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DIRECT PRODUCT DECOMPOSITIONS OF ELATION GROUPS

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1. **Introduction.** Let G be a collineation group of a projective plane π. Let E be the subgroup generated by all elations in G. In the case that π is finite and G fixes no point or line, F. Piper [6; 7] has proved that if G contains certain combinations of perspectivities, then E is isomorphic to $PSL(3, \mathfrak{F})$ for some finite field $\mathfrak{F}$. The isomorphism is geometrically significant in the sense that there exists a Desarguesian subplane π_1 and E acts as the little projective group of π_1 in the natural way.

In the case that π is finite and G fixes a line ℓ, let S be the subgroup of G generated by all elations in G which fix a fixed point $P i \ell$. C. Hering [5] has determined the structure of S under the hypothesis that G contains certain elations with axis ℓ.

We allow π to be finite or infinite, we consider the case where G fixes a line ℓ, and we study $E_{(\ell)}(G)$, the subgroup of all elations in G which have axis ℓ. It is well known that if $E_{(\ell)}(G)$ contains non-identity elations with distinct centers then $E_{(\ell)}(G)$ is elementary abelian and therefore is usually a direct product of many subgroups. But there may be no decomposition into two factors in which each factor is the set of all elations in $E_{(\ell)}(G)$ which have a fixed point as center. (See Examples 4.3 and 4.4.) In Theorems 3.2 and 3.3 we find sufficient conditions, in terms of the existence of perspectives in G and the finiteness of certain subgroups of $E_{(\ell)}(G)$ (or of G), for the existence of such a geometrically significant direct product decomposition into two factors. Examples 4.2, 4.3, and 4.5 demonstrate the necessity of the finiteness hypotheses of Theorems 3.2 and 3.3.

2. **Notation.** For any point X and line y we let (X) denote the set of all lines through X and we let (y) denote the set of all points on y. Thus, for example, for any collineation group G, $G_{(y)}$ is the subgroup of all collineations in G which have axis y, $G_{(X)(y)}$ is the subgroup of all collineations in G which have center X and axis y, and $G_{(X),A}$ is the subgroup of all collineations in G which have center X and which fix A.

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It is well known that the set of all elations in a group G which have a fixed axis ℓ forms a subgroup of G. This subgroup cannot be described concisely using the above conventions, so we will use the notation $E_{(\ell)}(G)$ for this subgroup. In order to have a uniform notation in some direct product equalities, we will sometimes use $E_{(\ell)(A)}(G)$ for the group $G_{(\ell)(A)}$ when $AI\ell$. When there is no chance of confusion, the "(G)" will sometimes be omitted from the above notations.

For any two points A, B on a line ℓ, the product $E_{(\ell)(A)}(G)\cdot E_{(\ell)(B)}(G)$ is a direct product $E_{(\ell)(A)}(G)\times E_{(\ell)(B)}(G)$. (This follows from the uniqueness of the center of a non-identity elation.) So to derive conclusions of the form $E_{(\ell)}(G)= E_{(\ell)(A)}(G)\times E_{(\ell)(B)}(G)$ in Theorems 3.2 and 3.3, it is sufficient merely to show $E_{(\ell)}(G)=E_{(\ell)(A)}(G)\cdot E_{(\ell)(B)}(G)$.

3. **Structure theorems.** The following theorem was proved by J. André [2, p. 31] for finite planes.

THEOREM 3.1. *Let G be a collineation group of a projective plane π and let a be a line of π. If $E_{(a)}(G)$ is finite and non-trivial or if $G_{(a)}$ is finite, then either the set of centers of non-identity homologies in $G_{(a)}$ is an $E_{(a)}(G)$-orbit or it is empty.*

Proof. First we show that $G_{(a)}$ is finite whenever $E_{(a)}$ is finite and non-trivial by showing that if $1\neq e\in E_{(a)}$ then e has only finitely many $G_{(a)}$-conjugates and $C_{G_{(a)}}(e)$ (the centralizer in $G_{(a)}$ of e) is finite. The elation e has only finitely many $G_{(a)}$-conjugates because all such conjugates belong to $E_{(a)}$ which is finite by hypothesis. If $1\neq g\in G_{(a)}$ and g has center $X\not{I}a$, then $g^{-1}eg(X)= g^{-1}e(X)\neq e(X)$ because the fixed points of g^{-1} consist only of X and the points of a. So $g\notin C_{G_{(a)}}(e)$. Thus $C_{G_{(a)}}(e)\subseteq E_{(a)}$ which is finite.

