J. Appl. Probab. **61,** 716–738 (2024)
*doi:*10.1017/jpr.2023.70

ON A TIME-CHANGED VARIANT OF THE GENERALIZED COUNTING PROCESS

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Abstract

In this paper, we time-change the generalized counting process (GCP) by an independent inverse mixed stable subordinator to obtain a fractional version of the GCP. We call it the mixed fractional counting process (MFCP). The system of fractional differential equations that governs its state probabilities is obtained using the Z transform method. Its one-dimensional distribution, mean, variance, covariance, probability generating function, and factorial moments are obtained. It is shown that the MFCP exhibits the long-range dependence property whereas its increment process has the short-range dependence property. As an application we consider a risk process in which the claims are modelled using the MFCP. For this risk process, we obtain an asymptotic behaviour of its finite-time ruin probability when the claim sizes are subexponentially distributed and the initial capital is arbitrarily large. Later, we discuss some distributional properties of a compound version of the GCP.

Keywords: Mixed fractional Poisson process; inverse mixed stable subordinator; long-range dependence property; risk process

2020 Mathematics Subject Classification: Primary 60G22
Secondary 60G55

1. Introduction

Di Crescenzo *et al.* [10] introduced and studied a Lévy process, namely the generalized counting process (GCP), which performs k kinds of jumps of amplitude $1, 2, \dots, k$ with positive rates $\lambda_1, \lambda_2, \dots, \lambda_k$, respectively. We denote it by $\{M(t)\}_{t\geq 0}$. Its transition probabilities are given by

$$\mathbb{P}\{M(t+h)=n \mid M(t)=l\} = \begin{cases} 1-\Lambda h+\mathrm{o}(h), & n=l, \\ \lambda_j h+\mathrm{o}(h), & n=l+j,\ j=1,2,\dots,k, \\ \mathrm{o}(h), & n>l+k, \end{cases}$$

where $\Lambda=\lambda_1+\lambda_2+\cdots+\lambda_k$ for a fixed positive integer k and $\mathrm{o}(h)\to 0$ as $h\to 0$. Moreover, Di Crescenzo *et al.* [10] studied the fractional version of the GCP, namely the generalized

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fractional counting process (GFCP), which we denote by $\{M^{\alpha}(t)\}_{t\geq 0}$, $0<\alpha\leq 1$. It is a time-changed GCP in which the time-change is done using an independent inverse stable subordinator $\{Y_{\alpha}(t)\}_{t\geq 0}$, i.e. $\{M(Y_{\alpha}(t))\}_{t\geq 0}$. Its state probabilities $p^{\alpha}(n, t)=\mathbb{P}\{M^{\alpha}(t)=n\}$ satisfy the following system of fractional differential equations:

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}p^{\alpha}(n, t)=-\Lambda p^{\alpha}(n, t)+\sum_{j=1}^{\min\{n,k\}}\lambda_j p^{\alpha}(n-j, t), \quad n\geq 0, \tag{1.1}$$

with the initial conditions

$$p^{\alpha}(n, 0)=\begin{cases}1, & n=0,\\ 0, & n\geq 1.\end{cases}$$

Here $\mathrm{d}^{\alpha}/\mathrm{d}t^{\alpha}$ is the Caputo fractional derivative defined as (see [20])

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}f(t):=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^t (t-s)^{-\alpha}f'(s)\,\mathrm{d}s, & 0<\alpha<1,\\ f'(t), & \alpha=1.\end{cases}$$

Its Laplace transform is given by (see [20, eq. (5.3.3)])

$$\mathcal{L}\left(\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}}f(t); s\right)=s^{\alpha}\mathcal{L}(f(t); s)-s^{\alpha-1}f(0), \quad s>0, \tag{1.2}$$

where

$$\mathcal{L}(f(t);s)=\int_0^{\infty}\mathrm{e}^{-st}f(t)\,\mathrm{d}t$$

is the Laplace transform of $f(t)$.

The involvement of the fractional derivative induces a global memory in the system. For $\alpha=1$, the GFCP reduces to the GCP $\{M(t)\}_{t\geq 0}$. The system of governing differential equations for its state probabilities can be obtained from (1.1). For $k=1$, the GFCP and GCP reduce to the time-fractional Poisson process (TFPP) (see [5]) and the Poisson process, respectively. The GFCP exhibits overdispersion and it has the long-range dependence (LRD) property. Di Crescenzo *et al.* [10] showed that the ratio of a positive integer power of the GFCP over its mean tends to 1 in probability, that is, it exhibits cut-off behaviour at mean times. In other words, it converges abruptly to equilibrium.

