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Base matrices of various heights

Jörg Brendle

Abstract. A classical theorem of Balcar, Pelant, and Simon says that there is a base matrix of height $\mathfrak{h}$, where $\mathfrak{h}$ is the distributivity number of $\mathcal{P}(\omega)/\text{fin}$. We show that if the continuum $\mathfrak{c}$ is regular, then there is a base matrix of height $\mathfrak{c}$, and that there are base matrices of any regular uncountable height $\leq \mathfrak{c}$ in the Cohen and random models. This answers questions of Fischer, Koelbing, and Wohofsky.

1 Introduction

A collection $\mathfrak{A} = \{\mathcal{A}_\gamma : \gamma < \vartheta\}$ of mad (maximal almost disjoint) families of subsets of the natural numbers ω is called a *refining matrix of height* ϑ if:

- $\mathcal{A}_\delta$ *refines* $\mathcal{A}_\gamma$ for $\delta \geq \gamma$, i.e., for all $A \in \mathcal{A}_\delta$, there is $B \in \mathcal{A}_\gamma$ with $A \subseteq^* B$, and
- there is no *common refinement* of the $\mathcal{A}_\gamma$, i.e., no mad family $\mathcal{A}$ refining all the $\mathcal{A}_\gamma$.

$\mathfrak{A}$ is a *base matrix* if it is a refining matrix and $\bigcup_{\gamma<\vartheta} \mathcal{A}_\gamma$ is *dense* in $\mathcal{P}(\omega)/\text{fin}$, i.e., for all $B \in [\omega]^\omega$, there are $\gamma < \vartheta$ and $A \in \mathcal{A}_\gamma$ with $A \subseteq^* B$. The *distributivity number* $\mathfrak{h}$ of $\mathcal{P}(\omega)/\text{fin}$ is the least cardinal κ such that $\mathcal{P}(\omega)/\text{fin}$ as a forcing notion is not κ-distributive; equivalently, it is the least κ such that there is a collection $\mathfrak{A}$ of size κ of mad families without common refinement. Clearly, a refining matrix must have height at least $\mathfrak{h}$, and it is easy to see that there is one of height $\mathfrak{h}$ and none of regular height $> \mathfrak{c}$. Furthermore, if there is a refining matrix of height ϑ, then there is one of height $cf(\vartheta)$ so that it suffices to consider regular heights. A famous theorem of Balcar, Pelant, and Simon [BPS] (see also [Bl, Theorem 6.20]) says that there is even a base matrix of height $\mathfrak{h}$. It is natural to ask whether there can consistently be refining (base) matrices of other heights, and in interesting recent work, Fischer, Koelbing, and Wohofsky [FKW1] proved that it is consistent that $\mathfrak{h} = \omega_1$ and there is a refining matrix of height $\vartheta \leq \mathfrak{c}$, where $\vartheta > \omega_1$ is regular, all of whose maximal branches are cofinal (see Section 2 for a formal definition). We show the following theorems.

Theorem A *If $\mathfrak{c}$ is regular, then there is a base matrix of height $\mathfrak{c}$.*

Theorem B *In the Cohen and random models, there are base matrices of any regular uncountable height $\leq \mathfrak{c}$.*

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This answers Questions 7.5 and 7.7 of [FKW1]. Note that our results are incomparable with the one of the latter work. Their construction does not give a base matrix (in fact, by another result of Fischer, Koelbing, and Wohofsky [FKW2], a base matrix of height $> \mathfrak{h}$ always has some non-cofinal maximal branches, though one may still ask whether one can get such a base matrix in which some maximal branches are cofinal), whereas ours necessarily gives non-cofinal maximal branches. In fact, in the Cohen and random models, $\mathfrak{h} = \omega_1$ is the only cardinal ϑ for which there is a refining (base) matrix of height ϑ all of whose maximal branches are cofinal, and higher refining matrices have no cofinal branches at all (this follows from Fact 1).

2 Preliminaries

The *Cohen model* (resp. *random model*) is the model obtained by adding at least ω_2 many Cohen (resp. random) reals to a model of the continuum hypothesis CH [BJ].

