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SELF-AFFINE PROCESSES AND THE ERGODIC THEOREM

WIM VERVAAT

ABSTRACT. Known results for strictly stable motions as finiteness of moments and local boundedness of sample-path variation are generalized to self-affine processes, *i.e.*, self-similar processes with stationary increments. The proofs are new, even for stable motions, and are obtained by applying the ergodic theorem to powers of the (one-sided) increments.

0. **Introduction.** In this paper all stochastic processes have time domain $\mathbb{R}_+ := [0,\infty)$ and state space $\bar{\mathbb{R}} := [-\infty,\infty]$. We assume processes to be *almost finite-valued* in the sense that all marginal distributions are concentrated on $\mathbb{R}$. This does not exclude infinite values for the sample paths. The distribution of a process is the family of all finite-dimensional distributions. Equality in distribution is expressed by $=_d$.

A process $X = \big(X(t)\big)_{t\geq 0}$ is H *self-affine* ($H \in \mathbb{R}$ fixed) if it is H *self-similar*:

$$X(a\cdot) =_d a^H X(\cdot) \quad \text{for all } a > 0, \tag{0.1}$$

and in addition has *stationary increments*:

$$X(b+\cdot) - X(b) =_d X(\cdot) - X(0) \quad \text{for all } b \geq 0 \tag{0.2}$$

regarded as processes with time domain $\mathbb{R}_+$. Ignoring trivial or pathological cases we may restrict our attention to the case $H > 0$ and $X(0) = 0$ (*cf.* Vervaat (1985)). In this case $t \longmapsto X(t)$ is easily seen to be continuous in probability, which allows us to select a smooth version of the following type.

A function f from a real interval T to $\bar{\mathbb{R}}$ is called *separable* with separating set D if D is a dense subset of T and the closure in $T \times \bar{\mathbb{R}}$ of the graph of $f|_D$ contains the graph of f. We call f *universally separable* if each dense subset of T is separating for f. A process that is continuous in probability possesses a version which is universally separable wp1 (Neveu (1964), Proposition III.4.2). Henceforth we assume self-affine processes to be selected in this way. As a consequence we cannot exclude infinite values for the sample paths. Another consequence is, for instance, that $X \equiv 0$ wp1 in case $X(t) = 0$ wp1 for each t separately. Nevertheless, sample paths of self-affine processes can be very ill-behaved (*cf.* Maejima (1983)).

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An important subclass of the H self-affine processes are those with *independent increments*. They are the strictly stable motions with exponent H^{-1}. Such processes exist only for $H \geq \frac{1}{2}$. We have Brownian motion for $H = \frac{1}{2}$. The characteristic function of the symmetrized distribution of $X(1)$ has the form $\lambda \longmapsto \exp -c|\lambda|^{1/H}$ for some $c > 0$.

The contents of this paper can be described from two different viewpoints. Here is the first one. The distribution of an H self-affine process over the space of all functions $f\colon \mathbb{R}_+ \longrightarrow \bar{\mathbb{R}}$ such that $f(0) = 0$ is invariant under the transformations

$$f \longmapsto a^{-H} f(a\cdot) \qquad (a > 0) \tag{0.3}$$

$$f \longmapsto f(b+\cdot) - f(b) \qquad (b \geq 0). \tag{0.4}$$

We explore what can be derived if we apply the ergodic theorem to these probability preserving transformations, which leads us to several results around the second line of invariance. Because of this, discussions prior to the results often replace formal proofs afterwards.

For an alternative description, consider the collection of known properties of strictly stable motions, as listed in the following table:

<table>
<tr><th>range of H:</th><th>$\mathbb{E}|X(1)|^p = \infty$:</th><th>sample path variation:</th></tr>
<tr><td>$H = \frac{1}{2}$</td><td>for no $p > 0$</td><td rowspan="2">nowhere bounded</td></tr>
<tr><td>$\frac{1}{2} < H \leq 1$</td><td rowspan="2">iff $p \geq \frac{1}{H}$</td></tr>
<tr><td>$1 < H$</td><td>locally bounded</td></tr>
</table>

TABLE. PROPERTIES OF STRICTLY STABLE MOTIONS.

See Zolotarev (1986) and Fristedt (1974) for these results. We want to generalize them as much as possible to all self-affine processes. Complete generalization is impossible, as the following list of examples shows.