By the finiteness of $G_{(a)}$, the $G_{(a)}$-orbits of centers of non-identity homologies in $G_{(a)}$ are finite in number and size. Denote these orbits by $\mathcal{M}_1$, $\mathcal{M}_2,\ldots,\mathcal{M}_k$. Let $m_i=|\mathcal{M}_i|$, $g=|G_{(a)}|$, $e=|E_{(a)}|$, and $g_i=|G_{(a)(A_i)}|$ for some $A_i\in\mathcal{M}_i$.

We can now apply the argument of J. André [2]. By the semiregularity of the action of $E_{(a)}$ on the set of points off a, it is sufficient to show either that $k=1$ and $|\mathcal{M}_1|=|E_{(a)}|$ or that $k=0$.

Now $|G_{(a)}|=|G_{(a)}(A_i)|\,|G_{(a),A_i}|=|\mathcal{M}_i|\,|G_{(a)(A_i)}|$ (by the definition of $\mathcal{M}_i$ and by the fact that A_iIa). Thus

$$g=m_ig_i. \tag{3.11}$$

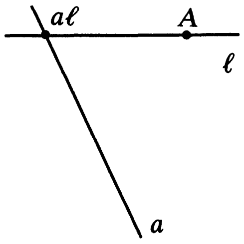
Every element of $G_{(a)}$ belongs to $G_{(a)(X)}$ for some X and this X is unique if the element of $G_{(a)}$ is not the identity. Thus

$$
\begin{aligned}
g &= e + \sum_{XIa} (|G_{(a)(X)}| - 1) \\
&= e + \sum_{i=1}^{k} |\mathcal{M}_i| (|G_{(a)(A_i)}| - 1) \\
&= e + \sum_{i=1}^{k} m_i (g_i - 1) \\
&= e + g \sum_{i=1}^{k} (1 - g_i^{-1}) \quad \text{(by 3.11).}
\end{aligned} \tag{3.12}
$$

Now each $g_i \geq 2$. Thus $1 - g_i^{-1} \geq \frac{1}{2}$. So, by 3.12, $g \geq e + gk(\frac{1}{2})$. But $e > 0$, thus $g > gk(\frac{1}{2})$. So we must have $k = 1$ or 0. If $k = 1$, then, by 3.12, $g = e + g - gg_1^{-1}$ or $g = g_1 e$. This and 3.11 show that $|\mathcal{M}_1| = m_1 = e = |E_{(a)}|$.

REMARK 3.13. The conclusion of this theorem is false if the hypothesis that $E_{(a)}$ is finite is replaced by $G_{(a)(A)}$ is finite for some (or all) $A I a$. See Example 4.1. The author has been unable to establish the necessity of the hypothesis that $E_{(a)}$ is non-trivial.

THEOREM 3.2. *Let G be a collineation group of a projective plane π, let ℓ and a be distinct lines of π, and let A be a point on ℓ but not on a. If G contains a non-identity homology with center A and axis a (i.e., if $G_{(A)(a)} \neq \{1\}$) and if either $E_{(\ell)(A)}(G)$ is finite and non-trivial or $G_{(A),a\ell}$ is finite, then $E_{(\ell)}(G) = E_{(\ell)(A)}(G) \times E_{(\ell)(a\ell)}(G)$.*



Proof. We first apply the dual of Theorem 3.1 to the group $G_{(A),a\ell}$. To verify that the hypotheses of this dual hold for $G_{(A),a\ell}$, we must note that

$$E_{(A)}(G_{(A),a\ell}) = E_{(A)(\ell)}(G)$$

and

$$(G_{(A),a\ell})_{(A)} = G_{(A),a\ell}$$

so that, by hypothesis, either the first of these subgroups is finite and non-trivial or the second subgroup is finite. Thus, by the dual of Theorem 3.1, the

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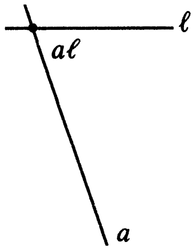
set $E_{(A)(\ell)}(a)$ coincides with the set of lines which are axes of non-identity homologies in $G_{(A),a\ell}$. But this last set contains $E_{(\ell)}(a)$, because if $1 \neq \alpha \in G_{(A)(a)}$ and $e \in E_{(\ell)}$ then $e(a)$ is the axis of the non-identity homology $e\alpha e^{-1}$ (which is in $G_{(A),a\ell}$ because e fixes A and $a\ell$). Thus $E_{(A)(\ell)}(a) \supseteqq E_{(\ell)}(a)$.