For $A, B \subseteq \omega$, we say A is *almost contained* in B, and write $A \subseteq^* B$, if $A \setminus B$ is finite. $A \subsetneq^* B$ if $A \subseteq^* B$ and $B \setminus A$ is infinite. For an ordinal ϑ_0, $\{A_\gamma : \gamma < \vartheta_0\}$ is a $\subseteq^*$-*decreasing chain* of length ϑ_0 if $A_\delta \subseteq^* A_\gamma$ for all $\gamma < \delta < \vartheta_0$. $\subsetneq^*$-*decreasing chains* are defined analogously. For a refining matrix $\mathfrak{A} = \{\mathcal{A}_\gamma : \gamma < \vartheta\}$ and an ordinal $\vartheta_0 \leq \vartheta$, $\{A_\gamma : \gamma < \vartheta_0\}$ is a *branch* in $\mathfrak{A}$ if it is a $\subseteq^*$-decreasing chain and $A_\gamma \in \mathcal{A}_\gamma$ for $\gamma < \vartheta_0$. A branch is *maximal* if it cannot be properly extended to a longer branch. A branch is *cofinal* if $\vartheta_0 = \vartheta$. Every cofinal branch is maximal, but there may be maximal branches that are not cofinal.

Fact 1 (Folklore) There are no $\subsetneq^*$-decreasing chains of length ω_2 in $\mathcal{P}(\omega)$ in the Cohen and random models.

This is proved by an isomorphism-of-names argument using the homogeneity of the Cohen or random algebra.

For $A, B \in [\omega]^\omega$, A *splits* B if both $A \cap B$ and $B \setminus A$ are infinite. $\mathcal{X} \subseteq [\omega]^\omega$ is a *splitting family* if every $B \in [\omega]^\omega$ is split by a member of $\mathcal{X}$. The *splitting number* $\mathfrak{s}$ is the least size of a splitting family. It is well known that $\mathfrak{h} \leq \mathfrak{s}$ ([Bl] or [Ha]).

Fact 2 (Folklore [see [Bl]; see also [Ha, Proposition 22.13] for Cohen forcing]) After adding at least ω_1 Cohen or random reals to a model of ZFC, $\mathfrak{s} = \omega_1$. (In fact, the first ω_1 generics are a witness for $\mathfrak{s}$.)

We will prove the following.

Main Theorem 3 Assume $\vartheta \leq \mathfrak{c}$ is a regular cardinal and

(A) either there is no $\subsetneq^*$-decreasing chain of length ϑ in $\mathcal{P}(\omega)$,
(B) or $\mathfrak{s} \leq \vartheta$.

Then there is a base matrix of height ϑ.

Clearly, Theorem A follows from part (B) of the main theorem. (We note, however, that splitting families and $\mathfrak{s} \leq \mathfrak{c}$ are not needed in this case [see the comment at the beginning of the proof of Main Claim 5].) Theorem B follows from either (A) or (B)

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in view of Facts 1 and 2. Note that part (B) implies that in many other models of set theory there are base matrices of height ϑ for any regular ϑ between $\mathfrak{h}$ and $\mathfrak{c}$, e.g., in the Hechler model (this satisfies $\mathfrak{s} = \omega_1$ by [BD]; see also [Bl]), or in *any* extension by at least ω_1 Cohen or random reals (Fact 2). The former is, and the latter may be (depending on the ground model), a model for the failure of (A). We do not know whether (A) $+\neg$ (B) is consistent but conjecture that it is. This clearly implies $\mathfrak{s} \geq \mathfrak{b}^{++}$, where $\mathfrak{b}$ is the unbounding number (which is known to be consistent; see [BF]).

