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REAL HYPERSURFACES WITH ϕ-INVARIANT SHAPE OPERATOR IN A COMPLEX PROJECTIVE SPACE

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Abstract. We characterize real hypersurfaces of type (A) and ruled real hypersurfaces in a complex projective space in terms of two ϕ-invariances of their shape operators, and give geometric meanings of these real hypersurfaces by observing their some geodesics.

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1. Introduction. The theory of Riemannian submanifolds in a Euclidean sphere is one of the most interesting objects in differential geometry. It is known that an isometric immersion f of a Kähler manifold M with Kähler structure J into a sphere has parallel second fundamental form σ if and only if σ is J-invariant, that is $\sigma(JX, JY) = \sigma(X, Y)$ holds for each vector X, Y on M (Proposition 3).

In this context, we consider a real hypersurface M^{2n-1} in an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature $c(> 0)$, furnished with the almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ on M induced from the Kähler structure J of the ambient space $\mathbb{C}P^n(c)$. In this case the structure tensor ϕ behaves on M similarly to a Kähler structure on a Kähler manifold, and on the other hand there exists no real hypersurface with parallel second fundamental form in $\mathbb{C}P^n(c)$. So, we introduce the following conditions concerning ϕ-invariances of the shape operator A of M.

The shape operator A of M is called *strongly ϕ-invariant* if A satisfies

$$\langle A\phi X, \phi Y\rangle = \langle AX, Y\rangle, \text{ i.e., } \sigma(\phi X, \phi Y) = \sigma(X, Y) \tag{1.1}$$

for all vectors X and Y on M. Also, it is called *weakly ϕ-invariant* if A satisfies

$$\langle A\phi X, \phi Y\rangle = \langle AX, Y\rangle, \text{ i.e., } \sigma(\phi X, \phi Y) = \sigma(X, Y) \tag{1.2}$$

for all vectors X and Y orthogonal to the characteristic vector ξ on M.

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Dedicated to Professor Hiroshi Asano on the occasion of his 75th birthday

We here note that there exist real hypersurfaces satisfying these conditions. Indeed, the real hypersurfaces which are called of type (A) with radius $\pi/(2\sqrt{c}\,)$ have strongly ϕ-invariant shape operator, and all of the real hypersurfaces of type (A) and the ruled real hypersurfaces have weakly ϕ-invariant shape operator, which are known as examples which enrich the theory of real hypersurfaces in $\mathbb{C}P^n(c)$.

The main purpose of this paper is to characterize real hypersurfaces of type (A) and ruled real hypersurfaces in $\mathbb{C}P^n(c)$ by these ϕ-invariances of shape operators (Theorems 1 and 2).

2. Real hypersurfaces of type (A) in $\mathbb{C}P^n(c)$. Let M^{2n-1} be a real hypersurface with unit normal local vector field $\mathcal{N}$ of an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c. The Riemannian connections $\widetilde{\nabla}$ of $\mathbb{C}P^n(c)$ and ∇ of M are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_X\mathcal{N} = -AX \tag{2.1}$$

for vector fields X and Y tangent to M, where $\langle\ ,\ \rangle$ denotes the metric of M induced from the standard Riemannian metric of $\mathbb{C}P^n(c)$ and A is the shape operator of M in $\mathbb{C}P^n(c)$. It is known that M admits an almost contact metric structure $(\phi, \xi, \eta, \langle\ ,\ \rangle)$ induced from the Kähler structure J of $\mathbb{C}P^n(c)$. The characteristic vector field ξ of M is defined as $\xi = -J\mathcal{N}$ and this structure satisfies

$$\phi^2 = -I + \eta\otimes\xi,\ \eta(\xi) = 1 \quad \text{and} \quad \langle\phi X, \phi Y\rangle = \langle X, Y\rangle - \eta(X)\eta(Y),$$

where I denotes the identity map of the tangent bundle TM of M. It follows from the fact that $\widetilde{\nabla}J = 0$ and Equations (2.1) that

$$\nabla_X\xi = \phi AX. \tag{2.2}$$

Here, for later use we recall the Codazzi equation of M^{2n-1} in $\mathbb{C}P^n(c)$.

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle\phi X, Y\rangle\xi\}. \tag{2.3}$$

The eigenvalues and eigenvectors of the shape operator A are called *principal curvatures* and *principal curvature vectors* of M in $\mathbb{C}P^n(c)$, respectively. In the following, we denote by V_λ the eigenspace associated with the principal curvature λ, namely we set $V_\lambda = \{v \in TM \mid Av = \lambda v\}$.

We usually call M a *Hopf hypersurface* if the characteristic vector ξ is a principal curvature vector at each point of M. It is known that every tube of sufficiently small constant radius around each Kähler submanifold of $\mathbb{C}P^n(c)$ is a Hopf hypersurface. This fact tells us that the notion of Hopf hypersurfaces is natural in the theory of real hypersurfaces in $\mathbb{C}P^n(c)$.

The following lemma is a useful tool in the theory of Hopf hypersurfaces in $\mathbb{C}P^n(c)$, $n \geqq 2$.

Lemma 1. *For a Hopf hypersurface M^{2n-1} $(n \geqq 2)$ with principal curvature α corresponding to the characteristic vector field ξ in $\mathbb{C}P^n(c)$, we have the following*:

1. *α is locally constant on M*;
2. *If X is a tangent vector of M perpendicular to ξ with $AX = \lambda X$, then $A\phi X = \frac{\alpha\lambda + (c/2)}{2\lambda - \alpha}\phi X$.*

REMARK 1. In Lemma 1(2), we note that $2\lambda - \alpha \neq 0$ because $c > 0$.

The following real hypersurfaces are so-called real hypersurfaces of type (A_1) and type (A_2), respectively.