Let $e \in E_{(\ell)}$. By the conclusion of the above paragraph, there is an $e_A \in E_{(A)(\ell)}$ with $e_A(a) = e(a)$. Then $e_A^{-1}e \in E_{(\ell)(a\ell)}$. Thus $e = e_A e_{a\ell}$ for some $e_{a\ell} \in E_{(\ell)(a\ell)}$.

REMARK 3.21. The conclusion of this theorem is false if the finiteness of $E_{(A)(\ell)}$ is replaced by the finiteness of $G_{(A)(a)}$ or by the finiteness of $E_{(X)(\ell)}$ for any $XI\ell$ other than $X = A$. See Examples 4.2 and 4.3.

LEMMA 3.22. *Let G be a collineation group of a projective plane π and let a and ℓ be distinct lines of π. If G contains a non-identity elation with center $a\ell$ and axis a (i.e., if $E_{(a)(a\ell)}(G) \neq \{1\}$) and if $E_{(\ell)(a\ell)}(G)$ is finite, then $E_{(\ell)}(G)$ is finite and $|E_{(\ell)}(G)| \leq |E_{(\ell)(a\ell)}(G)|^2$.*



Proof. Let $\{\tau_i\}$ be a set of representatives of the cosets of $E_{(\ell)(a\ell)}$ in $E_{(\ell)}$. Let $1 \neq \lambda \in E_{(a)(a\ell)}$. Let $\rho_i = \tau_i^{-1}\lambda\tau_i\lambda^{-1}$.

As pointed out by C. Hering in [4, Lemma 3.1], $\rho_i = (\tau_i^{-1}\lambda\tau_i)\lambda^{-1}$ has center $a\ell$ (because it is the product of two collineations with center $a\ell$) and similarly $\rho_i = \tau_i^{-1}(\lambda\tau_i\lambda^{-1})$ has axis ℓ. Thus all $\rho_i \in E_{(\ell)(a\ell)}$.

Now we will show that the ρ_i are all distinct. We have $\rho_i = \rho_j \Leftrightarrow \tau_i^{-1}\lambda\tau_i = \tau_j^{-1}\lambda\tau_j \Leftrightarrow \tau_i\tau_j^{-1}$ commutes with λ. Now, if $i \neq j$, then $\tau_i\tau_j^{-1} \in E_{(\ell)} \setminus E_{(\ell)(a\ell)}$. But the only elements of $E_{(\ell)}$ which commute with a non-identity $\lambda \in E_{(a)(a\ell)}$ are those of $E_{(\ell),a} = E_{(\ell)(a\ell)}$. Thus $\rho_i \neq \rho_j$ if $i \neq j$.

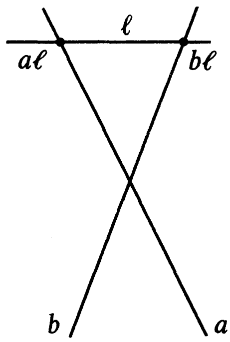
So finally we have:

$$|E_{(\ell)}| = [E_{(\ell)} : E_{(\ell)(a\ell)}]\,|E_{(\ell)(a\ell)}| = |\{\tau_i\}|\,|E_{(\ell)(a\ell)}| = |\{\rho_i\}|\,|E_{(\ell)(a\ell)}| \leq |E_{(\ell)(a\ell)}|^2.$$

REMARK 3.23. It is not possible, even with stronger finiteness conditions, to replace the inequality in the conclusion by an equality, nor to deduce that $E_{(\ell)} = E_{(\ell)(a\ell)} \times E_{(\ell)(X)}$ for some $XI\ell$ in the above lemma. See Example 4.4.