The explicit expression for the state probabilities of the GFCP was obtained by Di Crescenzo *et al.* [10]. Let $\mathbb{N}_0$ denote the set of non-negative integers. The following expression for the state probabilities $p(n, t)=\mathbb{P}\{M(t)=n\}$ of the GCP will be used later:

$$p(n, t)=\sum_{\Omega(k,n)}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}t^{x_1+x_2+\cdots+x_k}\mathrm{e}^{-\Lambda t}, \quad n\geq 0, \tag{1.3}$$

where

$$\Omega(k, n)=\{(x_1, x_2, \ldots, x_k)\colon x_1+2x_2+\cdots+kx_k=n,\ x_j\in\mathbb{N}_0\}.$$

For more properties of the GFCP and an application of the GCP in risk theory, we refer the reader to Kataria and Khandakar [19]. Also, for other counting processes that perform

https://doi.org/10.1017/jpr.2023.70 Published online by Cambridge University Press

jumps of amplitude larger than 1, we refer to Orsingher and Polito [24] and Orsingher and Toaldo [25].

A subordinator is a one-dimensional Lévy process with non-decreasing sample paths. A stable subordinator is a driftless subordinator. The mixed stable subordinator $\{D_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ is a subordinator that is characterized by the following Laplace transform (see [1]):

$$\mathbb{E}\big(\mathrm{e}^{-sD_{\alpha_1,\alpha_2}(t)}\big) = \mathrm{e}^{-t\phi(s)}, \quad s > 0, \tag{1.4}$$

where $0 < \alpha_2 < \alpha_1 < 1$ and $\phi(s) = C_1 s^{\alpha_1} + C_2 s^{\alpha_2}$ is the Laplace exponent with $C_1 + C_2 = 1$, $C_1 \geq 0$, $C_2 \geq 0$. The first passage time of a mixed stable subordinator is called the inverse mixed stable subordinator (IMSS) $\{Y_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$. It is defined as

$$Y_{\alpha_1,\alpha_2}(t) = \inf\{s \geq 0 \colon D_{\alpha_1,\alpha_2}(s) > t\}, \quad t \geq 0.$$

For $C_i = 1$, the IMSS reduces to the inverse stable subordinator Y_{α_i}, $i = 1, 2$.

In this paper, we consider the GCP time-changed by an independent IMSS, that is,

$$\mathcal{M}_{\alpha_1,\alpha_2}(t) := M(Y_{\alpha_1,\alpha_2}(t)), \quad t \geq 0.$$

We call it the mixed fractional counting process (MFCP). For $k = 1$, it reduces to the mixed fractional Poisson process (MFPP) (see [1], [3]). We obtain its state probabilities in terms of convolution of two suitable functions. The Z transform method is used to obtain the system of fractional differential equations that governs its state probabilities. Some of its distributional properties are obtained, such as mean, variance, covariance, probability generating function (PGF), and factorial moments. It is observed that the MFCP has the overdispersion property. Also, it is shown that the MFCP has the LRD property whereas its increment has the short-range dependence (SRD) property. Moreover, we consider a risk process in which the claims are modelled using the MFCP. We obtain an asymptotic behaviour of its finite-time ruin probability when the claim sizes are subexponentially distributed and the initial capital is arbitrarily large.

Later, we study the following process:

$$X(t) := \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0,$$

where $\{Y_i\}_{i\geq 1}$ is a sequence of independent and identically distributed (i.i.d.) random variables which are independent of the GCP $\{M(t)\}_{t\geq 0}$. We refer to the process $\{X(t)\}_{t\geq 0}$ as the compound generalized counting process (CGCP). Its finite-dimensional distribution function, mean, and variance are obtained. A particular case of the CGCP is discussed by taking discrete distribution for Y_1 with non-negative support.

2. Preliminaries

Here, we give some known results and definitions related to the Mittag–Leffler function, the LRD property, the Z transform, and the IMSS.