3 Proof of main theorem

By recursion on $\alpha < \mathfrak{c}$, we shall construct sets $\Omega_\gamma \subseteq \mathfrak{c}$ and families $\mathcal{A}_\gamma = \{A_{\gamma,\alpha} : \alpha \in \Omega_\gamma\}$, $\gamma < \vartheta$, such that:

(I) All $\mathcal{A}_\gamma$ are mad.
(II) If $\gamma < \delta < \vartheta$ and $\beta \in \Omega_\delta$, then there is $\alpha \leq \beta$ in Ω_γ such that $A_{\delta,\beta} \subseteq^* A_{\gamma,\alpha}$.
(III) For all $B \in [\omega]^\omega$, there are $\gamma < \vartheta$ and $\alpha \in \Omega_\gamma$ such that $A_{\gamma,\alpha} \subseteq^* B$.

This is clearly sufficient. In case (B), let $\{S_\zeta : \zeta < \nu\}$ be a splitting family with $\nu \leq \vartheta$. Let $\{(X_\alpha, \xi_\alpha) : \alpha < \mathfrak{c}\}$ list all pairs $(X, \xi) \in [\omega]^\omega \times \vartheta$. At stage α of the construction, we will have sets $\{\Omega_\gamma \cap \alpha : \gamma < \vartheta\}$, ordinals $\{\eta_\beta : \beta < \alpha\}$ below ϑ, and families $\{\{A_{\gamma,\beta} : \beta \in \Omega_\gamma \cap \alpha\} : \gamma < \vartheta\}$ such that:

(i_α) $\mathcal{A}_\gamma^\alpha := \{A_{\gamma,\beta} : \beta \in \Omega_\gamma \cap \alpha\}$ is almost disjoint for $\gamma < \vartheta$.
(ii_α) For all $\beta < \alpha$, the set $\{\gamma : \beta \in \Omega_\gamma\}$ is the interval of ordinals $[\eta_\beta, \max(\eta_\beta, \xi_\beta)]$ and
  - for $\gamma \in [\eta_\beta, \max(\eta_\beta, \xi_\beta)]$, $A_{\gamma,\beta} = A_{\eta_\beta,\beta}$, and
  - for $\gamma < \eta_\beta$, there is $\beta' < \beta$ in Ω_γ such that $A_{\eta_\beta,\beta} \subsetneq^* A_{\gamma,\beta'}$.

(iii_α) For all $\beta < \alpha$, $A_{\eta_\beta,\beta} \subsetneq^* X_\beta$ and, in case (B), $A_{\eta_\beta,\beta} \subseteq^* S_\zeta$ or $A_{\eta_\beta,\beta} \subseteq^* \omega \setminus S_\zeta$, where ζ is minimal such that S_ζ splits $A_{\gamma,\beta'}$ whenever $\gamma < \eta_\beta$ and $\beta' \in \Omega_\gamma \cap \beta$ are such that $A_{\eta_\beta,\beta} \subsetneq^* A_{\gamma,\beta'}$.

Let us first see that this suffices for completing the proof: indeed, (II) and (III) follow from (ii_α) and (iii_α), respectively. To see (I), fix $\gamma < \vartheta$ and $Y \in [\omega]^\omega$. Then there is $\alpha < \mathfrak{c}$ such that $(Y, \gamma) = (X_\alpha, \xi_\alpha)$. So $A_{\max(\eta_\alpha,\xi_\alpha),\alpha} = A_{\eta_\alpha,\alpha} \subseteq^* Y$ by ($\mathrm{ii}_{\alpha+1}$) and ($\mathrm{iii}_{\alpha+1}$) and $A_{\max(\eta_\alpha,\xi_\alpha),\alpha} \subseteq^* A_{\gamma,\beta}$ for some $\beta \leq \alpha$ by ($\mathrm{ii}_{\alpha+1}$). Thus $Y \cap A_{\gamma,\beta}$ is infinite, as required.