(A_1) A geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$;

(A_2) A tube of radius r $(0 < r < \pi/\sqrt{c}\,)$ around a totally geodesic Kähler submanifold $\mathbb{C}P^\ell(c)$ in $\mathbb{C}P^n(c)$ with $1 \leqq \ell \leqq n-2$.

In this paper, summing up the real hypersurfaces of type (A_1) and type (A_2), we call them *the real hypersurfaces of type* (A). The real hypersurfaces of type (A) are known as typical examples of Hopf hypersurfaces. The tangent bundle TM of real hypersurfaces M of type (A_1) with radius r $(0 < r < \pi/\sqrt{c}\,)$ is decomposed as $TM = \{\xi\}_{\mathbb{R}} \oplus V_\lambda$ with $\alpha = \sqrt{c}\,\cot(\sqrt{c}\,r)$, $\lambda = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, $\dim_{\mathbb{R}} V_\lambda = 2n-2$ and $\phi V_\lambda = V_\lambda$. The tangent bundle TM of real hypersurfaces M of type (A_2) with radius r $(0 < r < \pi/\sqrt{c}\,)$ is decomposed as $TM = \{\xi\}_{\mathbb{R}} \oplus V_{\lambda_1} \oplus V_{\lambda_2}$ with $\alpha = \sqrt{c}\,\cot(\sqrt{c}\,r)$, $\lambda_1 = (\sqrt{c}/2)\cot(\sqrt{c}\,r/2)$, $\lambda_2 = (-\sqrt{c}/2)\tan(\sqrt{c}\,r/2)$, $\dim_{\mathbb{R}} V_{\lambda_1} = 2n-2\ell-2$, $\dim_{\mathbb{R}} V_{\lambda_2} = 2\ell$ and $\phi V_{\lambda_i} = V_{\lambda_i}$ $(i = 1, 2)$. Note that a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ is congruent to a tube of radius $(\pi/\sqrt{c}) - r$ around totally geodesic $\mathbb{C}P^{n-1}(c)$ in $\mathbb{C}P^n(c)$.

We prepare the following which is a characterization of the real hypersurfaces of type (A) (see [**10**]).

LEMMA 2. *Let M be a real hypersurface in $\mathbb{C}P^n(c)$ $(n \geqq 2)$. Then the following conditions are mutually equivalent*:

1. *M is locally congruent to a real hypersurface of type* (A);
2. $\phi A = A\phi$;
3. $\langle(\nabla_X A)Y, Z\rangle = (c/4)(-\eta(Y)\langle\phi X, Z\rangle - \eta(Z)\langle\phi X, Y\rangle)$ *for arbitrary vectors X, Y and Z on M.*

At the end of this section we recall the definition of circles in Riemannian geometry. Let $\gamma = \gamma(s)$ be a smooth real curve parametrized by its arclength s on a Riemannian manifold M. If the curve γ satisfies the following ordinary differential equations with some constant $k(\geqq 0)$

$$\nabla_{\dot\gamma}\dot\gamma = kY_s \quad \text{and} \quad \nabla_{\dot\gamma} Y_s = -k\dot\gamma, \tag{2.4}$$

where $\nabla_{\dot\gamma}$ is the covariant differentiation along γ with respect to the Riemannian connection ∇ of M and Y_s is so-called the unit principal normal vector of γ, we call γ a *circle* of curvature k on M. We regard a geodesic as a circle of null curvature. It is known that Equations (2.4) are equivalent to the equation

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\dot\gamma\rangle\dot\gamma = 0, \tag{2.5}$$

where $\langle\ ,\ \rangle$ is the Riemannian metric of M.

3. Ruled real hypersurfaces in $\mathbb{C}P^n(c)$.

We recall ruled real hypersurfaces in $\mathbb{C}P^n(c)$, which are typical examples of non-Hopf hypersurfaces. A real hypersurface M is called a *ruled real hypersurface* of $\mathbb{C}P^n(c)$ $(n \geqq 2)$ if the holomorphic distribution T^0 defined by $T^0(x) = \{X \in T_xM \mid X \perp \xi\}$ for $x \in M$ is integrable and each of its maximal integral manifolds is a totally geodesic complex hyperplane $\mathbb{C}P^{n-1}(c)$ of $\mathbb{C}P^n(c)$. A ruled real hypersurface is constructed in the following manner. Given an arbitrary regular real curve γ in $\mathbb{C}P^n(c)$ which is defined on an interval I we have at each fixed point $\gamma(t)$ ($t \in$ I) a totally geodesic complex hyperplane $\mathbb{C}P_t^{n-1}(c)$ that is orthogonal to the plane spanned by $\{\dot\gamma(t), J\dot\gamma(t)\}$. Then we see that $M = \bigcup_{t\in I} \mathbb{C}P_t^{n-1}(c)$ is a ruled real hypersurface in $\mathbb{C}P^n(c)$. The following is a well-known characterization of ruled real hypersurfaces in terms of the shape operator A.

LEMMA 3. *For a real hypersurface M in $\mathbb{C}P^n(c)$ $(n \geqq 2)$, the following conditions* (1), (2) and (3) *are mutually equivalent*:

1. *M is a ruled real hypersurface.*
2. *Let $\mu = \langle A\xi, \xi\rangle$ and $\nu = \|A\xi - \mu\xi\|$. Then the subset $M_1 = \{x \in M | \nu(x) \neq 0\}$ of M is open dense and there exists a unit vector field U on M_1 such that it is orthogonal to ξ and satisfies that $A\xi = \mu\xi + \nu U$, $AU = \nu\xi$ and $AX = 0$ for an arbitrary tangent vector X orthogonal to ξ and U.*
3. *The shape operator A of M satisfies $\langle Av, w\rangle = 0$ for arbitrary tangent vectors $v, w \in T_xM$ orthogonal to ξ_x at each point $x \in M$.*

We treat a ruled real hypersurface locally, because generally this hypersurface has singularities. When we study ruled real hypersurfaces, we usually omit points where ξ is principal and suppose that ν does not vanish everywhere, namely a ruled hypersurface M is usually supposed $M_1 = M$.