EXAMPLES 0.1. (a) There exist Gaussian self-affine processes (fractional Brownian motions, *cf.* Mandelbrot and Van Ness (1968)) for $0 < H < 1$, and for them all moments are finite.

(b) For all combinations of H and p not allowed in the middle column there are self-affine processes X such that $\mathbb{E}|X(1)|^p = \infty$. See Kôno and Maejima (1991).

(c) For $H > 1$, there are H self-affine processes whose sample paths have nowhere bounded variation wp1. See Vervaat (1985, §§5.3, 6.5).

1. **Auxiliary results.** We first formulate the ergodic theorem as we are going to apply it. It deviates from the standard formulation by the possibility of an infinite expectation. This version is known in the folklore but rare in the books. See Vervaat (1985, Theorem 3.7) for a proof starting from the finite-expectation case. Recall that a sequence $(\xi_k)_{k=1}^{\infty}$ of random variables in $\mathbb{R}$ is called *stationary* if its distribution is invariant under the time shift, *i.e.*, $(\xi_{n+k})_{k=1}^{\infty}$ has the same distribution for $n = 0, 1, 2, \ldots$.

THEOREM 1.1. *Let $(\xi_k)_{k=1}^{\infty}$ be a stationary sequence of random variables in $\mathbb{R}$ and let φ be a Borel measurable function $\mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathbb{E}\varphi(\xi_1)$ exists, finite or infinite (i.e., $\big(\mathbb{E}\varphi^+(\xi_1), \mathbb{E}\varphi^-(\xi_1)\big) \neq (\infty, \infty)$). Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi(\xi_k) = \mathbb{E}^I \varphi(\xi_1) \text{ wp1}, \tag{1.1}$$

where $\mathbb{E}^I$ denotes conditional expectation with respect to the σ-algebra I of events that are almost invariant under the shift of (ξ_k).

If X has stationary increments, then $\big(X(k) - X(k-1)\big)_{k=1}^{\infty}$ is a stationary sequence. Its first element equals $X(1)$ if $X(0) = 0$ wp1, which is the case if X is self-affine. Then we have by Theorem 1.1 for all Borel functions φ such that $\mathbb{E}\varphi\big(X(1)\big)$ exists, finite or infinite:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi\big(X(k) - X(k-1)\big) = \mathbb{E}^I \varphi\big(X(1)\big) \text{ wp1}, \tag{1.2}$$

where I is the σ-algebra of events that are almost invariant under the shift of $\big(X(k) - X(k-1)\big)$. We will apply (1.2) in the next sections for $\varphi(x) = (x^+)^p$.

The following theorem presents instances of propagation of sample-path behavior of self-affine processes over their entire time domain.

THEOREM 1.2. *Let X be H self-affine with $H > 0$. Then the following hold.*
(a) The events $[X(1) = 0]$ and $[X \equiv 0]$ are equal up to a null event.
(b) If $X(1) \geq 0$ wp1, then X has increasing sample paths wp1.
(c) Wp1 X is either nowhere of bounded variation or of bounded variation on all bounded time intervals.

PROOF. For (a), see O'Brien and Vervaat (1983, Lemma 3), for (b) Vervaat (1985, Theorem 2.2), and for (c) Vervaat (1985, Theorem 2.4(b)). ■

For the following theorem and the sequel we define stochastic order. Let ξ and η be two random variables in $\bar{\mathbb{R}}$. We say that ξ is *stochastically smaller* than η, notation $\xi \leq_d \eta$, if $\mathbb{P}[\xi > x] \leq \mathbb{P}[\eta > x]$ for all $x \in \mathbb{R}$. Then $\mathbb{E}f(\xi) \leq \mathbb{E}f(\eta)$ for all nondecreasing functions $f \colon \bar{\mathbb{R}} \longrightarrow [0, \infty]$.

THEOREM 1.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let ξ be an $\bar{\mathbb{R}}$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}\xi^+ < \infty$. If $\mathbb{E}^{\mathcal{G}}\xi \geq_d \xi$, then $\mathbb{E}^{\mathcal{G}}\xi = \xi$* wp1, *i.e., ξ differs from a $\mathcal{G}$ measurable function by a null function.*

The theorem has been proved by Smit (1983). The next proof is due to M.S. Keane (personal communication).

