n \to G/G_n$ such that θ_{n+1} induces θ_n. A group is *para-free* if it has the same lower central sequence as a free group and is residually nilpotent.

The constructions of this section are like those of the preceding sections except we need different embeddings. We want an embedding of a free group K into F which has the properties of $F(\kappa)$ but which becomes trivial when we mod out by F_n.

4.2. *Definition*. A group, G, is *nilpotent of class n* (nil-n) if $G_{n+1} = \langle 1 \rangle$. There are free nil-$n$ groups and a nil-n free product. We say κ has the embedding property $F_1(\kappa)$ if there exists a free group F, and a smooth chain of subgroups K^ν $(\nu < \kappa)$, such that: $F \supseteq K = \bigcup_{\nu<\kappa} K^\nu$ witness $F(\kappa)$; and there exists $\{f_i | i \in I\} \subseteq F$ such that for all $n > 0$ $\bar{K}$ is a nil-n free factor of F and $\{\bar{f}_i | i \in I\}$ freely generate a complementary summand of $\bar{K}$. (The bar indicates image mod F_{n+1}.)

4.3. Lemma. (1) *$F_1(\omega)$ holds.*

(2) *Suppose κ is a regular cardinal and there is a sparse stationary subset of κ all of whose elements are of cofinality ν. If $F_1(\nu)$ holds then $F_1(\kappa)$ holds.*

(3) *For all n, $F_1(\omega_n)$ holds.*

Proof. (1) Let F be the free group on generators $\{a_m | m < \omega\}$. Let

$$K^0 = \langle a_0[a_1, a_2] \rangle,\ K^1 = \langle a_0[a_1, a_2], a_1[[a_3, a_4], [a_5, a_6]] \rangle,$$

and so on. For each n, $K^n * \langle a_m \rangle_{m>n} = F$. For any n, $K + F_n = F$ so we can take $\{f_i | i \in I\} = \phi$. Finally since $a_0 \notin F$ and $K + F_2 = F$, K is not a free factor of F. We will omit proving (2), for the same reason we didn't prove Lemma 2.9. An induction argument establishes (3).

Note that the example which established (1) has the properties used in Theorem 2.4 to construct κ-separable groups. We can now add the word para-free to most of our previous results.

4.4. Theorem. (1) *Assume κ is a regular cardinal and $E(\kappa)$ holds. There exist 2^κ κ-separable para-free groups.*

(2) *Assume $(V = L)$. For all regular non-weakly compact cardinals κ and $E \subseteq \kappa - W$ where $W = \{\nu < \kappa | \mathrm{cf}(\nu)$ is weakly compact$\}$ there is a para-free strongly κ-free group, A, such that $\Gamma(A) = \tilde{E}$.*

(3) *For all $n > 0$, there exist 2^{ω_n} para-free strongly ω_n-free groups of cardinality ω_n.*

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(4) *Suppose κ is regular, $F_1(\kappa)$ holds and there is a Canadian tree of height κ. There is a para-free κ^+-free group which is not strongly κ^+-free.*

Proof. In each case we use the same construction as was used to prove the result without the requiring the groups to be para-free, except we use an embedding which witnesses $F_1(\kappa)$. We will prove (4) as that case illustrates the modifications to the previous proofs.

We can view the group constructed in Theorem 3.6 as the union of a smooth chain of free groups. Using the notation of Theorem 3.6, let G_ν be the group generated by G_0 and $\{A_\beta|_{\beta<\nu}\}$. To obtain $G_{\nu+1}$ from G_ν, we first write G_ν as $(H_{B_\nu} * G')$ where

$$H_{B_\nu} = *(H_{B_\nu(\sigma)})_{\sigma<\kappa}.$$

We then embed H_{B_ν} into A_ν so that

$$A_\nu \supseteq H_{B_\nu} = \bigcup_{\sigma<\kappa} *(H_{B_\nu(\beta)})_{\beta\leqq\sigma} \text{ and } \{f_{\nu i}|i \in I\}$$

witness $F_1(\kappa)$. Let $G_{\nu+1} = A_\nu * G'$. For any n,

$$\bar{G} = \bar{G}_0 * \langle \bar{f}_{\nu i}|i \in I\rangle_{\nu<\kappa},$$

where the bars denote the image modG_n and $*$ is the nil-n free product. (G_0 is as in Theorem 3.6, but G_n is the n^{th} element of the lower central series.) So G has the same lower central sequence as a free group. Since any ω_1-free group is residually nilpotent (this can be expressed as a sentence in $L_{\infty\omega}$), G is para-free.