We clarify a fundamental property on some geodesics of ruled real hypersurfaces in $\mathbb{C}P^n(c)$. In the following, for a curve γ on a submanifold M^n isometrically immersed into an arbitrary Riemannian manifold $\widetilde{M}^{n+p}$ through f, we call γ an *extrinsic geodesic* if the curve $f \circ \gamma$ is a geodesic in $\widetilde{M}^{n+p}$.

LEMMA 4. *On a ruled real hypersurface M in $\mathbb{C}P^n(c)$ $(n \geqq 2)$, every geodesic γ whose initial vector $\dot\gamma(0)$ is orthogonal to the characteristic vector $\xi_{\gamma(0)}$ is an extrinsic geodesic.*

Proof. Let M_0 be the leaf through the point $\gamma(0)$ for the holomorphic distribution T^0M. We here take a geodesic γ_1 on M_0 with the same initial condition that $\gamma_1(0) = \gamma(0)$ and $\dot\gamma_1(0) = \dot\gamma(0)$. Since M_0 is locally congruent to a totally geodesic complex hyperplane $\mathbb{C}P^{n-1}(c)$ of $\mathbb{C}P^n(c)$, we see that the curve γ_1 is also a geodesic in the ambient space $\mathbb{C}P^n(c)$, which implies that the curve γ_1 is a geodesic on our ruled real hypersurface M. Hence the uniqueness theorem on geodesics tells us that these two curves γ and γ_1 are coincidental. Thus we get the desired conclusion. □

We should note that the tangent vector $\dot\gamma(s)$ of a geodesic γ in this lemma is orthogonal to $\xi_{\gamma(s)}$ at each point $\gamma(s)$.

The following is fundamental on ruled real hypersurfaces in $\mathbb{C}P^n(c)$.

PROPOSITION 1. *Every ruled real hypersurface in $\mathbb{C}P^n(c)$ $(n \geqq 2)$ is not complete.*

Proof. By direct computation we find that every integral curve γ of the vector field ϕU is a geodesic on a ruled real hypersurface M and the function ν satisfies the differential equation on the curve γ: $\phi U\nu = \nu^2 + \frac{c}{4}$ (for details, see **[4]**). Then, solving this equation, we have $\nu(s) = \frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}s + C)$ with some constant C. These imply that every geodesic $\gamma = \gamma(s)$ with initial vector $\dot\gamma(0) = (\phi U)_{\gamma(0)}$ on our ruled real hypersurface M is defined on the open interval $I = (-\frac{2}{\sqrt{c}}(\frac{\pi}{2} + C), \frac{2}{\sqrt{c}}(\frac{\pi}{2} - C))$. Thus we get the conclusion. □

REMARK 2. In $\mathbb{C}H^n(c)$, we also consider ruled real hypersurfaces. We emphasize that there exist many *complete* ruled real hypersurface in $\mathbb{C}H^n(c)$ (for details, see **[7]**).

4. Statements of results. The following is a classification theorem of real hypersurfaces in $\mathbb{C}P^n(c)$ with strongly ϕ-invariant shape operator.

THEOREM 1. *Let M^{2n-1} $(n \geqq 2)$ be a real hypersurface of $\mathbb{C}P^n(c)$. Then the following conditions* (1), (2) *and* (3) *are mutually equivalent.*

1. *M is locally congruent to a real hypersurface of type (A) with radius $\pi/(2\sqrt{c}\,)$.*
2. *The shape operator A of M is strongly ϕ-invariant.*
3. *M satisfies the following*:
   - (3i) *At each fixed point $p \in M$, there exist orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ_p of M such that all geodesics $\gamma_i = \gamma_i(s)$ on M with $\dot{\gamma}_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$ $(1 \leqq i \leqq 2n-2)$ are mapped to circles of the same positive curvature in $\mathbb{C}P^n(c)$;*
   - (3ii) *There exists at least one integral curve of the characteristic vector field ξ of M which is mapped to a geodesic in $\mathbb{C}P^n(c)$.*

Proof. We shall show that Condition (1) implies both Conditions (2) and (3). We first consider the case of type (A_2) with radius $\pi/(2\sqrt{c}\,)$. Let M be a real hypersurface of type (A_2) with radius $\pi/(2\sqrt{c}\,)$ around totally geodesic $\mathbb{C}P^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$. Then M has three distinct constant principal curvatures 0 (with multiplicity 1), $\sqrt{c}/2$ (with multiplicity $2n-2\ell-2$) and $-\sqrt{c}/2$ (with multiplicity 2ℓ). We here remark that $A\xi = 0$. Moreover, Lemma 1 tells us that $\phi V_{\sqrt{c}/2} = V_{\sqrt{c}/2}$ and $\phi V_{-\sqrt{c}/2} = V_{-\sqrt{c}/2}$. Hence we see that $-\phi A\phi\xi = 0 = A\xi$, $-\phi A\phi u = (\sqrt{c}/2)u = Au$ for each $u \in V_{\sqrt{c}/2}$ and $-\phi A\phi v = (-\sqrt{c}/2)v = Av$ for each $v \in V_{-\sqrt{c}/2}$, so that

$$-\phi A\phi X = AX \quad \text{for all vectors } X \in TM, \tag{4.1}$$

which is equivalent to the definition (1.1) of strongly ϕ-invariance of the shape operator A of M. Thus we can see that Condition (1) implies Condition (2) in the case of type (A_2) with radius $\pi/(2\sqrt{c}\,)$.