PROOF. Let f be a strictly increasing, strictly convex function on $\mathbb{R}$ such that $f(x) \longrightarrow 0$ as $x \longrightarrow -\infty$ and $f(x) \sim x$ as $x \longrightarrow \infty$ (e.g., $f(x) = \log(1 + e^x)$). Set $f(-\infty) := 0$ and $f(\infty) := \infty$. Note that $\mathbb{E}f(\xi) < \infty$ because $\mathbb{E}\xi^+ < \infty$ and $f(x) \sim x$ as $x \longrightarrow \infty$. By Jensen's inequality for conditional expectations we have

$$\mathbb{E}^{\mathcal{G}} f(\xi) \geq f(\mathbb{E}^{\mathcal{G}}\xi) \text{ wp1}, \tag{1.3}$$

with equality iff $\xi = \mathbb{E}^{\mathcal{G}}\xi$ wp1 (also in case $\mathbb{P}[\xi=-\infty] > 0$, to be checked separately). However,

$$\infty > \mathbb{E}f(\xi) = \mathbb{E}\mathbb{E}^{\mathcal{G}}f(\xi) \geq \mathbb{E}f(\mathbb{E}^{\mathcal{G}}\xi) \geq \mathbb{E}f(\xi)$$

(the last inequality because $\mathbb{E}^{\mathcal{G}}\xi \geq_d \xi$), so the second last inequality is in fact an equality, which in presence of (1.3) implies $\mathbb{E}^{\mathcal{G}}f(\xi) = f(\mathbb{E}^{\mathcal{G}}\xi)$ wp1. So $\xi = \mathbb{E}^{\mathcal{G}}\xi$ wp1 by the line after (1.3). ∎

COROLLARY 1.4. *If $\mathbb{E}\xi$ exists, finite or infinite, and $\mathbb{E}^{\mathcal{G}}\xi =_d \xi$, then $\mathbb{E}^{\mathcal{G}}\xi = \xi$* wp1.

2. **Locally bounded variation.** We are going to apply (1.2) with $\varphi(x) = x^+$. This case already occurs in Vervaat (1985, §3), the germ of the present paper.

Let X be H self-affine ($H > 0$). We are aiming at the case of X having locally bounded variation wp1, but first we assume that X has nondecreasing sample paths wp1. We confront (1.1):

$$\frac{1}{n}\sum_{k=1}^{n}\big(X(k)-X(k-1)\big) = \frac{1}{n}X(n) \to \mathbb{E}^I X(1) \text{ wp1}$$

with

$$\frac{1}{n}X(n) =_d n^{H-1}X(1) \begin{cases} \to 0 \text{ wp1} & \text{if } 0 < H < 1; \\ = X(1) & \text{if } H = 1; \\ \to \infty \cdot X(1) & \text{if } 1 < H. \end{cases}$$

Comparing the limits, which must be equal in distribution, we find

$$\begin{array}{ll} \mathbb{E}^I X(1) = 0 \text{ wp1} & \text{if } 0 < H < 1; \\ \mathbb{E}^I X(1) =_d X(1) & \text{if } H = 1; \\ \mathbb{P}[\mathbb{E}^I X(1)=\infty] = 1 - \mathbb{P}[\mathbb{E}^I X(1)=0] = \mathbb{P}[X(1)>0] & \text{if } 1 < H. \end{array}$$

For $0 < H < 1$ it follows subsequently that $\mathbb{E}X(1) = \mathbb{E}\mathbb{E}^I X(1) = 0$, $X(1) = 0$ wp1 (since $X(1) \geq 0$ wp1), so that $X \equiv 0$ wp1 by Theorem 1.2(a).

For $H = 1$ it follows by Corollary 1.4 that $X(1)$ differs by a null function from a I measurable random variable, so that $X(k) - X(k-1) = X(1)$ wp1 for all k. Consequently, $X(t) = tX(1)$ wp1 for $t \in \mathbb{N}$. By self-similarity we find the same for $t \in 2^{-n}\mathbb{N}$, $n = 1, 2, \ldots$ (apply (0.1) for $H = 1$ and $a = 2^{-n}$). By universal separability it follows that $X(t) = tX(1)$ for all real $t \geq 0$ simultaneously wp1.