2.1. Mittag–Leffler function

The three-parameter Mittag–Leffler function is defined as (see [20, p. 45])

$$E^{\gamma}_{\alpha,\beta}(x) := \sum_{j=0}^{\infty} \frac{\gamma(\gamma+1)\cdots(\gamma+j-1)x^j}{j!\Gamma(j\alpha+\beta)}, \quad x \in \mathbb{R},$$

where $\alpha > 0$, $\beta > 0$, and $\gamma > 0$.

It reduces to the two-parameter Mittag–Leffler function for $\gamma = 1$. It further reduces to the Mittag–Leffler function for $\gamma = \beta = 1$.

For $n \geq 0$, we have (see [20, eq. (1.9.5)])

$$\big(E^{\gamma}_{\alpha,\beta}(x)\big)^{(n)} = \gamma(\gamma+1)\cdots(\gamma+n-1)E^{\gamma+n}_{\alpha,n\alpha+\beta}(x), \tag{2.1}$$

where $\big(E^{\gamma}_{\alpha,\beta}(x)\big)^{(n)}$ denotes the nth derivative of three-parameter Mittag–Leffler function, that is,

$$\big(E^{\gamma}_{\alpha,\beta}(x)\big)^{(n)} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}E^{\gamma}_{\alpha,\beta}(x).$$

The following asymptotic result follows from equation (2.44) of [3]:

$$E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}) \sim \frac{\lambda^{-\gamma}t^{-\alpha\gamma}}{\Gamma(\beta-\alpha\gamma)}, \quad t \to \infty, \tag{2.2}$$

where $\lambda > 0$ and $\beta \neq \alpha\gamma$.

The following result holds (see [14, eq. (17.6)]):

$$\mathcal{L}^{-1}\left(\frac{s^{\rho-1}}{s^{\alpha}+as^{\beta}+b};t\right) = t^{\alpha-\rho}\sum_{m=0}^{\infty}(-a)^m t^{(\alpha-\beta)m}E^{m+1}_{\alpha,\alpha+(\alpha-\beta)m-\rho+1}(-bt^{\alpha}), \tag{2.3}$$

along with the conditions $\alpha > \beta > 0$, $\alpha - \rho + 1 > 0$, and $|as^{\beta}/(s^{\alpha}+b)| < 1$.

2.2. *Z* transform

The unilateral Z transform of a function $f(k)$, $k \in \mathbb{N}_0$, is defined by (see [9, eq. (12.3.3)])

$$F(z) = Zf(k) = \sum_{k=0}^{\infty} f(k)z^{-k}, \quad z \in \mathbb{R}. \tag{2.4}$$

The following translation property holds (see [9, eq. (12.4.1)]):

$$Zf(k-m) = z^{-m}\left(F(z) + \sum_{r=-m}^{-1} f(r)z^{-r}\right), \quad m \geq 0. \tag{2.5}$$

Note that the coefficient of z^{-k} in (2.4) is the inverse Z transform, i.e. $f(k) = Z^{-1}(F(z))$. Let $f(k)$ be a probability mass function (PMF) whose support is $\mathbb{N}_0$. Then its PGF is given by

$$G(u) = \sum_{k=0}^{\infty} u^k f(k), \quad |u| \leq 1.$$

It is related to the unilateral Z transform as follows:

$$G(z^{-1}) = F(z). \tag{2.6}$$

2.3. Inverse mixed stable subordinator

Let $f_{\alpha_1,\alpha_2}(t, x)$ be the density function of the IMSS. Its Laplace transform is given by (see [1])

$$\mathcal{L}(f_{\alpha_1,\alpha_2}(t, x);s) = \frac{\phi(s)}{s}\mathrm{e}^{-x\phi(s)}, \tag{2.7}$$

where $\phi(s)$ is given in (1.4).

The mean $U_{\alpha_1,\alpha_2}(t) = \mathbb{E}(Y_{\alpha_1,\alpha_2}(t))$ of the IMSS is given by (see [22])

$$U_{\alpha_1,\alpha_2}(t) = \frac{t^{\alpha_1}}{C_1}E_{\alpha_1-\alpha_2,\alpha_1+1}(-C_2t^{\alpha_1-\alpha_2}/C_1), \tag{2.8}$$

where $E_{\alpha_1-\alpha_2,\alpha_1+1}(\cdot)$ is the two-parameter Mittag–Leffler function.