5. Almost free abelian groups. In this section by "group" we shall mean "abelian group". The major results of this section are: The description of Γ for ω_n, namely that it can take any value; and the construction of ω_n-separable groups. The construction will require a more complicated embedding and a more elaborate criterion for freeness. Hill [**10**] defined, for each non-negative integer n, a class of group $\mathscr{F}_n$. The class $\mathscr{F}_0$ consists of all countable torsion free groups. Inductively, $G \in \mathscr{F}_{n+1}$ if $G = \bigcup_{\nu<\mu} G_\nu$ (smooth) where $\mu \leqq \omega_{n+1}$, each G_ν is free, and $G_{\nu+1}/G_\nu \in \mathscr{F}_n$. The following criterion for freeness holds.

5.1. THEOREM. ([**10**]) *For every positive n if $G \in \mathscr{F}_n$, then G is ω_n-free.*

This yields an alternate construction of an ω_n-free not strongly ω_n-free group.

5.2. THEOREM. *Assume, for some $n < \omega$, $2^{\omega_n} = \omega_{n+1}$. There exists an ω_{n+1}-free group which is not strongly ω_{n+1}-free.*

Proof. We define a group, G, whose underlying set is ω_{n+1}. Let $\{X_\nu|\nu < \omega_{n+1}\}$ be an enumeration of the subsets of ω_{n+1} whose cardinality is ω_n. Further, assume each set occurs ω_{n+1} times in this enumeration. Let $G_0 \subseteq \omega_{n+1}$ be a free group on ω_n generators. More exactly the set underlying G_0 is contained in ω_{n+1}. Suppose $X_\nu \subseteq G_\nu$ and G_ν/X_ν is free. Choose $G_\mu \cup \{\nu\} \subseteq G_{\nu+1} \subseteq \alpha < \omega_{n+1}$, such that $G_{\nu+1}/G_\nu \in \mathscr{F}_n$ and $G_{\nu+1}/X_\nu$ is not free. This is possible since there is $A \in \mathscr{F}_n$ with $|A| = \omega_n$ and A is not free (cf. Theorem 2.10). Otherwise given G_ν, choose $G_\nu \cup \{\nu\} \subseteq G_{\nu+1} \subseteq \alpha < \omega_{n+1}$ such that $G_{\nu+1}/G_\nu \in \mathscr{F}_n$. If λ is a limit ordinal, let $G_\lambda = \bigcup_{\nu<\lambda} G_\nu$. Let $G = \bigcup_{\nu<\omega_{n+1}} G_\nu$. By Theorem 5.1, G is ω_{n+1}-free. G is not strongly ω_{n+1}-free, since G has no ω_{n+1}-pure subgroup of size ω_n.

5.3. COROLLARY. *Assume, for some $n < \omega$, that $2^{\omega_n} = \omega_{n+1}$. There are* 4 *$\omega_{n+1}$-free groups of cardinality ω_{n+1} which are pairwise not $L_{\infty\omega_{n+1}}$-equivalent.*

Proof. Let G be the group constructed above. Choose F and F_1, free groups of cardinality ω_n and ω_{n+1} respectively. Then F_1, G, $G \oplus F$, and $G \oplus F_1$ are the required groups.

We will distill from Hill's proof of Theorem 5.1 a weaker result which will suffice in what follows and can be generalized.

5.4. *Definition.* Suppose ω_α is a regular cardinal. Define inductively, for each $n < \omega$, a class of groups $\mathscr{E}_{\alpha,n}$. We will usually suppress the α and write E_n. The class E_0 consists of the ω_α-free groups of cardinality ω_α. In particular if $\alpha = 0$, $\mathscr{E}_0$ is the class of countable torsion free groups. Inductively $A \in \mathscr{E}_{n+1}$ if there exists, $\{A_\nu | \nu < \omega_{\alpha+n+1}\}$, an $\omega_{\alpha+n+1}$-filtration of A be free groups such that:

(i) if $\mathrm{cf}(\nu) < \omega_\alpha$ then A_τ is $\omega_{\alpha+n+1}$-pure; and

(ii) $A_{\nu+1}/A_\nu \in \mathscr{E}_n$, for all ν.