Next, we notice that, for $\alpha = 0$ and for limit α, there is nothing to show. Hence it suffices to describe the successor step, that is, the construction at stage $\alpha + 1$, and to prove that ($\mathrm{i}_{\alpha+1}$) through ($\mathrm{iii}_{\alpha+1}$) still hold. Assume $Y \subseteq^* X_\alpha \cap A_{\gamma,\beta}$ for some $\gamma < \vartheta$ and $\beta \in \Omega_\gamma \cap \alpha$, and let δ be such that $\gamma < \delta < \vartheta$. We say that Y *splits at* δ if:

- for all γ' with $\gamma \leq \gamma' < \delta$, there is $\beta \in \Omega_{\gamma'} \cap \alpha$ such that $Y \subseteq^* A_{\gamma',\beta}$, and
- there is no $\beta \in \Omega_\delta \cap \alpha$ such that $Y \subseteq^* A_{\delta,\beta}$.

We say Y *splits below* $\gamma_0 > \gamma$ if there is δ with $\gamma < \delta < \gamma_0$ such that Y splits at δ. For infinite $Y \subseteq X_\alpha$, call $\mathcal{A}_\gamma^\alpha \restriction Y$ *mad* if $\{Y \cap A_{\gamma,\beta} : \beta \in \Omega_\gamma \cap \alpha \text{ and } |Y \cap A_{\gamma,\beta}| = \aleph_0\}$ is a mad family below Y. The following is crucial for our construction.

Crucial Lemma 4 Let $\gamma_0 \leq \vartheta$ be an ordinal, and let $Y_0 \subseteq X_\alpha$ be infinite. Assume

(mad) $\mathcal{A}_\gamma^\alpha \restriction Y_0$ is mad for all $\gamma < \gamma_0$.

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Then there are $\gamma < \gamma_0$, $\beta \in \Omega_\gamma \cap \alpha$, and an infinite $Y \subseteq^* Y_0 \cap A_{\gamma,\beta}$ that does not split below γ_0.

Proof We make a proof by contradiction. Assume

(split) if $Z \subseteq^* Y_0 \cap A_{\gamma,\beta}$, for some $\gamma < \gamma_0$ and $\beta \in \Omega_\gamma \cap \alpha$, then Z splits below γ_0.

By recursion on $n \in \omega$, we construct infinite sets $(Y_s^0 : s \in 2^{<\omega})$ and $(Y_s : s \in 2^{<\omega})$, as well as ordinals $(\delta_s^0 : s \in 2^{<\omega})$ and $(\delta_n : n \in \omega)$ such that:

(a) $Y_s \subseteq Y_s^0$ and $Y^0_{s\hat{}i} \subseteq Y_s$ for $i \in \{0,1\}$.
(b) $\delta_n = \max\{\delta_s^0 : |s| = n\} < \gamma_0$ and $\delta^0_{s\hat{}i} > \delta_{|s|}$ for $i \in \{0,1\}$.
(c) Y_s splits at δ_s^0 and there are distinct $\beta, \beta' \in \Omega_{\delta_s^0} \cap \alpha$ such that $Y^0_{s\hat{}0} = Y_s \cap A_{\delta_s^0,\beta}$ and $Y^0_{s\hat{}1} = Y_s \cap A_{\delta_s^0,\beta'}$ (in particular, $Y^0_{s\hat{}0} \cap Y^0_{s\hat{}1}$ is finite).
(d) $Y_{s\hat{}i} = Y^0_{s\hat{}i} \cap A_{\delta_{|s|},\beta}$ for some $\beta \in \Omega_{\delta_{|s|}} \cap \alpha$, for $i \in \{0,1\}$.

We verify that we can carry out the construction. In the basic step $n = 0$ and $s = \langle\rangle$, by **(mad)**, let $Y_{\langle\rangle} = Y^0_{\langle\rangle} := Y_0 \cap A_{0,\beta}$ for some $\beta \in \Omega_0 \cap \alpha$ such that this intersection is infinite. By clause **(split)**, we know that there is $\delta_0 = \delta^0_{\langle\rangle}$ with $0 < \delta_0 < \gamma_0$ such that $Y_{\langle\rangle}$ splits at δ_0.