The following asymptotic result holds (see [28, eq. (48)]):

$$U_{\alpha_1,\alpha_2}(t) \sim \begin{cases} \dfrac{t^{\alpha_1}}{C_1\Gamma(\alpha_1+1)}, & t \to 0, \\ \dfrac{t^{\alpha_2}}{C_2\Gamma(\alpha_2+1)}, & t \to \infty. \end{cases} \tag{2.9}$$

For fixed $s > 0$ and large t, the variance and covariance of the IMSS have the following limiting behaviour (see [17, eqs (22) and (29)]):

$$\mathrm{Var}(Y_{\alpha_1,\alpha_2}(t)) \sim \frac{t^{2\alpha_2}}{C_2^2}\left(\frac{2}{\Gamma(2\alpha_2+1)} - \frac{1}{(\Gamma(\alpha_2+1))^2}\right)$$

and

$$\mathrm{Cov}(Y_{\alpha_1,\alpha_2}(s), Y_{\alpha_1,\alpha_2}(t)) \sim C_1^{-2}s^{2\alpha_1}E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2s^{\alpha_1-\alpha_2}/C_1) - t^{\alpha_2-1}K(s), \tag{2.10}$$

where

$$K(s) = \frac{s^{\alpha_1+1}}{C_1C_2\Gamma(\alpha_2)}\sum_{m=0}^{\infty}\frac{(m(\alpha_1-\alpha_2)+\alpha_1)(-C_2s^{\alpha_1-\alpha_2}/C_1)^m}{\Gamma(m(\alpha_1-\alpha_2)+\alpha_1+2)}. \tag{2.11}$$

2.4. The LRD and SRD properties

We will use the following definition (see [13], [23]).

Definition 2.1. Let $s > 0$ be fixed and let $\{X(t)\}_{t\geq 0}$ be a stochastic process such that its correlation function has the following asymptotic behaviour:

$$\mathrm{Corr}\,(X(s), X(t)) \sim c(s)t^{-\gamma}, \quad \text{as } t \to \infty,$$

where the value of c(s) depends on s. The process $\{X(t)\}_{t\geq 0}$ is said to exhibit the LRD property if $\gamma \in (0, 1)$ and it has the SRD property if $\gamma \in (1, 2)$.

3. Mixed fractional counting process

In this section we introduce a time-changed version of the GCP, namely the mixed fractional counting process (MFCP). It is obtained by time-changing the GCP by an independent IMSS. We denote it by $\{\mathcal{M}_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ and define it as follows:

$$\mathcal{M}_{\alpha_1,\alpha_2}(t) := M(Y_{\alpha_1,\alpha_2}(t)), \tag{3.1}$$

where the GCP $\{M(t)\}_{t\geq 0}$ and the IMSS $\{Y_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ are independent.

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For $k=1$, the MFCP reduces to the MFPP (see [1], [3]) as in this case the GCP reduces to the Poisson process. On taking $\lambda_j=\lambda$ for all $j=1, 2, \ldots, k$, the GCP reduces to the Poisson process of order k (see [19, Section 4.1]). Thus, for such λ_j, the MFCP reduces to a fractional version of the Poisson process of order k. Also, for $\lambda_j=\lambda(1-\rho)\rho^{j-1}/(1-\rho^k)$, $0\leq\rho<1$, $j=1, 2, \ldots, k$, the GCP reduces to the Pólya–Aeppli process of order k (see [19, Section 4.2]). So, for such λ_j, the MFCP reduces to a fractional version of the Pólya–Aeppli process of order k.

3.1. Some properties of the MFCP

Here we discuss some distributional properties of the MFCP. Let $0<s\leq t<\infty$ and

$$q_1=\sum_{j=1}^{k} j\lambda_j, \quad q_2=\sum_{j=1}^{k} j^2\lambda_j.$$

The mean and variance of the GCP are given by (see [10])

$$\mathbb{E}(M(t))=tq_1 \quad \text{and} \quad \operatorname{Var}(M(t))=tq_2. \tag{3.2}$$

The mean, variance, and covariance of the MFCP can be obtained using (3.2) and Theorem 2.1 of [22] in the following form:

$$\mathbb{E}(\mathcal{M}_{\alpha_1,\alpha_2}(t))=q_1U_{\alpha_1,\alpha_2}(t), \tag{3.3}$$

$$\operatorname{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(t))=q_2U_{\alpha_1,\alpha_2}(t)+q_1^2\operatorname{Var}(Y_{\alpha_1,\alpha_2}(t)), \tag{3.4}$$

$$\operatorname{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s), \mathcal{M}_{\alpha_1,\alpha_2}(t))=q_2U_{\alpha_1,\alpha_2}(s)+q_1^2\operatorname{Cov}(Y_{\alpha_1,\alpha_2}(s), Y_{\alpha_1,\alpha_2}(t)), \tag{3.5}$$

where $U_{\alpha_1,\alpha_2}(\cdot)$ is given in (2.8).