5.4. LEMMA. (Criterion for freeness) *Fix n. Suppose $B = \bigcup_{\nu<\lambda} B_\nu$* (*smooth*) *and each B_ν is a free group on $\omega_{\alpha+n}$ generators. Further suppose B_ν satisfies* (i) *and* (ii) *of Definition* 5.3. *If $\lambda < \omega_{\alpha+n+1}$, then B is free.*

Proof. We can assume that λ is a cardinal. The lemma is proved by induction on n. If $n = 0$, condition (i) gives B as the union of a smooth chain of $\omega_{\alpha+1}$-pure free groups. Hence B is free.

Assume the lemma is true for m and suppose $n = m + 1$. For each $\nu < \lambda$, choose a set of free generators $\{x_i | i \in I_\nu\}$ for B_ν. Impose the further condition that if $\mathrm{cf}(\nu) < \omega_\alpha$, then

$$\{x_i | i \in I_{\nu+1}\} \supseteq \{x_i | i \in I_\nu\}.$$

If ν is a limit ordinal choose $\{B_{\nu\tau} | \tau < \omega_{\alpha+n}\}$, an $\omega_{\alpha+n}$-filtration of $B_{\nu+1}/B_\nu$ satisfying (i) and (ii) (relative to m). There are now two cases.

Case 0. Assume $\lambda = \omega_{\alpha+n}$. Choose $\{C_\beta | \beta < \omega_{\alpha+n}\}$ an $\omega_{\alpha+n}$-filtration of B such that:

(1) $C_\beta \subseteq B_\beta$;

(2) $C_\beta \cap B = \langle x_i | i \in J(\beta, \beta) \subseteq I_\beta \rangle$, if β is a successor ordinal;

(3) $C_\beta \cap B_\nu = \langle x_i | i \in J(\nu, \beta) \subseteq I_\nu \rangle$, if $\nu < \beta$;

(4) $(C_\beta \cap B_{\nu+1}) + B_\nu/B_\nu = B_{\nu\tau}$, for some τ when ν is a limit ordinal and $\nu < \beta$.

It is not hard to see such a filtration exists. Since $C_\beta \subseteq B_\beta$, C_β is free. When we have shown $C_{\beta+1}/C_\beta$ is free, we will be done.

If β is a successor ordinal, then clause (2) and our choice of generating sets guarantees $C_{\beta+1}/C_\beta$ is free. Suppose now that β is a limit ordinal. Since for some τ

$$C_{\beta+1}/(C_{\beta+1} \cap B_\beta) \simeq B_{\beta\tau}$$

which is a free group, it is enough to show $(C_{\beta+1} \cap B_\beta)/C_\beta$ is free. Now

$$(C_{\beta+1} \cap B_\beta)/C_\beta = \bigcup_{\nu<\beta}((C_{\beta+1} \cap B_\nu) + C_\beta)/C_\beta.$$

Note that

$$((C_{\beta+1} \cap B_\nu) + C_\beta)/C_\beta \simeq (C_{\beta+1} \cap B_\nu)/(C_\beta \cap B_\nu)$$

which by (3) is free. So $(C_{\beta+1} \cap B_\beta)/C_\beta$ is the union of a smooth chain of free groups.

Next consider the following isomorphisms:

$$\begin{aligned}
&(((C_{\beta+1} \cap B_{\nu+1}) + C_\beta)/C_\beta))/(((C_{\beta+1} \cap B_\nu) + C_\beta)/C_\beta)\\
&\simeq ((C_{\beta+1} \cap B_{\nu+1}) + C_\beta)/((C_{\beta+1} \cap B_\nu) + C_\beta)\\
&\simeq ((C_{\beta+1} \cap B_{\nu+1}) + (C_\beta \cap B_{\nu+1}) + B_\nu)/\\
&\qquad\qquad\qquad\qquad ((C_{\beta+1} \cap B_\nu) + (C_\beta \cap B_{\nu+1}) + B_\nu)\\
&\simeq ((C_{\beta+1} \cap B_{\nu+1}) + B_\nu)/((C_\beta \cap B_{\nu+1}) + B_\nu)\\
&\simeq (((C_{\beta+1} \cap B_{\nu+1}) + B_\nu)/B_\nu)/(((C_\beta \cap B_{\nu+1}) + B_\nu)/B_\nu).
\end{aligned}$$

This group is free if $\mathrm{cf}(\nu) < \omega_\alpha$ and is isomorphic to $B_{\nu\tau}/B_{\nu\sigma}$ for some $\tau > \sigma$ otherwise. Since $B_{\nu\tau}/B_{\nu\sigma} \in \mathscr{E}_m$, $C_{\beta+1} \cap B_\beta/C_\beta$ is free.