Suppose Y_s^0, Y_s, and δ_s^0 have been constructed for $|s| = n$ and $\delta_n = \max\{\delta_s^0 : |s| = n\} < \gamma_0$ are such that (a) through (d) hold. We thus know that Y_s splits at δ_s^0 and, by the definition of splitting and clause **(mad)**, we can find distinct $\beta, \beta' \in \Omega_{\delta_s^0} \cap \alpha$ such that $Y^0_{s\hat{}0} := Y_s \cap A_{\delta_s^0,\beta}$ and $Y^0_{s\hat{}1} := Y_s \cap A_{\delta_s^0,\beta'}$ are infinite. Using again **(mad)**, we see that for $i \in \{0,1\}$ there is $\beta \in \Omega_{\delta_n} \cap \alpha$ such that $Y_{s\hat{}i} := Y^0_{s\hat{}i} \cap A_{\delta_n,\beta}$ is infinite. Again by **(split)**, there is $\delta^0_{s\hat{}i}$, $i \in \{0,1\}$, with $\delta_n < \delta^0_{s\hat{}i} < \gamma_0$ such that $Y_{s\hat{}i}$ splits at $\delta^0_{s\hat{}i}$. Finally, let $\delta_{n+1} := \max\{\delta^0_{s\hat{}i} : |s| = n \text{ and } i \in \{0,1\}\} < \gamma_0$. This completes the construction.

Let $\delta_\omega = \bigcup_n \delta_n$. Clearly $\delta_\omega \leq \gamma_0$ is a limit ordinal of countable cofinality. Next, for $f \in 2^\omega$, let Y_f be a pseudointersection of the $Y_{f\restriction n}$, $n \in \omega$. If possible, choose $\beta_f \in \Omega_{\delta_\omega} \cap \alpha$ such that $Y_f \cap A_{\delta_\omega,\beta_f}$ is infinite. By (a) and (c) in this construction and by (ii$_\alpha$), we see that if $f \neq f'$ then $\beta_f \neq \beta_{f'}$. However, $\Omega_{\delta_\omega} \cap \alpha$ has size strictly less than $\mathfrak{c}$, and therefore there is $f \in 2^\omega$ for which there is no such β_f. Since $Y_f \subseteq^* Y_0$ by construction, this implies that $\mathcal{A}^\alpha_{\delta_\omega} \restriction Y_0$ is not mad and, by **(mad)**, $\gamma_0 = \delta_\omega$. This means, however, that any Y_f contradicts **(split)**. This completes the proof of the crucial lemma. ∎

We next show:

Main Claim 5 There is $\gamma < \vartheta$ such that $\mathcal{A}^\alpha_\gamma \restriction X_\alpha$ is not mad.

Proof Note that, in case $\vartheta = \mathfrak{c}$, there is nothing to show because by (ii$_\alpha$) we see that a tail of the sequence $(\Omega_\gamma \cap \alpha : \gamma < \vartheta)$ is empty, and therefore so is $\mathcal{A}^\alpha_\gamma$ (in fact, the proof of Theorem A is quite a bit simpler than the general argument: there is no need to list the ξ_α, we may simply let $\xi_\alpha = \alpha$, η_α will always be $\leq \alpha$, and the splitting family is unnecessary).

Hence assume $\vartheta < \mathfrak{c}$. By way of contradiction, suppose all $\mathcal{A}^\alpha_\gamma \restriction X_\alpha$ are mad. By the crucial lemma with $\gamma_0 = \vartheta$ and $Y_0 = X_\alpha$, we know that there are $\gamma < \vartheta$, $\beta \in \Omega_\gamma \cap \alpha$ and an infinite $Y \subseteq^* X_\alpha \cap A_{\gamma,\beta}$ that does not split below ϑ. This means for all δ with $\gamma \leq \delta < \vartheta$ there is $\beta \in \Omega_\delta \cap \alpha$ such that $Y \subseteq^* A_{\delta,\beta}$. By (i$_\alpha$) and (ii$_\alpha$), we see that there

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must be a strictly increasing sequence $(\beta_\varepsilon : \varepsilon < \vartheta)$ of ordinals below α such that for $\varepsilon' > \varepsilon$,

- $\eta_{\beta_{\varepsilon'}} > \max(\eta_{\beta_\varepsilon}, \xi_{\beta_\varepsilon})$ and $Y \subsetneq^* A_{\eta_{\beta_{\varepsilon'}},\beta_{\varepsilon'}} \subsetneq^* A_{\eta_{\beta_\varepsilon},\beta_\varepsilon}$.