Remark 3.1. A stochastic process $\{X(t)\}_{t\geq 0}$ is said to exhibit overdispersion if

$$\operatorname{Var}(X(t))-\mathbb{E}(X(t))>0, \quad t>0.$$

Observe that

$$\operatorname{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(t))-\mathbb{E}(\mathcal{M}_{\alpha_1,\alpha_2}(t))=(q_2-q_1)U_{\alpha_1,\alpha_2}(t)+q_1^2\operatorname{Var}(Y_{\alpha_1,\alpha_2}(t))>0, \quad t>0.$$

Thus the MFCP exhibits overdispersion.

Theorem 3.1. *The MFCP has the LRD property.*

Proof. Let $s>0$ be fixed. From (3.4) and (3.5), we get

$$\operatorname{Corr}(\mathcal{M}_{\alpha_1,\alpha_2}(s), \mathcal{M}_{\alpha_1,\alpha_2}(t))=\frac{q_2U_{\alpha_1,\alpha_2}(s)+q_1^2\operatorname{Cov}(Y_{\alpha_1,\alpha_2}(s), Y_{\alpha_1,\alpha_2}(t))}{\sqrt{\operatorname{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(s))}\sqrt{q_2U_{\alpha_1,\alpha_2}(t)+q_1^2\operatorname{Var}(Y_{\alpha_1,\alpha_2}(t))}},$$

which on using (2.9)–(2.10) reduces to the following as $t\to\infty$:

$$\begin{aligned}&\operatorname{Corr}(\mathcal{M}_{\alpha_1,\alpha_2}(s), \mathcal{M}_{\alpha_1,\alpha_2}(t))\\ &\sim\frac{q_2U_{\alpha_1,\alpha_2}(s)+q_1^2\big(C_1^{-2}s^{2\alpha_1}E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2s^{\alpha_1-\alpha_2}/C_1)-t^{\alpha_2-1}K(s)\big)}{\sqrt{\operatorname{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(s))}\sqrt{\frac{q_2t^{\alpha_2}}{C_2\Gamma(\alpha_2+1)}+q_1^2\frac{t^{2\alpha_2}}{C_2^2}\big(\frac{2}{\Gamma(2\alpha_2+1)}-\frac{1}{(\Gamma(\alpha_2+1))^2}\big)}}\\ &\sim c(s)t^{-\alpha_2},\end{aligned}$$

where

$$c(s)=\frac{q_2U_{\alpha_1,\alpha_2}(s)+q_1^2C_1^{-2}s^{2\alpha_1}E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2s^{\alpha_1-\alpha_2}/C_1)}{\sqrt{\mathrm{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(s))}\sqrt{\frac{q_1^2}{C_2^2}\left(\frac{2}{\Gamma(2\alpha_2+1)}-\frac{1}{(\Gamma(\alpha_2+1))^2}\right)}}.$$

As $0<\alpha_2<1$, the MFCP exhibits the LRD property. □

Remark 3.2. It is known that the processes that exhibit the LRD property have applications in areas like finance (such as [11], [21]), econometrics (see [26]), hydrology (see [12, pp. 461–472]), and internet data traffic modelling (see [16]). As the MFCP possesses the LRD property, it has potential applications in these areas. Its application to risk theory is discussed later in this paper.