We next take orthonormal vectors $v_1, \ldots, v_{2n-2}$ perpendicular to the characteristic vector ξ_p at an arbitrary fixed point p of M in such a way that $v_1, \ldots, v_{2n-2\ell-2}$ and $v_{2n-2\ell-1}, \ldots, v_{2n-2}$ are orthonormal bases of $V_{\sqrt{c}/2}$ and $V_{-\sqrt{c}/2}$, respectively. Let $\gamma_i = \gamma_i(s)$ $(1 \leqq i \leqq 2n-2\ell-2)$ be a geodesic on M with initial condition that $\gamma_i(0) = p$ and $\dot{\gamma}_i(0) = v_i$. Then

$$\begin{aligned}
\nabla_{\dot{\gamma}_i(s)}\langle\dot{\gamma}_i(s), \xi_{\gamma_i(s)}\rangle &= \langle\dot{\gamma}_i(s), \nabla_{\dot{\gamma}_i(s)}\xi_{\gamma_i(s)}\rangle = \langle\dot{\gamma}_i(s), \phi A\dot{\gamma}_i(s)\rangle \quad \text{(from (2.2))}\\
&= \langle\dot{\gamma}_i(s), A\phi\dot{\gamma}_i(s)\rangle \quad \text{(from Lemma 2(2))}\\
&= \langle A\dot{\gamma}_i(s), \phi\dot{\gamma}_i(s)\rangle = -\langle\phi A\dot{\gamma}_i(s), \dot{\gamma}_i(s)\rangle = 0,
\end{aligned}$$

which, together with $\langle\dot{\gamma}_i(0), \xi_p\rangle = \langle v_i, \xi_p\rangle = 0$, implies that $\langle\dot{\gamma}_i(s), \xi_{\gamma_i(s)}\rangle = 0$ for each s. Hence, using Lemma 2(3), we get

$$\begin{aligned}
&\nabla_{\dot{\gamma}_i(s)}\|A\dot{\gamma}_i(s) - (\sqrt{c}/2)\dot{\gamma}_i(s)\|^2\\
&\quad = 2\langle(\nabla_{\dot{\gamma}_i(s)}A)\dot{\gamma}_i(s), A\dot{\gamma}_i(s) - (\sqrt{c}/2)\dot{\gamma}_i(s)\rangle\\
&\quad = 2\langle(\nabla_{\dot{\gamma}_i(s)}A)\dot{\gamma}_i(s), A\dot{\gamma}_i(s)\rangle - \sqrt{c}\,\langle(\nabla_{\dot{\gamma}_i(s)}A)\dot{\gamma}_i(s), \dot{\gamma}_i(s)\rangle = 0,
\end{aligned}$$

which, combined with $A\dot{\gamma}_i(0) - (\sqrt{c}/2)\dot{\gamma}_i(0) = Av_i - (\sqrt{c}/2)v_i = 0$, shows that $A\dot{\gamma}_i(s) = (\sqrt{c}/2)\dot{\gamma}_i(s)$ for every s. So, in view of (2.1) we know that the geodesic $\gamma_i = \gamma_i(s)$

on M satisfies the following differential equations in the ambient $\mathbb{C}P^n(c)$:

$$\widetilde{\nabla}_{\dot{\gamma}_i(s)}\dot{\gamma}_i(s) = \frac{\sqrt{c}}{2}\mathcal{N} \quad \text{and} \quad \widetilde{\nabla}_{\dot{\gamma}_i(s)}\mathcal{N} = -\frac{\sqrt{c}}{2}\dot{\gamma}_i(s)$$

for each s. That is, all geodesics $\gamma_i = \gamma_i(s)$ ($1 \leqq i \leqq 2n - 2\ell - 2$) on M are mapped to circles of the same positive curvature $\sqrt{c}/2$ in $\mathbb{C}P^n(c)$. Also, by the same discussion as above we find that all geodesics $\gamma_j = \gamma_j(s)$ ($2n - 2\ell - 1 \leqq j \leqq 2n - 2$) on M with initial vector $\dot{\gamma}_j(0) = v_j \in V_{-\sqrt{c}/2}$ are mapped to circles of the same positive curvature $\sqrt{c}/2$ in $\mathbb{C}P^n(c)$. Hence we obtain Condition (3i). We here recall that the characteristic vector field ξ on our real hypersurface M satisfies $A\xi = 0$. This, together with the first equality in (2.1) and (2.2), yields that every integral curve of ξ is mapped to a geodesic in $\mathbb{C}P^n(c)$. Then we know that Condition (1) implies Condition (3) in the case of type (A_2) with radius $\pi/(2\sqrt{c})$. The above discussion holds good even in the case of type (A_1) with radius $\pi/(2\sqrt{c})$. Therefore, we can see that Condition (1) implies both Conditions (2) and (3).

Conversely, we show that Condition (2) implies Condition (1). Setting $X = \xi$ in Equation (4.1), we see that $A\xi = 0$. We next take a principal curvature vector X orthogonal to ξ with principal curvature λ. Then it follows from Lemma 1(2) and (4.1) that $\lambda = \pm\sqrt{c}/2$. Again, by using Lemma 1(2) we see that each of $V_{\sqrt{c}/2}$ and $V_{-\sqrt{c}/2}$ is invariant by ϕ, so that $\phi A = A\phi$ holds on our real hypersurface M. This, combined with Lemma 2(2), gives us Condition (1).