THEOREM 3.3. *Let G be a collineation group of a projective plane π and let a, b, and ℓ be distinct non-current lines of π. If G contains non-identity elations with axes a and b and centers $a\ell$ and $b\ell$, respectively (i.e., if $E_{(a)(a\ell)}(G) \neq \{1\}$*

and $E_{(b)(b\ell)}(G) \neq \{1\}$), *and if* $E_{(\ell)(a\ell)}(G)$ *or* $E_{(\ell)(b\ell)}(G)$ *is finite, then* $E_{(\ell)}(G) = E_{(\ell)(a\ell)}(G) \times E_{(\ell)(b\ell)}(G)$.



Proof. We may assume that $E_{(\ell)(a\ell)}$ is finite. Then, by Lemma 3.22, $E_{(\ell)}$ is finite and thus so is $E_{(\ell)(b\ell)} \subseteq E_{(\ell)}$. Then:

$$\begin{aligned} |E_{(\ell)(a\ell)}|^2\,|E_{(\ell)(b\ell)}|^2 &= |E_{(\ell)(a\ell)} \times E_{(\ell)(b\ell)}|^2 \\ &\leq |E_{(\ell)}|^2 \\ &\leq |E_{(\ell)(a\ell)}|^2\,|E_{(\ell)(b\ell)}|^2 \end{aligned}$$

by Lemma 3.22 applied twice (with a replaced by b the second time). Thus equality holds: $|E_{(\ell)}|^2 = |E_{(\ell)(a\ell)} \times E_{(\ell)(b\ell)}|^2$. But these groups are finite, so $E_{(\ell)} = E_{(\ell)(a\ell)} \times E_{(\ell)(b\ell)}$.

REMARK 3.31. The hypothesis that $E_{(\ell)(a\ell)}$ or $E_{(\ell)(b\ell)}$ is finite cannot be removed from this theorem. See Example 4.5.

4. **Counterexamples.** In each of our examples, the projective plane will be $PG(2, \mathfrak{F})$ for a field $\mathfrak{F}$ which will be specified. (For details, see e.g., A. Albert and R. Sandler [1, p. 32–42] or H. S. M. Coxeter [3, p. 111–122].) The notations $P(x, y, z)$ and $L\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ will denote the point represented by the row vector (x, y, z) and the line represented by the column vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, respectively. The collineation group G will be the group of collineations induced by a group of matrices of the form $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{bmatrix}$. Because of this special form of the matrices, G will be isomorphic to the group of matrices. So we will indulge in the abuse of language and notation and we will regard G as the same as a group of matrices.

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EXAMPLE 4.1. Let $\mathfrak{F}$ be any field of characteristic 0, let $\pi = PG(2, \mathfrak{F})$. Throughout this example let $d = \pm 1$ and let b range over the *even* integers. Let $G = \left\{ \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ b & 0 & 1 \end{bmatrix} \right\}$. This set G forms a group. Let $a = L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Then $G = G_{(a)}$ and $E_{(a)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \right\}$. The center of $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ b & 0 & 1 \end{bmatrix}$ is $P(b/2, 0, 1)$. So the set of centers of non-identity homologies is $\{P(n, 0, 1) \mid n \text{ an integer}\}$. Thus $P(1, 0, 1)$ and $P(2, 0, 1)$ belong to this set of centers. But $P(2, 0, 1) \notin \{P(b+1, 0, 1)\} = E_{(a)}(P(1, 0, 1)$ (because all b are even). This verifies Remark 3.13.

EXAMPLE 4.2. Let $\mathfrak{F}$ be any field of characteristic 0, let $\pi = PG(2, \mathfrak{F})$. In this example b and c range over the integers and $d = \pm 1$. Let

$$G = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ b & c & 1 \end{bmatrix} \middle| \, b \equiv c \pmod 2 \right\}.$$

This set G is a group. Let

$$\ell = L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad a = L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

$A = P(0, 1, 0)$. Then

$$G_{(A)(a)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

is the group of order 2 and so it is non-trivial and finite. However

$$E_{(\ell)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1 \end{bmatrix} \middle| \, b \equiv c \pmod 2 \right\},$$

while

$$E_{(\ell)(A)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix} \middle| \, c \equiv 0 \pmod 2 \right\},$$

and

$$E_{(\ell)(a\ell)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix} \middle| \, b \equiv 0 \pmod 2 \right\}.$$

Thus $E_{(\ell)} \neq E_{(\ell)(A)} \times E_{(\ell)(a\ell)}$. This verifies the first part of the Remark 3.21.