Let $\overset{d}{=}$ denote the equality in distribution. Di Crescenzo *et al.* [10] showed that the GCP is equal in distribution to a compound Poisson process, that is,

$$M(t)\overset{d}{=}\sum_{i=1}^{N(t)}X_i,\qquad t\geq 0. \tag{3.6}$$

Here $\{N(t)\}_{t\geq 0}$ is the Poisson process with intensity parameter Λ which is independent of the sequence of i.i.d. random variables $\{X_i\}_{i\geq 1}$ such that

$$\mathbb{P}\{X_1=j\}=\frac{\lambda_j}{\Lambda},\quad j=1,2,\ldots,k.$$

From (3.1) and (3.6), we get

$$\mathcal{M}_{\alpha_1,\alpha_2}(t)\overset{d}{=}\sum_{i=1}^{N_{\alpha_1,\alpha_2}(t)}X_i, \tag{3.7}$$

where $\{N_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ is the MFPP. Thus the MFCP is equal in distribution to a compound MFPP studied in [18].

In the next result, we obtain the PGF $G^{\alpha_1,\alpha_2}(u,t)=\mathbb{E}(u^{\mathcal{M}_{\alpha_1,\alpha_2}(t)})$ of the MFCP.

Proposition 3.1. *The PGF of the MFCP is given by*

$$G^{\alpha_1,\alpha_2}(u,t)=\sum_{m=0}^{\infty}(-C_2t^{\alpha_1-\alpha_2}/C_1)^mE^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}\left(\frac{t^{\alpha_1}}{C_1}\sum_{j=1}^{k}(1-u^j)\lambda_j\right)$$
$$-\sum_{m=0}^{\infty}(-C_2t^{\alpha_1-\alpha_2}/C_1)^{m+1}E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)(m+1)+1}\left(\frac{t^{\alpha_1}}{C_1}\sum_{j=1}^{k}(1-u^j)\lambda_j\right). \tag{3.8}$$

Proof. We have

$$G_{X_1}(u)=E(u^{X_1})=\frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_ju^j. \tag{3.9}$$

Using (3.7) and (3.9), the proof follows from the PGF of the MFPP (see [3, eq. (2.28)]). □

Let $p^{\alpha_1,\alpha_2}(n, t)=\mathbb{P}\{\mathcal{M}_{\alpha_1,\alpha_2}(t)=n\}$, $n\geq 0$ be the PMF of the MFCP. On substituting $u=0$ in (3.8), we get

$$p^{\alpha_1,\alpha_2}(0, t)=\sum_{m=0}^{\infty}(-C_2t^{\alpha_1-\alpha_2}/C_1)^m E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}(-\Lambda t^{\alpha_1}/C_1)$$
$$-\sum_{m=0}^{\infty}(-C_2t^{\alpha_1-\alpha_2}/C_1)^{m+1} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)(m+1)+1}(-\Lambda t^{\alpha_1}/C_1),$$

which reduces to the zero state probability of the MFPP for $k=1$, i.e. $\Lambda=\lambda_1$ (see [3, eq. (2.13)]).

Remark 3.3. On substituting $u=1$ in (3.8), we get

$$\begin{aligned}G^{\alpha_1,\alpha_2}(u, t)\big|_{u=1}&=\sum_{n=0}^{\infty}p^{\alpha_1,\alpha_2}(n, t)\\&=\sum_{m=0}^{\infty}\frac{(-C_2t^{\alpha_1-\alpha_2}/C_1)^m}{\Gamma((\alpha_1-\alpha_2)m+1)}-\sum_{m=0}^{\infty}\frac{(-C_2t^{\alpha_1-\alpha_2}/C_1)^{m+1}}{\Gamma((\alpha_1-\alpha_2)(m+1)+1)}\\&=1.\end{aligned}$$

Thus $p^{\alpha_1,\alpha_2}(n, t)$ is indeed a PMF.

Next we use the Z transform technique to obtain the system of fractional differential equations that governs the state probabilities of the MFCP.

Proposition 3.2. *For $n\geq 0$, the state probabilities of the MFCP satisfy*

$$C_1\frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}}p^{\alpha_1,\alpha_2}(n, t)+C_2\frac{\mathrm{d}^{\alpha_2}}{\mathrm{d}t^{\alpha_2}}p^{\alpha_1,\alpha_2}(n, t)=-\Lambda p^{\alpha_1,\alpha_2}(n, t)+\sum_{j=1}^{\min\{n,k\}}\lambda_j p^{\alpha_1,\alpha_2}(n-j, t), \quad (3.10)$$

with the initial conditions $p^{\alpha_1,\alpha_2}(0, 0)=1$ and $p^{\alpha_1,\alpha_2}(n, 0)=0$, $n\geq 1$.