Case 1. Assume $\lambda < \omega_{\alpha+n}$. In this case we again choose an $\omega_{\alpha+n}$-filtration, $\{C_\beta | \beta < \omega_{\alpha+n}\}$, of B. Here we require:

(1) $C_\beta \cap B_\nu = \langle x_i | i \in J(\nu, \beta) \subseteq I_\nu \rangle$; and
(2) $((C_\beta \cap B_{\nu+1}) + B_\nu)/B_\nu = B_{\nu\tau}$, for some τ when $\mathrm{cf}(\nu) \geqq \omega_\alpha$.

Since $C_\beta = \bigcup_{\nu<\lambda} B_\nu \cap C_\beta$, C_β is the union of a smooth chain of free groups. Further since $B_{\nu+1}/B_\nu$ is $\omega_{\alpha+n}$-free and $B_\nu/(B_\nu \cap C_\beta)$ is free, $B_{\nu+1}/(B_\nu \cap C_\beta)$ is free. Hence $(B_{\nu+1} \cap C_\beta)/(B_\nu \cap C_\beta)$ is free. So C_β is free. Since

$$C_{\beta+1}/C_\beta = \bigcup_{\nu<\lambda}((C_{\beta+1} \cap B_\nu) + C_\beta)/C_\beta,$$

the same proof as that of Case 0 shows B is free.

We now turn to the embeddings.

5.5. Lemma. *Suppose $G \in \mathscr{E}_n$ and $\omega_\alpha \leqq \kappa \leqq \omega_{\alpha+n}$. There exist free groups $A \supseteq B = \bigcup_{\nu<\kappa} B_\nu$ such that A/B_ν is free and $A/B \simeq G$.*

Proof. (By induction) If $n = 0$ any free resolution by groups of cardinality ω_α can be made to work (cf. Lemma 2.8). Next we suppose the lemma is true for n, and attempt to prove it for $n + 1$. There are two cases, $\kappa \leqq \omega_{\alpha+n}$ and $\kappa = \omega_{\alpha+n+1}$.

Assume $\kappa \leqq \omega_{\alpha+n}$ and $G \in \mathscr{E}_{n+1}$. As in [**14**] we can view G as

$$\bigoplus (F_\tau)_{\tau<\omega_{\alpha+n+1}}/\bigoplus (K_\tau)_{\tau<\omega_{\alpha+n+1}}$$

where each F_τ is a free group and K_τ is either $\langle 0 \rangle$ or K_τ embeds a complementary summand C_τ of $\bigoplus_{\nu<\tau} K_\nu$ (as a subgroup of $\bigoplus_{\nu<\tau} F_\nu$) into F_τ. That is K_τ is generated by elements of the form $c - f$, where $f \in F_\tau$ is the image of c under the embedding. Further (using a bar to indicate images) $F_\tau/\bar{C}_\tau$ is an $\mathscr{E}_n$ group.

Suppose $K_\tau \neq \langle 0 \rangle$. By the induction hypothesis, we can find free groups A, $B = \bigcup_{\beta<\kappa} B_\beta$ such that $A/B \simeq F_\tau/\bar{C}_\tau$ and A/B_β is free. There is an isomorphism $\theta: A \rightarrow F_\tau$ such that $\theta(B) = \bar{C}_\tau$. Let $\bar{C}_{\tau\beta} = \theta(B_\beta)$. Let $K_{\tau\beta}$ be the subgroup of K_τ which associates $C_{\tau\beta}$ with $\bar{C}_{\tau\beta}$ (i.e. $K_{\tau\beta} = (C_\tau + \bar{C}_{\tau\beta}) \cap K_\tau$). If $K_\tau = \langle 0 \rangle$ then let $K_{\tau\beta} = \langle 0 \rangle$.

For each β, we wish to show $\bigoplus (K_{\tau\beta})_{\tau<\kappa}$ is a direct summand of $\bigoplus (F_\tau)_{\tau<\kappa}$. This is easy. Let $\bigoplus (K_{\tau\beta})_{\tau<\kappa} = K^\beta$. Then

$$\bigoplus (F_\tau)_{\tau<\kappa}/K^\beta = \bigcup_{\nu<\kappa}(\bigoplus (F_\tau)_{\tau<\nu} + K^\beta)/K^\beta.$$

Since

$$(\bigoplus (F_\tau)_{\tau<\nu} + K^\beta)/K^\beta \simeq \bigoplus (F_\tau)_{\tau<\nu}/\bigoplus (K_{\beta\tau})_{\tau<\nu},$$

we have a smooth chain of free groups. Also,

$$((\bigoplus (F_\tau)_{\tau<\nu+1} + K^\beta)/K^\beta)/((\bigoplus (F_\tau)_{\tau<\nu} + K^\beta)/K^\beta) \simeq F_\nu/\bar{C}_{\beta\nu}$$

which is a free group. So $\bigoplus (F_\tau)_{\tau<\kappa}/K^\beta$ is free.