In case (A), this contradicts the initial assumption that there are no $\subsetneq^*$-decreasing chains of length ϑ in $\mathcal{P}(\omega)$. So assume we are in case (B). Define a sequence $(\zeta_\varepsilon : \varepsilon < \vartheta)$ of ordinals below ν such that

- ζ_ε is minimal such that S_{ζ_ε} splits all $A_{\eta_{\beta_{\varepsilon'}},\beta_{\varepsilon'}}$ for $\varepsilon' < \varepsilon$.

Using (iii$_\alpha$), we see that S_{ζ_ε} does not split $A_{\eta_{\beta_\varepsilon},\beta_\varepsilon}$. Therefore, the sequence must be strictly increasing, which is impossible (and thus contradictory) in case $\nu < \vartheta$. If $\nu = \vartheta$ note that there cannot be any ζ such that S_ζ splits Y, contradicting the initial assumption that the S_ζ form a splitting family. This final contradiction establishes the main claim. ■

We now let $\eta_\alpha := \min\{\gamma : \mathcal{A}^\alpha_\gamma \restriction X_\alpha \text{ is not mad}\} < \vartheta$. Choose $Y_0 \subseteq X_\alpha$ infinite and almost disjoint from all members of $\mathcal{A}^\alpha_{\eta_\alpha}$. Note that $\mathcal{A}^\alpha_\gamma \restriction Y_0$ is mad for all $\gamma < \eta_\alpha$. Thus, by the crucial lemma with $\gamma_0 = \eta_\alpha$, we know there are $\gamma < \eta_\alpha$, $\beta \in \Omega_\gamma \cap \alpha$, and an infinite $Y \subseteq^* Y_0 \cap A_{\gamma,\beta}$ that does not split below η_α. Then,

($\star$) for all δ with $\gamma \le \delta < \eta_\alpha$, there is $\beta = \beta_\delta \in \Omega_\delta \cap \alpha$ such that $Y \subseteq^* A_{\delta,\beta}$.

Choose infinite $A_{\eta_\alpha,\alpha} \subsetneq^* Y$. In case (B), choose $\zeta < \nu$ minimal such that S_ζ splits all A_{δ,β_δ} with $\gamma \le \delta < \eta_\alpha$. If $Y \cap S_\zeta$ is infinite, additionally require $A_{\eta_\alpha,\alpha} \subsetneq^* Y \cap S_\zeta$. (If not, we will automatically have $A_{\eta_\alpha,\alpha} \subsetneq^* \omega \backslash S_\zeta$.)

Next, for all γ with $\eta_\alpha \le \gamma \le \max(\eta_\alpha, \xi_\alpha)$, we let $A_{\gamma,\alpha} = A_{\eta_\alpha,\alpha}$. Also put

$$\Omega_\gamma \cap (\alpha+1) = \begin{cases} \Omega_\gamma \cap \alpha, & \text{if } \gamma < \eta_\alpha \text{ or } \gamma > \max(\eta_\alpha, \xi_\alpha), \\ (\Omega_\gamma \cap \alpha) \cup \{\alpha\}, & \text{if } \eta_\alpha \le \gamma \le \max(\eta_\alpha, \xi_\alpha). \end{cases}$$

Then clauses (i$_{\alpha+1}$) and (iii$_{\alpha+1}$) are immediate, and (ii$_{\alpha+1}$) follows from ($\star$). This completes the proof of the main theorem.

4 Further remarks and questions

Obviously, the main remaining problem is whether the spectrum of heights of base matrices can be non-convex on regular cardinals.

Question 6 Is it consistent that for some regular ϑ with $\mathfrak{h} < \vartheta < \mathfrak{c}$ there is no base (refining) matrix of height ϑ?