Finally, we verify that Condition (3) implies Condition (1). We take orthonormal vectors $v_1, v_2, \ldots, v_{2n-2}$ at an arbitrary fixed point p of a real hypersurface M satisfying Condition (3i). Then, from (2.5) they satisfy

$$\widetilde{\nabla}_{\dot{\gamma}_i}\widetilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = -k^2\dot{\gamma}_i \tag{4.2}$$

for some positive constant k. On the other hand, from (2.1) we have

$$\widetilde{\nabla}_{\dot{\gamma}_i}\widetilde{\nabla}_{\dot{\gamma}_i}\dot{\gamma}_i = \langle(\nabla_{\dot{\gamma}_i}A)\dot{\gamma}_i, \dot{\gamma}_i\rangle\mathcal{N} - \langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i. \tag{4.3}$$

Comparing the tangential components of (4.2) and (4.3), we see that

$$\langle A\dot{\gamma}_i, \dot{\gamma}_i\rangle A\dot{\gamma}_i = k^2\dot{\gamma}_i,$$

so that at $s = 0$ we get

$$\langle Av_i, v_i\rangle Av_i = k^2 v_i \quad \text{for } 1 \leqq i \leqq 2n-2.$$

Since $k \neq 0$, we obtain

$$Av_i = kv_i \quad \text{or} \quad Av_i = -kv_i \quad \text{for } 1 \leqq i \leqq 2n-2. \tag{4.4}$$

So we find that ξ is a principal curvature vector, because $\langle A\xi, v_i\rangle = \langle \xi, Av_i\rangle = 0$ for $1 \leqq i \leqq 2n - 2$. This, together with Condition (3ii), implies that $A\xi = 0$. Then the real hypersurface M is a Hopf hypersurface which has at most three distinct principal curvatures $k(= k(p))$, $-k$ and $0(= \langle A\xi, \xi\rangle)$ at the point p. Thus, from Lemma 1(2) and $c > 0$ we know that $c/(4k) = k$, so that $k = \sqrt{c}/2$. Hence M is a Hopf hypersurface with at the most three distinct constant principal curvatures $\sqrt{c}/2, -\sqrt{c}/2$ and $\alpha = \langle A\xi, \xi\rangle = 0$ at its each point, so that $\phi A = A\phi$ holds on M. Therefore we can conclude that our real hypersurface M is a hypersurface of type (A) with radius $\pi/(2\sqrt{c})$. □

REMARK 3. (1) In Condition (3i) we do not need to suppose that we take the vectors $v_1, \ldots, v_{2n-2}$ as a local field of orthonormal frames on M. However, for all real hypersurfaces M in Theorem 1 we can take a local field of orthonormal frames $v_1, \ldots, v_{2n-2}$ on M satisfying Condition (3i).

(2) If we omit Condition (3ii), Theorem 1 is no longer true. The discussion in the proof of Theorem 1 tells us that a real hypersurface M in $\mathbb{C}P^n(c)$ satisfies Condition (3i) if and only if M is locally congruent to either a real hypersurface of type (A_1) with radius r $(0 < r < \pi/\sqrt{c}\,)$ or a real hypersurface of type (A_2) with radius $r = \pi/(2\sqrt{c}\,)$.

Inspired by Condition (3ii), we are interested in the number of *extrinsic geodesics* (i.e., geodesics of $\mathbb{C}P^n(c)$ lying on this hypersurface) on real hypersurfaces of type (A) with radius $\pi/(2\sqrt{c}\,)$. To do this, we review congruence theorems on geodesics on real hypersurfaces of type (A) in $\mathbb{C}P^n(c)$.

For a geodesic γ on a real hypersurface M of type (A) in $\mathbb{C}P^n(c)$, we define its *structure torsion* ρ_γ by $\rho_\gamma = \langle \dot\gamma, \xi_\gamma \rangle$. Clearly, it satisfies $-1 \leqq \rho_\gamma \leqq 1$. Moreover, for each geodesic γ on M, from the discussion in the proof of Theorem 1 we know that the structure torsion ρ_γ is constant along γ.

For geodesics on a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$, we can classify them by means of their structure torsions (see proposition 2.3 in [**2**]):

LEMMA 5. *On a geodesic sphere $G(r)$ of radius r $(0 < r < \pi/\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ $(n \geqq 2)$, two geodesics γ_1, γ_2 are congruent to each other with respect to the isometry group* $\mathrm{I}(G(r))$ *of $G(r)$, namely there exists an isometry φ of $G(r)$ with $\gamma_2(s) = (\varphi \circ \gamma_1)(s)$ for each s if and only if their structure torsions ρ_{γ_1} and ρ_{γ_2} satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$.*

To obtain a congruence theorem for geodesics on real hypersurfaces of type (A_2) in $\mathbb{C}P^n(c)$, we need another invariant. For a geodesic γ on a real hypersurface of type (A) in $\mathbb{C}P^n(c)$ we define its *normal curvature* κ_γ by $\kappa_\gamma = \langle A\dot\gamma, \dot\gamma \rangle$. By Lemma 2 we have

$$\nabla_{\dot\gamma} \kappa_\gamma(s) = \langle (\nabla_{\dot\gamma(s)} A)\dot\gamma(s), \dot\gamma(s) \rangle = 0,$$

which shows that κ_γ is constant along γ.

Geodesics on a real hypersurface of type (A_2) are classified by means of their structure torsions and normal curvatures (see theorem 2 in [**1**]):

LEMMA 6. *On a real hypersurface M of type* (A_2) *in $\mathbb{C}P^n(c)$ $(n \geqq 2)$, two geodesics γ_1, γ_2 are congruent to each other with respect to the isometry group* $\mathrm{I}(M)$ *of M if and only if their structure torsions and normal curvatures satisfy $|\rho_{\gamma_1}| = |\rho_{\gamma_2}|$ and $\kappa_{\gamma_1} = \kappa_{\gamma_2}$.*

The following proposition implies that by the number of extrinsic geodesics we can distinguish between the real hypersurface of type (A_1) with radius $\pi/(2\sqrt{c}\,)$ and the real hypersurfaces of type (A_2) with radius $\pi/(2\sqrt{c}\,)$.