EXAMPLE 4.3. Let $\mathcal{K}$ be a field and let τ be transcendental over $\mathcal{K}$. Let $\mathfrak{F}=\mathcal{K}(\tau)$, $\pi=PG(2,\mathfrak{F})$. In this example all d_j, d, b range over $\mathcal{K}$, all i, j range over the integers and all summations are over the integers. Let

$$G=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \tau^i & 0 \\ b\tau+\sum_j d_j & \sum_j d_j\tau^j & 1 \end{bmatrix}\right\}$$

with almost all $d_j=0$. This set G of collineations forms a group. Let

$$\ell=L\begin{pmatrix}0\\0\\1\end{pmatrix},\qquad A=P(0,1,0),\qquad a=L\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Then

$$G_{(A)(a)}=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \tau^i & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}\neq\{1\}.$$

Also

$$E_{(A)(\ell)}=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sum_j d_j\tau^j & 1 \end{bmatrix}\right\}$$

with almost all $d_j=0$, and $\sum_j d_j=0$,

$$E_{(a\ell)(\ell)}=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b\tau & 0 & 1 \end{bmatrix}\right\},$$

and

$$E_{(\ell)}=\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b\tau+\sum_j d_j & \sum_j d_j\tau^j & 1 \end{bmatrix}\right\}$$

with almost all $d_j=0$. Clearly the conclusions of Theorem 3.2 are false.

An element

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b\tau+\sum_j d_j & \sum_j d_j\tau^j & 1 \end{bmatrix}$$

of $E_{(\ell)}$ has center $P(b\tau+\sum_j d_j, \sum_j d_j\tau^j, 0)$. So two of these elements with centers other than $A=P(0,1,0)$ have the same centers if and only if the $(3,1)$ and $(3,2)$ entries of one are the same $\mathcal{K}$-multiple of the $(3,1)$ and $(3,2)$ entries of the other. Thus for $XI\ell$, $X\neq A$, either $|E_{(X)(\ell)}|=|\mathcal{K}|$ or $|E_{(X)(\ell)}|=1$ (according as X can or cannot be expressed as $X=P(b\tau+\sum_j d_j, \sum_j d_j\tau^j, 0)$ for some $b, d_j\in\mathcal{K}$ with almost all $d_j=0$). Now let $\mathcal{K}$ be finite; then $E_{(X)(\ell)}$ is finite for $XI\ell$, $X\neq A$. This verifies the second part of Remark 3.21.

EXAMPLE 4.4. Let $\mathfrak{L}$, $\mathcal{K}$, $\mathfrak{F}$ be distinct finite fields with $\mathfrak{L}\subset\mathcal{K}\subset\mathfrak{F}$. Let $\mathcal{K}=\mathfrak{L}\oplus\mathfrak{B}$ (as a vector space over $\mathfrak{L}$) and let $\mathfrak{F}=\mathcal{K}\oplus\mathcal{D}$ (as a vector space over $\mathcal{K}$). Let $\pi=PG(2,\mathfrak{F})$. Let $d_0\in\mathcal{D}$, $d_0\neq 0$. In this example let d range over $\mathcal{D}$, let b range over $\mathfrak{B}$ and let n and m range over $\mathfrak{L}$. Let

$$G=\left\{\begin{bmatrix}1 & 0 & 0\\ nd_0 & 1 & 0\\ d+b & b+m & 1\end{bmatrix}\right\}.$$

This set G forms a group. Let

$$\ell=L\begin{pmatrix}0\\0\\1\end{pmatrix}\quad\text{and let}\quad a=L\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Then

$$E_{(a)(a\ell)}=\left\{\begin{bmatrix}1 & 0 & 0\\ nd_0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}\neq\{1\}.$$

Also

$$E_{(\ell)(a\ell)}=\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ d & 0 & 1\end{bmatrix}\right\}$$

and

$$E_{(\ell)}=\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ d+b & b+m & 1\end{bmatrix}\right\}.$$