Proof. See Appendix A.1. □

Remark 3.4. In view of (3.7), Proposition 3.2 follows as a particular case of Proposition 3.2 of [18].

Using (3.10), it can be shown that the PGF of the MFCP solves the following fractional differential equation:

$$C_1\frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}}G^{\alpha_1,\alpha_2}(u, t)+C_2\frac{\mathrm{d}^{\alpha_2}}{\mathrm{d}t^{\alpha_2}}G^{\alpha_1,\alpha_2}(u, t)=-\sum_{j=1}^{k}(1-u^j)\lambda_j G^{\alpha_1,\alpha_2}(u, t),$$

with initial condition $G^{\alpha_1,\alpha_2}(u, 0)=1$.

Theorem 3.2. *Let*

$$\Omega(k, n)=\{(x_1, x_2, \ldots, x_k)\colon x_1+2x_2+\cdots+kx_k=n,\ x_j\in\mathbb{N}_0\},$$
$$z_k=x_1+x_2+\cdots+x_k.$$

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The state probabilities of the MFCP are given by

$$p^{\alpha_1,\alpha_2}(n,t)=\sum_{\Omega(k,n)}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\frac{z_k!}{C_1^{z_k+1}}\left(C_1 f^{*(z_k+1)}(t)+C_2 g^{*(z_k+1)}(t)\right),\quad n\geq 0, \tag{3.11}$$

where $f^{*(z_k+1)}$ and $g^{*(z_k+1)}$ are (z_k+1)-fold convolutions of

$$f(t)=t^{\frac{z_k(\alpha_1-1)}{z_k+1}}\sum_{m=0}^{\infty}(-C_2t^{\alpha_1-\alpha_2}/C_1)^m E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+\frac{z_k\alpha_1+1}{z_k+1}}(-\Lambda t^{\alpha_1}/C_1)$$

and

$$g(t)=t^{\frac{z_k(\alpha_1-1)+\alpha_1-\alpha_2}{z_k+1}}\sum_{m=0}^{\infty}(-C_2t^{\alpha_1-\alpha_2}/C_1)^m E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+\frac{z_k\alpha_1+\alpha_1-\alpha_2+1}{z_k+1}}(-\Lambda t^{\alpha_1}/C_1),$$

respectively.

Proof. See Appendix A.2. □

Remark 3.5. On substituting $k=1$ in (3.11), we get the state probabilities of the MFPP (see [18, eq. (3.1)]).

Remark 3.6. In view of (3.7), the PMF of the MFCP can be obtained as follows:

$$\begin{aligned}p^{\alpha_1,\alpha_2}(n,t)&=\sum_{r=0}^{n}\mathbb{P}\{X_1+X_2+\cdots+X_r=n\}\mathbb{P}\{N_{\alpha_1,\alpha_2}(t)=r\}\\&=\sum_{r=0}^{n}\sum_{\substack{x_1+x_2+\cdots+x_k=r\\x_1+2x_2+\cdots+kx_k=n}}\frac{r!}{\Lambda^r}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\mathbb{P}\{N_{\alpha_1,\alpha_2}(t)=r\}\\&=\sum_{\Omega(k,n)}\frac{z_k!}{\Lambda^{z_k}}\prod_{j=1}^{k}\frac{\lambda_j^{x_j}}{x_j!}\mathbb{P}\{N_{\alpha_1,\alpha_2}(t)=z_k\},\end{aligned}$$

where in the penultimate step we used the explicit expression for the PMF of $X_1+X_2+\cdots+X_r$ (see [10, p. 297]). The result follows on using the PMF of the MFPP (see [18, eq. (3.1)]).

The rth factorial moment

$$\psi^{\alpha_1,\alpha_2}(r,t)=\mathbb{E}(\mathcal{M}_{\alpha_1,\alpha_2}(t)(\mathcal{M}_{\alpha_1,\alpha_2}(t)-1)\cdots(\mathcal{M}_{\alpha_1,\alpha_2}(t)-r+1)),\quad r\geq 1,$$

of the MFCP can be obtained using its PGF as follows.