PROPOSITION 2. (1) *The geodesic sphere $G(\pi/(2\sqrt{c}\,))$ of radius $\pi/(2\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ $(n \geqq 2)$ has just one congruence class of extrinsic geodesics with respect to the isometry group* $\mathrm{I}(G(\pi/(2\sqrt{c}\,)))$ *of $G(\pi/(2\sqrt{c}\,))$. This extrinsic geodesic is an integral curve of the characteristic vector field ξ on $G(\pi/(2\sqrt{c}\,))$.*
(2) *Every real hypersurface M of type* (A_2) *with radius $\pi/(2\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ has uncountably infinite congruence classes of extrinsic geodesics with respect to the isometry group* $\mathrm{I}(M)$ *of M. These extrinsic geodesics are expressed as a one-parameter family of geodesics $\gamma_a = \gamma_a(s)$ $(0 \leqq a \leqq 1/\sqrt{2}\,)$ on M with initial vector*

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$\dot{\gamma}(0) = \sqrt{1-2a^2}\,\xi_{\gamma(0)} + au + av$, *where* u, v *are unit vectors orthogonal to* $\xi_{\gamma(0)}$ *with* $Au = (\sqrt{c}/2)u$, $Av = (-\sqrt{c}/2)v$.

Proof. Note that a curve $\gamma = \gamma(s)$ on a real hypersurface M of type (A) in $\mathbb{C}P^n(c)$ is an extrinsic geodesic if and only if the curve γ is a geodesic of M and the following equation holds (see Lemma 2(3)):

$$\langle A\dot{\gamma}(0), \dot{\gamma}(0)\rangle = 0. \tag{4.5}$$

For the claim (1). For a geodesic $\gamma = \gamma(s)$ of $G(\pi/(2\sqrt{c}\,))$, we can set

$$\dot{\gamma}(0) = \rho_\gamma \xi_{\gamma(0)} + \sqrt{1-\rho_\gamma^2}\,u, \tag{4.6}$$

where ρ_γ is the structure torsion of γ and u is a unit vector orthogonal to $\xi_{\gamma(0)}$. Then it follows from (4.5), (4.6) and equalities $A\xi_{\gamma(0)} = 0$, $Au = (\sqrt{c}/2)u$ that $\rho_\gamma = \pm 1$, so that the extrinsic geodesic γ is an integral curve of ξ. Furthermore, any integral curves of ξ are congruent to one another (see Lemma 5). Thus we get Statement (1).

For the claim (2). For a geodesic $\gamma = \gamma(s)$ of our real hypersurface M, we can set

$$\dot{\gamma}(0) = \rho_\gamma \xi_{\gamma(0)} + au + bv, \tag{4.7}$$

where a, b are nonnegative constants with $\rho_\gamma^2 + a^2 + b^2 = 1$, $A\xi_{\gamma(0)} = 0$ and u, v are unit vectors orthogonal to $\xi_{\gamma(0)}$ with $Au = (\sqrt{c}/2)u$, $Av = (-\sqrt{c}/2)v$. Hence, from (4.5) and (4.7) we know that the geodesic γ of M is an extrinsic geodesic if and only if the structure ρ_γ of γ satisfies $\rho_\gamma^2 = 1 - 2a^2$ $(0 \leqq a \leqq 1/\sqrt{2}\,)$. Therefore our real hypersurface M has uncountably infinite congruence classes of extrinsic geodesics (see Lemma 6). □

REMARK 4. (1) By virtue of Proposition 2(2) we see that real hypersurfaces M of type (A_2) with radius $\pi/(2\sqrt{c}\,)$ in $\mathbb{C}P^n(c)$ have a one-parameter family of closed geodesics $\gamma_a = \gamma_a(s)$ $(0 \leqq a \leqq 1/\sqrt{2}\,)$ with the same length $2\pi/\sqrt{c}$, which are *not* congruent to one another with respect to I(M). These curves γ_a $(0 \leqq a \leqq 1/\sqrt{2}\,)$ are mapped to geodesics of $\mathbb{C}P^n(c)$. We note that these curves γ_a, considered as curves in the ambient space $\mathbb{C}P^n(c)$, are congruent to one another with respect to the isometry group $SU(n+1)$ of $\mathbb{C}P^n(c)$ because all geodesics of $\mathbb{C}P^n(c)$ are congruent to one another.

(2) In an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c(<0)$, there exist also real hypersurfaces, so-called, of type (A). However, such a real hypersurface in $\mathbb{C}H^n(c)$ has no extrinsic geodesics (cf. [**8**]). So, an analogous result to Theorem 1 does not hold in the ambient space $\mathbb{C}H^n(c)$.

Next, under some conditions, we classify real hypersurfaces in $\mathbb{C}P^n(c)$ with weakly ϕ-invariant shape operator.

THEOREM 2. *For a real hypersurface* M^{2n-1} *of* $\mathbb{C}P^n(c)$ $(n \geqq 2)$ *we have the following two statements* (1), (2).

(1) *The following conditions* (1_a), (1_b), (1_c) *are mutually equivalent.*

(1_a) *M is a Hopf hypersurface with weakly* ϕ*-invariant shape operator.*

(1_b) *M is locally congruent to a real hypersurface of type* (*A*).

(1_c) *Every geodesic* γ *of M has constant normal curvature* κ_γ *along* γ.