If $\kappa = \omega_{n+1}$ then any free resolution of G, by groups of cardinality ω_{n+1}, can be made to work.

The above embedding lemma goes through without change for $\mathscr{F}_n$ groups. We now have the following theorems.

5.6. THEOREM. (1) *For all* $n > 0$ *and* $E \subseteq \omega_n$, *there is a strongly* ω_n*-free group* A *of cardinality* ω_n *such that* $\Gamma(A) = \tilde{E}$.

(2) *Suppose there is an* ω_α*-free not* $\omega_{\alpha+1}$*-free group. For all* $n > 0$ *and* $E \subseteq \{\nu < \omega_{\alpha+n} | \mathrm{cf}(\nu) \geqq \omega_\alpha\}$, *there is an* $\omega_{\alpha+n}$*-free group* A *of cardinality* $\omega_{\alpha+n}$ *such that* $\Gamma(A) = \tilde{E}$.

As with Theorem 2.15, the groups constructed by this method can be described.

5.7. THEOREM. (1) *For all* $n > 0$ *and* $\theta: \omega_n \rightarrow \mathscr{F}_{n-1}$, *there is a strongly* ω_n*-free*, A, *with an* ω_n*-filtration* $\{A_\nu | \nu < \omega_n\}$ *such that* $A_{\nu+1}/A_\nu \approx \theta(\nu)$.

(2) *Suppose ω_α is a regular cardinal. For all $n > 0$ and $\theta : \omega_{k+n} \to \mathscr{E}_{n-1}$ such that $\theta(\nu)$ is free when* cf $(\nu) < \omega_\alpha$; *there is an $\omega_{\alpha+n}$-free group A with an $\omega_{\alpha+n}$-filtration, $\{A_\nu | \nu < \omega_{\alpha+n}\}$ such that $A_{\nu+1}/A_\nu \approx \theta(\nu)$.*

Proof (of Theorems 5.6 and 5.7). We will content ourselves with proving Theorem 5.7 (1). Suppose n and θ are as in the hypothesis of the theorem. For each $m < n$ define a group A^m inductively by defining an ω_n-filtration, $\{A_\nu | \nu < \omega_n\}$, of A^m. This filtration will have the property that A_ν is ω_n-pure if $\mathrm{cf}(\nu) \neq \omega_m$ and $A_{\nu+1}/A_\nu \approx \theta(\nu)$ if $\mathrm{cf}(\nu) = \omega_m$.

Case 0. Let A_0 be a free group on ω_{n-1} generators.

Case 1. Assume $\mathrm{cf}(\nu) \neq \omega_m$. Let $A_{\nu+1}$ be the direct sum of A_ν and a free group on ω_{n-1} generators.

Case 2. Assume ν is a limit ordinal. Let $A_\nu = \bigcup_{\tau<\nu} A_\tau$. By Lemma 5.1, A_ν is free.

Case 3. Assume $\mathrm{cf}(\nu) = \omega_m$. Choose a continuous increasing function $f : \omega_m \to \nu$ such that $\mathrm{cf}(f(\tau)) = \omega_m$. So $A = \bigcup_{\tau<\omega_m} A_{f(\tau)}$ and $A/A_{f(\tau)}$ is free for each $\tau < \omega_m$. By Lemma 5.2 we can choose $A_{\nu+1}$ such that $A_{\nu+1}/A_\nu \approx \theta(\nu)$ and $A_{\nu+1}/A_{f(\tau)}$ is free.

Finally $A = \bigoplus (A_m)_{m<n}$ is the required group.

We will end this section with a construction of ω_n-separable groups. In the construction we combine the methods of Theorem 2.4 with those of this chapter. We begin with an embedding lemma.

5.8. LEMMA. *For each n and $G \in \mathscr{F}_n$, there exist free groups $A \supseteq B = \bigcup_{m<\omega} B_m$, C_m, C_{ms} $(m < s < \omega)$ satisfying:*

(i) $A/B = G$;

(ii) $B_m \oplus C_{mm+1} = B_{m+1}$;

(iii) $B_m \oplus C_m = A$;

(iv) $C_m = C_{ms} \oplus C_s$.