For $1 < H$, Theorem 1.2(a) sharpens the result to $\mathbb{E}^I X(1) = \infty \cdot 1_{[X\equiv 0]^c}$ wp1. It follows that $\mathbb{E}X(1) = \infty$ unless $X \equiv 0$ wp1.

We have found:

THEOREM 2.1. *If X is a nondecreasing self-affine process, then*
$X \equiv 0$ wp1 *if* $0 < H < 1$;
$X(t) \equiv tX(1)$ wp1 *if* $H = 1$;
$\mathbb{E}X(1) = \infty$ *if* $1 < H$ *unless* $X \equiv 0$ wp1.

Now we assume that X has sample paths of locally bounded variation wp1. In this case we can write $X = X_\uparrow - X_\downarrow$, where $X_\uparrow$ is the nondecreasing process obtained by $X_\uparrow(t) :=$

$\sup \sum_{k=1}^{n}\big(X(t_k)-X(t_{k-1})\big)^+$, the supremum taken over all $n \in \mathbb{N}$ and all sequences $0 = t_0 < t_1 < \cdots < t_n = t$, and where $X_\downarrow := (-X)_\uparrow$. From these expressions and the continuity in probability it is clear that both $X_\uparrow$ and $X_\downarrow$ are H self-affine as well. Applying Theorem 2.1 to $X_\uparrow$ and $X_\downarrow$ we find for $X = X_\uparrow - X_\downarrow$:

THEOREM 2.2. *If X is an H self-affine process whose sample paths have locally bounded variation* wp1, *then*

$X \equiv 0$ wp1 *if* $0 < H < 1$*;*
$X \equiv tX(1)$ wp1 *if* $H = 1$*;*
$\mathbb{E}X(1)^+ = \infty$ *in case* $1 < H$ *unless X is nonincreasing* wp1*;*
$\mathbb{E}X(1)^- = \infty$ *in case* $1 < H$ *unless X in nondecreasing* wp1.

REMARKS 2.3. (a) By Theorem 1.2(c) it follows for the case $H \leq 1$ that wp1 either X has nowhere bounded variation or $X(t) \equiv tX(1)$. This is a successful generalization of the top part in the right column of the table. No further generalization is possible for the bottom part of the same column, because of Examples 0.1(c).

(b) Let $A := [X$ has locally bounded variation$]$. Then A is invariant under the transformations (0.3) and (0.4), so $X1_A$ and $X1_{A^c}$ are also H self-affine. It follows that $H > 1$ and $\mathbb{E}|X(1)| = \infty$ if $\mathbb{P}(A) > 0$ unless $X(t)1_A \equiv tX(1)1_A$ wp1. Similar extensions are possible for the results in the next sections.

The following result builds on the arguments used above for the case $H = 1$. It will be needed in the sequel.

LEMMA 2.4. *Let X be an H self-affine ($H > 0$) universally separable process. If $X(1)^+$ differs by a null function from a $\mathcal{I}$ measurable function, then $X(t)^+ \equiv tX(1)^+$* wp1*, and $H \geq 1$ unless $X \equiv 0$* wp1. *If in addition $\mathbb{P}[X(1){>}0] > 0$, then $H = 1$ and $X(t) \equiv tX(1)$* wp1.

PROOF. We have $\big(X(k) - X(k{-}1)\big)^+ = X(1)^+$ wp1 for all k, which consumes all probability of the events $[X(t) > 0]$, a probability that does not depend on t because X is self-similar. So $[X(t) > 0]$ can only occur this way, and $X(t)^+ = tX(1)^+$ wp1 for all $t \in \mathbb{N}$, hence jointly for all real $t \geq 0$ by self-similarity and universal separability. The event $[X(1) \geq 0]$ turned out to be invariant wp1 for the transformations (0.3) and (0.4) applied to X, so $X1_{[X(1)\geq 0]}$ and $X1_{[X(1)<0]}$ are H self-affine, and monotone by Theorem 1.2(b). So $H \geq 1$ by Theorem 2.1, unless $X \equiv 0$ wp1. If $\mathbb{P}[X(1){>}0] > 0$ then $H = 1$, so that $X(t)1_{[X(1)<0]} \equiv tX(1)1_{[X(1)<0]}$ wp1 by Theorem 2.1. ∎