(2) *The following conditions* (2_a), (2_b), (2_c) *are mutually equivalent.*

(2_a) *The holomorphic distribution T^0M of M is integrable and the shape operator of M is weakly ϕ-invariant.*
(2_b) *M is a ruled real hypersurface.*
(2_c) *At each fixed point $p \in M$ there exist such orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to the characteristic vector ξ that all geodesics of M through p in the direction $v_i + v_j$ $(1 \leqq i \leqq j \leqq 2n-2)$ are mapped to geodesics in $\mathbb{C}P^n(c)$.*

Proof. (1) Suppose that Condition (1_a) holds. Then, using the property (1.2) and the assumption that M is a Hopf hypersurface, we see that $\phi A = A\phi$, so that by Lemma 2, M is a real hypersurface of type (A).

Conversely, we suppose that M is a real hypersurface of type (A). Then Equation (1.2) follows from the fact that $\phi A = A\phi$. Hence we can check the equivalency for Conditions (1_a) and (1_b).

Next, we shall show the equivalency for Conditions (1_b) and (1_c). It follows from our argument that Condition (1_b) implies Condition (1_c). We next suppose Condition (1_c). Then we see easily that

$$\langle (\nabla_X A)X, X\rangle = 0 \quad \text{for each vector } X \text{ on } M,$$

which is equivalent to saying that

$$\langle (\nabla_X A)Y, Z\rangle + \langle (\nabla_Y A)Z, X\rangle + \langle (\nabla_Z A)X, Y\rangle = 0 \tag{4.8}$$

for arbitrary vectors X, Y and Z on M. In consideration of the symmetry of the shape operator A, (4.8) and (2.3) we can see that Lemma 2(3) holds. Hence we get Condition (1_b). Thus we can check the equivalency for Conditions (1_b) and (1_c).

(2) It is obvious from Lemmas 3 and 4 that Condition (2_b) implies Conditions (2_a) and (2_c). Conversely, we suppose Condition (2_a). Then it follows from the integrability of the holomorphic distribution T^0M and (2.2) that

$$\langle (\phi A + A\phi)X, Y\rangle = 0 \quad \text{for arbitrary } X, Y \in T^0M \tag{4.9}$$

(see proposition 5 in [**5**]). Hence, in view of (1.2), (4.9) and the skew-symmetry of ϕ we see that

$$\begin{aligned}\langle AX, Y\rangle &= \langle A\phi X, \phi Y\rangle = -\langle \phi AX, \phi Y\rangle \\ &= \langle AX, \phi^2 Y\rangle = -\langle AX, Y\rangle = 0,\end{aligned}$$

so that by Lemma 3, M is a ruled real hypersurface. Hence we have Condition (2_b).

We suppose Condition (2_c). Then, from the first equaity in (2.1) we know that at each point $p \in M$ there exist orthonormal vectors $v_1, \ldots, v_{2n-2}$ orthogonal to ξ satisfying

$$\langle Av_i, v_j\rangle = 0 \quad \text{for } 1 \leqq i \leqq j \leqq 2n-2,$$

which yields Lemma 3(3). Thus we can see that M is a ruled real hypersurface, so that we obtain Condition (2_b). □

Remark 5. (1) An analogous result to Theorem 2 holds for real hypersurfaces in an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature $c(<0)$.

(2) Every geodesic of each real hypersurface of type (A) in $\mathbb{C}P^n(c)$ is mapped to a homogeneous curve in $\mathbb{C}P^n(c)$, namely it is represented by an orbit of a one-parameter subgroup of $\mathrm{SU}(n+1)$.

(3) The classification problem of real hypersurfaces with weakly ϕ-invariant shape operator in $\mathbb{C}P^n(c)$ is still open.

The following proposition was already seen in [**3**]. However, for readers we prove it again in order to guarantee the motivation of this paper.

PROPOSITION 3. *Let (M_n, J) be an n-dimensional Kähler manifold with Kähler structure J immersed into a $(2n+p)$-dimensional sphere $S^{2n+p}(c)$ of constant sectional curvature c through an isometric immersion f. Then f has parallel second fundamental form σ if and only if σ is J-invariant, namely $\sigma(JX, JY) = \sigma(X, Y)$ holds for all vectors X, Y on M_n.*

Proof. We suppose that σ is J-invariant. Our discussion here is due to [**3**]. We first recall the definition of the covariant derivative $\overline{\nabla}$ of the second fundamental form σ:

$$(\overline{\nabla}_X\sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z),$$

where D is the normal connection of f and ∇ is the Riemannian connection of the submanifold M_n. This, combined with the J-invariance of σ, implies

$$(\overline{\nabla}_Z\sigma)(JX, Y) = -(\overline{\nabla}_Z\sigma)(X, JY) \quad \text{for all vectors } X, Y \text{ and } Z \text{ on } M_n. \tag{4.10}$$

Using Equation (4.10) and the Codazzi equation $(\overline{\nabla}_X\sigma)(Y, Z) = (\overline{\nabla}_Y\sigma)(X, Z)$ for the sphere case repeatedly, we find the following:

$$\begin{aligned}(\overline{\nabla}_Z\sigma)(X, Y) &= (\overline{\nabla}_Y\sigma)(X, Z) = -(\overline{\nabla}_Y\sigma)(X, J^2Z)\\ &= (\overline{\nabla}_Y\sigma)(JX, JZ) = (\overline{\nabla}_{JZ}\sigma)(JX, Y)\\ &= -(\overline{\nabla}_{JZ}\sigma)(X, JY) = -(\overline{\nabla}_X\sigma)(JZ, JY)\\ &= (\overline{\nabla}_X\sigma)(Z, J^2Y) = -(\overline{\nabla}_X\sigma)(Z, Y)\\ &= -(\overline{\nabla}_Z\sigma)(X, Y) = 0.\end{aligned}$$

Next, we suppose that f has parallel second fundamental form. Then it is known that our Kähler manifold M_n is locally isometric to a compact Hermitian symmetric space and moreover this isometric immersion f of the compact Hermitian symmetric space into the ambient sphere $S^{2n+p}(c)$ is locally realized as a part of the embedding as the symmetric R-space.