Proof. By Lemma 5.5, there are free groups $A \supset B = \bigcup_{m<\omega} B_m$ such that $A/B \simeq G$ and A/B_m is free. Choose C_0 such that $B_0 \oplus C_0 = A$. In general if we have defined C_m and C_{ms} for $m, s < t$, let $C_{tt+1} = B_{t+1} \cap C_t$. Choose C_{t+1} such that $C_{tt+1} \oplus C_{t+1} = C_t$. Finally let $C_{mt+1} = C_{mt} \oplus C_{tt+1}$.

5.9. THEOREM. (1) *For each $n > 0$, there are 2^{ω_n} ω_n-separable groups of cardinality ω_n.*

(2) *Suppose $n > 0$ and $\theta : \omega_n \to \mathscr{F}_{n-1}$ is such that $\theta(\nu)$ is free if* $\mathrm{cf}(\nu) \neq \omega$. *There is an ω_n-separable group, A, with an ω_n-filtration, $\{A_\nu | \nu < \omega_n\}$, such that $A_{\nu+1}/A_\nu \approx \theta(\nu)$.*

Proof. It suffices to prove (2). We will define inductively an ω_n-filtration, $\{A_\nu | \nu < \omega_n\}$, of A and auxiliary groups, $B_{\nu\tau}$ ($\nu < \tau$ and $\theta(\nu)$ is free). These groups will satisfy:

(i) $A_\tau = A_\nu \oplus B_{\nu\tau}$, if $\nu < \tau$ and $\theta(\nu)$ is free;
(ii) $B_{\nu k} \oplus B_{\alpha\tau} = B_{\nu\tau}$, if $\nu < \alpha < \tau$ and $\theta(\nu)$ and $\theta(\alpha)$ are free;
(iii) if $\theta(\nu)$ is free and λ is a limit ordinal $> \nu$, then

$B_{\nu\lambda} = \bigcup_{\nu<\tau<\lambda} B_{\nu\tau}$; and

(iv) $B_{\nu k} \subseteq B_{\nu\tau}$, if $\nu < \alpha < \tau$ and $\theta(\nu)$ is free.

The proof again divides into 4 cases. We will only do the crucial case when $\theta(\nu)$ is not free. Since $\theta(\alpha)$ is free if $\mathrm{cf}(\alpha) \neq \omega$, we can choose τ_m such that $\theta(\tau_m)$ is free and $\bigcup_{m<\omega} \tau_m = \nu$. So

$$A = C \oplus \bigoplus (B_{\tau_m \tau_{m+1}})_{m<\omega},$$

for some free group C. Using Lemma 5.8, we can choose $A_{\nu+1}$, $B_{\tau_m \nu+1}$ such that:

$$A_{\tau_m} \oplus B_{\tau_m \nu+1} = A_{\nu+1};$$
$$B_{\tau_m \tau_s} \oplus B_{\tau_s \nu+1} = B_{\tau_m \nu+1}, \text{ for } m < s; \text{ and}$$
$$A_{\nu+1}/A_\nu \simeq \theta(\nu).$$

The rest of the proof follows that of Theorem 2.4.

6. Some questions.

6.1. Which of our results for abelian groups can be established for groups? Of relevance here is a result announced in [**18**] (p. 322) which says for any cardinal, κ, there is a κ-free not κ^+-free abelian group if and only if there exists a κ-free not κ^+-free group. More generally, it seems that we use only a minimal amount of specifically group theoretic methods. Where else can these methods be used?

6.2. If κ is a regular limit cardinal, what are the possible values for Γ on κ-free groups of cardinality κ? A possible result (which probably requires something like $(V = L)$) is that if $E \subseteq \kappa$ and $\{\nu | E \cap \nu$ is stationary in $\nu\} \subseteq \{\nu | \nu$ is not a regular cardinal$\}$, then there exists a κ-free group A of cardinality κ such that $\Gamma(A) = \tilde{E}$.

6.3. If κ is the successor of a singular cardinal, can there exist a κ-free (abelian) group which is not strongly κ-free?

6.4. Can we construct an ω_n-free not strongly ω_n-free (abelian) group without any set theoretic assumptions?

6.5. Is it possible for some κ that there is a κ-free not strongly κ-free group and no strongly κ-free non-free group of cardinality κ? If this can't happen this would rescue Theorem 1.1 of [**14**] (cf. Theorem 1.5).

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