3. **Infinite moments.**

We are going to apply (1.2) with $\varphi(x) = (x^+)^p$ for $p \leq 1$. Let X be H self-affine ($H > 0$) and universally separable. We confront (1.2):

$$\frac{1}{n}\sum_{k=1}^{n}\Big(\big(X(k) - X(k{-}1)\big)^+\Big)^p \longrightarrow \mathbb{E}^{\mathcal{I}}\big(X(1)^+\big)^p \text{ wp1} \tag{3.1}$$

with

$$\frac{1}{n}\sum_{k=1}^{n}\Big(\big(X(k){-}X(k{-}1)\big)^+\Big)^p \geq \frac{1}{n}\big(X(n)^+\big)^p =_d n^{pH-1}\big(X(1)^+\big)^p \begin{cases} \longrightarrow 0 \text{ wp1} & \text{if } H < \frac{1}{p}; \\ = (X(1)^+)^p & \text{if } H = \frac{1}{p}; \\ \longrightarrow \infty \cdot X(1)^+ & \text{if } H > \frac{1}{p}. \end{cases}$$

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The restriction $p \leq 1$ is needed for the inequality. The case $Hp < 1$ does not give interesting conclusions. We compare the two limits in the two other cases under the additional hypothesis that $\mathbb{E}\big(X(1)^+\big)^p < \infty$, so $\mathbb{E}^I\big(X(1)^+\big)^p < \infty$ wp1.

For the case $Hp = 1$ we find $\mathbb{E}^I\big(X(1)^+\big)^p \geq_d \big(X(1)^+\big)^p$, so $\big(X(1)^+\big)^p = \mathbb{E}^I\big(X(1)^+\big)^p$ wp1 by Theorem 1.3, *i.e.*, $X(1)^+$ differs by a null function from a I measurable random variable. By Lemma 2.4 it follows that $X(t)^+ \equiv tX(1)^+$ wp1, so that $X^+ \equiv 0$ wp1 in case $H \neq 1$.

For the case $Hp > 1$ we arrive at a contradiction unless $X(1)^+ = 0$ wp1, so unless X is nondecreasing wp1 (by Theorem 1.2(b)).

Combining all this with the restriction $p \leq 1$ we conclude:

THEOREM 3.1. *If X is a universally separable H self-affine process with $H \geq 1$, then $\mathbb{E}\big(X(1)^+\big)^p = \infty$ for $p \geq \frac{1}{H}$, unless X is decreasing* wp1 *or $H = 1$ and $X(t) \equiv tX(1)$* wp1.

We have generalized the middle column of the table as far as Examples 0.1(a,b) allow us.

4. **Infinite moments in presence of jumps.**

We are going to apply (1.2) with $\varphi(x) = (x^+)^p$ for general $p \geq 0$. However, our results are meaningful mainly for $p \geq 1$, and our motivation comes from this subcase. We try to confront (3.1) with the asymptotic behavior of

$$\frac{1}{n}\sum_{k=1}^{n}\Big(\big(X(k) - X(k-1)\big)^+\Big)^p =_d n^{pH-1}\sum_{k=1}^{n}\left(\left(X\Big(\frac{k}{n}\Big) - X\Big(\frac{k-1}{n}\Big)\right)^+\right)^p. \tag{4.1}$$

The following is obvious for the case $pH > 1$.

THEOREM 4.1. *Let X be an H self-affine process. If $pH > 1$ and the series on the right-hand side of (4.1) does not converge to 0 in probability as $n \longrightarrow \infty$, then $\mathbb{E}\big(X(1)^+\big)^p = \infty$.*

At this stage, results for $pH = 1$ seem hard to get at. Moreover, if X happens to be a pure jump process, then the series on the right-hand side of (4.1) converges wp1 to $\sum_{t\in[0,1]}\Big(\big(X(t+) - X(t-)\big)^+\Big)^p$, which is also a lower estimate for the lower limit of the same series in (4.1) if X happens to be càdlàg. Therefore we aim at asymptotic lower estimates of the same type, but in a context without assumptions about the smoothness of sample paths of X beyond separability.