The simplest instance would be $\mathfrak{h} = \omega_1$ and $\mathfrak{c} = \omega_3$ with no base (refining) matrix of height ω_2. By (B) in Main Theorem 3, this would imply $\mathfrak{s} = \omega_3$.

As the referee remarked, another constellation for a nontrivial spectrum, which would be convex, might be a model where $\mathfrak{s} = \mathfrak{c}$ is singular and there is a regular cardinal $\kappa \ge \mathfrak{h}$ with $\kappa < \mathfrak{c}$ such that the spectrum consists exactly of the regular cardinals in the interval $[\mathfrak{h}, \kappa]$. It is unknown, however, whether $\mathfrak{s} = \mathfrak{c}$ singular is consistent at all. The consistency of singular $\mathfrak{s}$ was shown by Dow and Shelah [DS], but in their model, $\mathfrak{c}$ is at least $\mathfrak{s}^+$.

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The proof of Main Theorem 3 may look a little like cheating because we do not refine our mad families everywhere when going to the next level. Thus, let us say $\mathfrak{A} = \{\mathcal{A}_\gamma : \gamma < \vartheta\}$ is a *strict* base (refining) matrix if it is a base (refining) matrix and for any $\gamma < \delta < \vartheta$ and any $A \in \mathcal{A}_\delta$ there is $B \in \mathcal{A}_\gamma$ with $A \subsetneq^* B$. We then obtain the following.

Proposition 7 *Assume $\vartheta \leq \mathfrak{c}$ is a regular cardinal such that there are $\subsetneq^*$-decreasing chains of length α in $\mathcal{P}(\omega)$ for any $\alpha < \vartheta$ and*

(A) *either there is no $\subsetneq^*$-decreasing chain of length ϑ in $\mathcal{P}(\omega)$,*
(B) *or $\mathfrak{s} \leq \vartheta$.*

Then there is a strict base matrix of height ϑ.

Proof sketch Modify the proof of Main Theorem 3 by attaching a $\subsetneq^*$-decreasing chain of length $\zeta_\beta + 1$ to the set $\{\gamma : \beta \in \Omega_\gamma\} = [\eta_\beta, \max(\eta_\beta, \xi_\beta)]$, where $\eta_\beta + \zeta_\beta = \max(\eta_\beta, \xi_\beta)$. This is clearly possible by assumption. ■

To analyze this a bit further, let $\mathfrak{ds}$ denote the least ordinal α such that there is no $\subsetneq^*$-decreasing chain of length α in $\mathcal{P}(\omega)$. It is easy to see that $\mathfrak{ds}$ is a regular cardinal with $\mathfrak{b}^+ \leq \mathfrak{ds} \leq \mathfrak{c}^+$. Put $\vartheta_0 = \min\{\mathfrak{ds}, \mathfrak{c}\}$, and assume ϑ_0 is regular. Then:

(1) there are strict base matrices of heights $\mathfrak{h}$ and ϑ_0, and
(2) all strict refining matrices have height between $\mathfrak{h}$ and ϑ_0.

To see (1), use the previous proposition for height ϑ_0, and note that the original construction of [BPS] gives a strict base matrix of height $\mathfrak{h}$. (2) is obvious. We leave it to the reader to verify that Proposition 7 implies the corresponding versions of Theorems A and B.

Corollary 8 *If $\mathfrak{c} \leq \omega_2$, then there is a strict base matrix of height $\mathfrak{c}$.*

Corollary 9 *Let ϑ be a regular uncountable cardinal. In the Cohen and random models, the following are equivalent:*

(i) $\vartheta \in \{\omega_1, \omega_2\}$.
(ii) *There is a strict base matrix of height ϑ.*
(iii) *There is a strict refining matrix of height ϑ.*

To see, e.g., Corollary 9, note that by Fact 1, $\mathfrak{ds} = \omega_2$ in either model, and use (1) and (2) above.

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Graduate School of System Informatics, Kobe University, Rokko-dai 1-1, Nada-ku, Kobe 657-8501, Japan
e-mail: brendle@kobe-u.ac.jp

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