Thus $|E_{(\ell)(a\ell)}|=|\mathcal{D}|$ and $|E_{(\ell)}|=|\mathcal{D}|\,|\mathfrak{B}|\,|\mathfrak{L}|=|\mathcal{D}|\,|\mathcal{K}|$. Hence, if $\deg\mathfrak{F}/\mathcal{K}>2$, it follows that,

$$|E_{(\ell)}(G)|<|E_{(\ell)(a\ell)}|^2.$$

Finally, if $XI\ell$ and $X\neq a\ell$, then $X=P(x,1,0)$ for some $x\in\mathfrak{F}$. Let $x=k_1+d_1$ for $k_1\in\mathcal{K}$, $d_1\in\mathcal{D}$. Then

$$E_{(X)(\ell)}=\begin{cases}\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ b(d_1+1) & b & 1\end{bmatrix}\right\} & \text{if } k_1=1.\\ \{1\} & \text{if } k_1\neq 1 \text{ and } k_1(1-k_1)^{-1}\notin\mathfrak{B}\\ \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m(1-k_1)^{-1}(d_1+k_1) & m(1-k_1)^{-1} & 1\end{bmatrix}\right\} & \text{if } k_1\neq 1 \text{ and } k_1(1-k_1)^{-1}\in\mathfrak{B}\end{cases}$$

Thus $|E_{(X)(\ell)}|=|\mathfrak{B}|$, 1 or $|\mathfrak{L}|$, and so $|E_{(X)(\ell)}|<|\mathcal{K}|$. It follows that

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$|E_{(X)(\ell)} \times E_{(a\ell)(\ell)}| < |\mathcal{K}|\,|\mathcal{D}| = |E_{(\ell)}(G)|$; hence $E_{(X)(\ell)} \times E_{(a\ell)(\ell)} \neq E_{(\ell)}$. This verifies Remark 3.23.

EXAMPLE 4.5. Let $\mathcal{K}$ be a field and let τ be a transcendental over $\mathcal{K}$. Let $\mathfrak{F} = \mathcal{K}(\tau)$ and let $\pi = PG(2, \mathfrak{F})$. In what follows let $a(\tau), b(\tau), \ldots, f(\tau)$ range over $\mathcal{K}[\tau]$, the ring of polynomials in τ with coefficients in $\mathcal{K}$. Let

$$G = \left\{ \begin{bmatrix} a(\tau) & b(\tau) & 0 \\ c(\tau) & d(\tau) & 0 \\ e(\tau) & f(\tau) & 1 \end{bmatrix} \right\}$$

with all matrices in G having determinant 1, with $a(0)+c(0) = b(0)+d(0)$, and with $e(0) = f(0)$. This set G of collineations forms a group. Let

$$a = L\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad b = L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \ell = L\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$E_{(a\ell)(a)} = \left\{ \begin{bmatrix} 1 & b(\tau) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; b(0) = 0 \right\} \neq \{1\},$$

and

$$E_{(b\ell)(b)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ c(\tau) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; c(0) = 0 \right\} \neq \{1\}.$$

Also

$$E_{(\ell)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e(\tau) & f(\tau) & 1 \end{bmatrix} \;\middle|\; e(0) = f(0) \right\},$$

$$E_{(a\ell)(\ell)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & f(\tau) & 1 \end{bmatrix} \;\middle|\; f(0) = 0 \right\},$$

and

$$E_{(b\ell)(\ell)} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e(\tau) & 0 & 1 \end{bmatrix} \;\middle|\; e(0) = 0 \right\}.$$

Clearly $E_{(\ell)} \neq E_{(a\ell)(\ell)} \times E_{(b\ell)(\ell)}$. This verifies Remark 3.31.

REFERENCES

1. A. A. Albert and R. Sandler, *An introduction to finite projective planes*, (Holt, Rinehart and Winston, New York, 1968).

https://doi.org/10.4153/CMB-1977-029-8 Published online by Cambridge University Press

2. J. André, *Über Perspektivitäten in endlichen projektiven Ebenen*, Arch. Math. **6** (1955), 29–32.

3. H. S. M. Coxeter, *Projective Geometry*, 2nd ed. (University of Toronto Press, Toronto, 1974).

4. C. Hering, *Eine Charakterisierung der endlichen zweidimensionalen projektiven Gruppen*, Math. Z. **82** (1963), 152–175.

5. C. Hering, *On shears of translation planes*, Abh. Math. Sem. Univ. Hamburg **37** (1972), 258–268.

6. F. Piper, *Collineation groups containing elations. I*, Math. Z. **89** (1965), 181–191.

7. F. Piper, *Collineation groups containing homologies*, J. Algebra **6** (1967), 256–269.

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