Let f be a function $T \longrightarrow \bar{\mathbb{R}}$ (T an interval in $\mathbb{R}$). By $\delta^+ f(t)$ we denote the *upward jump* of f at t:

$$\delta^+ f(t) := \big(\liminf_{s\downarrow t} f(s) - \limsup_{s\uparrow t} f(s)\big)^+ \quad \text{for } t \in T, \tag{4.2}$$

with the provision that $\liminf_{s\downarrow t} f(s)$ must be replaced by $f(t)$ in case $t = \max T$, and $\limsup_{s\uparrow t} f(s)$ by $f(t)$ in case $t = \min T$. Similarly we define $\delta^- f(t) := \delta^+(-f)(t)$ to be the *downward jump*. Note that $\delta^+ f(t) = \delta^- f(t) = 0$ does not imply that f is continuous at t (unless f is càdlàg). For instance, if f is nowhere bounded above or below (as are the sample paths of self-affine processes in Maejima (1983)), then f has no jumps at all according to our definition.

If f is separable with separating set D, then s on the right-hand side of (4.2) may be restricted to $s \in D$ without changing the left-hand side.

LEMMA 4.1. *(a) Let f and φ be a $\bar{\mathbb{R}}$-valued functions on* $[0,1]$ *and* $[0,\infty]$ *respectively, and suppose that φ is non-negative and lower semicontinuous. Then*

$$\sum_{t\in[0,1]} \varphi\big(\delta^+ f(t)\big) \le \liminf_{n\to\infty} \sum_{k=1}^{n} \varphi\left(\left(f\Big(\frac{k}{n}\Big) - f\Big(\frac{k-1}{n}\Big)\right)^+\right). \tag{4.3}$$

(b) If f is, in addition, separable with separating set D, then

$$\sum_{t\in[0,1]} \varphi\big(\delta^+ f(t)\big) = \lim_{m\to\infty} \inf \sum_{k=1}^{n} \varphi\Big(\big(f(u_k) - f(u_{k-1})\big)^+\Big), \tag{4.4}$$

where the infimum is taken over all finite sequences $0 = u_0 < u_1 < \cdots < u_n = 1$ *such that* $u_k \in D \cup \{0,1\}$ *and* $u_k - u_{k-1} < \frac{1}{m}$ *for* $1 \le k \le n$.

PROOF. (a) If $x <$ left-hand side of (4.3), then there is a finite subset F of $[0,1]$ such that already $x < \sum_{t\in F} \varphi\big(\delta^+ f(t)\big)$. By lower semicontinuity of φ and (4.2) there is an $\varepsilon > 0$ such that $x < \sum_{t\in F} \varphi\Big(\big(f(r_t) - f(l_t)\big)^+\Big)$ for all l_t and r_t that satisfy $0 \vee (t-\varepsilon) \le l_t < t < r_t \le (t+\varepsilon) \wedge 1$. So the sum on the right-hand side of (4.3) is also larger than x for all large n.

(b) The proof that the right-hand side of (4.4) is not smaller follows the methods of the proof of (a), supplemented by the argument that s in (4.2) may be restricted to D. To prove that the right-hand side is not larger, suppose it is larger than x. We must show that then the left-hand side is also larger than x. The right-hand side is still larger than some $y > x$ and there is an $m \in \mathbb{N}$ such that $\sum_{k=1}^{n} \varphi\big(f(u_k) - f(u_{k-1})^+\big)$ is larger than y for all sequences $(u_k)_{k=0}^{n}$ as indicated in the line after (4.4). Fix a finite sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ (not necessarily in D) such that $t_k - t_{k-1} < \frac{1}{m}$ for all k, and then select an increasing sequence $u_0, u_1, \ldots, u_{2n+1}$ such that $u_0 = 0$, $u_{2n+1} = 1$, $u_k \in D$ for the other k, $u_{2k} < t_k < u_{2k+1}$ (either inequality restricted to u having indices other than 0 and $2n+1$), and

$$\sum_{k=1}^{n} \varphi\big(\delta^+ f(t_k)\big) > \sum_{k=1}^{2n+1} \varphi\Big(\big(f(u_k) - f(u_{k-1})\big)^+\Big) - (y-x)$$

(which is possible by (4.2) and lower semicontinuity of φ). The left-hand side is not larger than the left-hand side of (4.4). The right-hand side is larger than $y-(y-x) = x$. ∎

We now are going to compare the right-hand side of (4.1) with functionals of $Y_p(1)$, where

$$Y_p(t) := \sum_{u\in[0,t]} \big(\delta^+ X(u)\big)^p \quad \text{for } t \ge 0. \tag{4.5}$$

Note that $Y_p(t)$ is indeed a random variable for each t, by (4.4) with countable D and our convention to take universally separable versions of X. Obviously, Y_p has nondecreasing sample paths, but its values may be ∞.