We here recall the embedding as symmetric R-spaces. Let $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ be a semisimple graded Lie algebra of the first kind and ν the characteristic element which defines its gradation, i.e. $\nu \in \mathfrak{g}_0$ and the eigenspaces of $\mathrm{ad}(\nu)$ with eigenvalues ± 1 and 0 are respectively given by $\mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_0$. Take a Cartan involution τ of $\mathfrak{g}$ such that $\tau(\nu) = -\nu$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition by τ, i.e., $\mathfrak{k}$ and $\mathfrak{p}$ are respectively the (± 1)-eigenspaces of τ. Furthermore, let G be the adjoint group of $\mathfrak{g}$ and K the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Then, under a suitable G-invariant metric, the homogeneous space G/K is a Riemannian symmetric space of noncompact type, and the orbit $K(\nu) \subset S \subset \mathfrak{p}$ is called a symmetric R-space, where S denotes the hypersphere in $\mathfrak{p}$ centred at the origin with radius $|\nu|$. Put $\theta = \exp \mathrm{ad}(\pi\sqrt{-1}\nu)$. Then the subspaces $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ are invariant by θ, and it gives an involution of $\mathfrak{g}$ such that $\theta \circ \tau = \tau \circ \theta$. Let $\mathfrak{k} = \mathfrak{k}_+ \oplus \mathfrak{k}_-$ and $\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ be the decompositions by θ, where $\mathfrak{k}_{\pm 1}$ and $\mathfrak{p}_{\pm 1}$ denote the (± 1)-eigenspaces of θ in $\mathfrak{k}$ and $\mathfrak{p}$, respectively. Then $\nu \in \mathfrak{p}_+$ and the subspaces $\mathfrak{k}_-$ and $\mathfrak{p}_{\pm 1}$ are $\mathfrak{k}_+$-modules satisfying

$$[\mathfrak{k}_-, \mathfrak{k}_-],\ [\mathfrak{p}_+, \mathfrak{p}_+],\ [\mathfrak{p}_-, \mathfrak{p}_-] \subset \mathfrak{k}_+,\ [\mathfrak{k}_-, \mathfrak{p}_-] \subset \mathfrak{p}_+,\ [\mathfrak{k}_-, \mathfrak{p}_+] \subset \mathfrak{p}_- \text{ and } [\mathfrak{p}_-, \mathfrak{p}_+] \subset \mathfrak{k}_-.$$

Let K_+ denote the isotropy subgroup of K at $\nu \in K(\nu)$ and put $M' = K/K_+$. Then M' is a compact symmetric space associated with the involution θ and the tangent space T_oM' at the origin o in K/K_+ is identified with the subspace $\mathfrak{k}_-$. Moreover the tangent space $T_\nu K(\nu)$ and the normal space $T_\nu^\perp K(\nu)$ in $\mathfrak{p}$ are respectively identified with $\mathfrak{p}_-$ and $\mathfrak{p}_+$. Let f' be the canonical embedding of M' into $\mathfrak{p}$ defined by $f'(kK_+) = k(\nu) \in K(\nu) \subset \mathfrak{p}$ where $k \in K$, and denote by σ_o the second fundamental form of f' at o. Then it follows

$$\sigma_0(X, Y) = [X, [Y, \nu]] \quad \text{for all } X, Y \in \mathfrak{k}_-.$$

We here refer to **[6]** for the semisimple graded Lie algebra and to **[11]** for the construction of symmetric R-spaces.

Now we assume that M' is a Hermitian symmetric space. Note that the Lie algebra of K_+ is $\mathfrak{k}_+$. Then, there exists an element $H \in \mathfrak{k}_+$ such that the almost complex structure J on T_oM' is given by the restriction of $\mathrm{ad}(H)$ to $\mathfrak{k}_-$, and moreover the element H is contained in the centre of the Lie algebra $\mathfrak{k}_+ \oplus \mathfrak{p}_+$ (for these facts we refer to **[9]**). Noting that $[H, \nu] = 0$ and $[\mathfrak{k}_-, [\mathfrak{k}_-, \mathfrak{p}_+]] \subset \mathfrak{p}_+$, we now get the following equalities:

$$\begin{aligned}\sigma_0(JX, JY) &= [JX, [JY, \nu]] = [\mathrm{ad}(H)X, [\mathrm{ad}(H)Y, \nu]]\\ &= [\mathrm{ad}(H)X, \mathrm{ad}(H)([Y, \nu])] = \mathrm{ad}(H)([\mathrm{ad}(H)X, [Y, \nu]]) - [\mathrm{ad}^2(H)X, [Y, \nu]]\\ &= 0 - [J^2X, [Y, \nu]] = [X, [Y, \nu]] = \sigma_0(X, Y)\end{aligned}$$

for $X, Y \in \mathfrak{k}_-$. Since the embedding $f' : M' \to \mathfrak{p}$ is K-equivariant, the second fundamental form of f' is J-invariant. Moreover, since the inclusion $S \hookrightarrow \mathfrak{p}$ is totally umbilical, the second fundamental form of the embedding $M' \to S$ is also J-invariant. By the classification theorem of parallel immersions (**[3]**), our parallel immersion $f : M_n \to S^{2n+p}(c)$ is locally constructed precisely as the composition of an embedding as the symmetric R-space $f' : M' \to S$ and a totally umbilical embedding $S \hookrightarrow S^{2n+p}(c)$. Hence the second fundamental form of f is also J-invariant. □

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