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LEMMA 4.3. Wp1 *either* $Y_p \equiv \infty$ *on* $(0,\infty)$ *or* Y_p *is finite-valued. If* $Hp \leq 1$ *and* Y_p *is finite-valued, then* $Y_p \equiv 0$ *(modulo null events).*

PROOF. Note that Y_p is pH self-affine in the more general sense that it need not be almost finite-valued. Because Y_p has increasing sample paths and $\mathbb{P}[Y_p(t) = \infty]$ does not depend on $t > 0$ by self-similarity, the implication $Y_p(t) = \infty$ (for a fixed $t > 0$) $\Rightarrow Y_p \equiv \infty$ on $(0,\infty)$ holds true wp1. This proves the first statement in the lemma.

Since $A := [Y_p$ finite-valued$]$ is invariant under the transformations (0.3) and (0.4), the nonincreasing process $Y_p 1_A$ is also pH self-affine, and finite-valued. If $pH < 1$, then $Y_p 1_A \equiv 0$ wp1 by Theorem 2.1. If $pH = 1$, then $Y_p(t)1_A \equiv tY_p(1)1_A$ wp1 by Theorem 2.1. However, if Y_p is finite-valued, then the sums on the right-hand side of (4.5) are finite, so their summands are positive only on a countable set of u's. So the sum cannot be of the form ct for varying t unless $c = 0$. We arrive at the same conclusion as for $pH < 1$. ∎

THEOREM 4.4. *Let* X *be* H *self-affine and universally separable. If* $pH \geq 1$ *and* X *makes positive jumps with positive probability, then* $\mathbb{E}\big(X(1)^+\big)^p = \infty$.

PROOF. If X makes positive jumps with positive probability, then $Y_p(1) > 0$ with positive probability. If $pH = 1$, then $Y_p \equiv \infty$ on $(0,\infty)$ with the same positive probability, by Lemma 4.3. From (3.1), (4.1) and Lemma 4.2(a) it follows that $\mathbb{E}^I\big(X(1)^+\big)^p \geq_d Y_p(1)$, which equals ∞ with positive probability. So $\mathbb{E}\big(X(1)^+\big)^p = \infty$ for $p = \frac{1}{H}$, and consequently also for larger p. ∎

We now check this result against what is known for strictly stable motions. We have obtained the 'if' part of the following variant of the middle column of the table. For strictly stable motions X (which are known to possess càdlàg versions) we have $\mathbb{E}\big(X(1)^+\big)^p = \infty$ iff X makes positive jumps with positive probability and $p \geq \frac{1}{H}$, for all $H \geq \frac{1}{2}$. Indeed, Brownian motion ($H = \frac{1}{2}$) does not make jumps at all, and for spectrally negative strictly stable motions with $H > \frac{1}{2}$ (those that make only negative jumps) all moments of $X(1)^+$ exist (*cf.* Zolotarev (1986)).

In Theorem 3.1, which applies to the case $H \geq 1$, there is no condition at all about positive jumps, which at first sight could contradict what is known for spectrally negative strictly stable motions. However, for $H > 1$ the spectrally negative strictly stable motions are nonincreasing (Theorem 3.1 could serve as a new proof for this, in presence of the finiteness of $\mathbb{E}X(1)^+$). For $H = 1$ the only strictly stable motion among the stable motions is the symmetric one (Kasahara, Maejima and Vervaat (1988) or O'Brien and Vervaat (1985, §7)), so Theorem 3.1 is not applicable here.

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V.M. Zolotarev (1986), *One-dimensional Stable Distributions*, Translations of Mathematical Monographs **65**, Amer. Math. Soc.

Mathématiques
Université Claude Bernard Lyon 1
43, Boulevard du 11 Novembre 1918
69622 Villeurbanne cedex
France
e-mail: vervaat@jonas.univ-lyon1.fr