Proposition 3.3. *For $r\geq 1$, the rth factorial moment of the MFCP is given by*

$$\psi^{\alpha_1,\alpha_2}(r,t)$$

$$=\sum_{m=0}^{\infty}\sum_{n=1}^{r}\frac{f_{m,n}(r,t)}{\Gamma(n\alpha_1+(\alpha_1-\alpha_2)m+1)}+\frac{C_2}{C_1}t^{\alpha_1-\alpha_2}\sum_{m=0}^{\infty}\sum_{n=1}^{r}\frac{f_{m,n}(r,t)}{\Gamma(n\alpha_1+(\alpha_1-\alpha_2)(m+1)+1)},$$

where

$$f_{m,n}(r,t)=\frac{(-1)^m r! C_2^m}{C_1^{m+n}}\binom{m+n}{n} t^{n\alpha_1+(\alpha_1-\alpha_2)m} \sum_{\substack{\sum_{i=1}^n m_i=r\\ m_i\in\mathbb{N}}} \prod_{\ell=1}^{n}\left(\frac{1}{m_\ell!}\sum_{j=1}^{k}(j)_{m_\ell}\lambda_j\right) \tag{3.12}$$

and $(j)_{m_\ell}=j(j-1)\cdots(j-m_\ell+1)$.

Proof. See Appendix A.3. □

The convergence of the series involved in Proposition 3.3 follows from Theorem 1.5 of [20].

Remark 3.7. On substituting $C_2=0$ and $C_1=1$ in Proposition 3.3, we get the rth factorial moment of the GFCP (see [19, eq. (19)]). Moreover, on substituting $k=1$ in Proposition 3.3, we get the rth factorial moment of the MFPP (see [3, eq. (2.34)]).

3.2. The increments of the MFCP

For a fixed $\delta>0$, the increment process $\{Z^{\delta}_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ of the MFCP is defined as

$$Z^{\delta}_{\alpha_1,\alpha_2}(t):=\mathcal{M}_{\alpha_1,\alpha_2}(t+\delta)-\mathcal{M}_{\alpha_1,\alpha_2}(t).$$

Next we show that $\{Z^{\delta}_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ exhibits the SRD property. First we give an asymptotic result for the covariance of the MFCP.

Proposition 3.4. *For fixed $s>0$, we have*

$$\mathrm{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s),\mathcal{M}_{\alpha_1,\alpha_2}(t))\sim L(s)-q_1^2 t^{\alpha_2-1}K(s),\quad \text{as } t\to\infty, \tag{3.13}$$

where L(s) and K(s) are some constants that depend on s.

Proof. For fixed $s>0$ and large t, we get the following on substituting (2.10) in (3.5):

$$\mathrm{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s),\mathcal{M}_{\alpha_1,\alpha_2}(t))\sim q_2 U_{\alpha_1,\alpha_2}(s)+q_1^2 C_1^{-2}s^{2\alpha_1}E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2 s^{\alpha_1-\alpha_2}/C_1)$$
$$-q_1^2 t^{\alpha_2-1}K(s),$$

where $K(s)$ is given in (2.11). On taking

$$L(s)=q_2 U_{\alpha_1,\alpha_2}(s)+q_1^2 C_1^{-2}s^{2\alpha_1}E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2 s^{\alpha_1-\alpha_2}/C_1),$$

the result follows. □

Theorem 3.3. *The increment process $\{Z^{\delta}_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ has the SRD property.*

Proof. The proof follows similar lines to that of Theorem 2 of [17]. Here we give a brief outline.

Let $s>0$ be fixed such that $0<s+\delta\leq t$. From (3.5) and (3.13) we have

$$\begin{aligned}
&\mathrm{Cov}\big(Z^{\delta}_{\alpha_1,\alpha_2}(s),Z^{\delta}_{\alpha_1,\alpha_2}(t)\big)\\
&=\mathrm{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s+\delta),\mathcal{M}_{\alpha_1,\alpha_2}(t+\delta))+\mathrm{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s),\mathcal{M}_{\alpha_1,\alpha_2}(t))\\
&\quad-\mathrm{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s+\delta),\mathcal{M}_{\alpha_1,\alpha_2}(t))-\mathrm{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(s),\mathcal{M}_{\alpha_1,\alpha_2}(t+\delta))\\
&\sim(1-\alpha_2)\delta q_1^2(K(s+\delta)-K(s))t^{\alpha_2-2}.
\end{aligned} \tag{3.14}$$

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From (3.5) we have

$$\begin{aligned}
&\operatorname{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(t), \mathcal{M}_{\alpha_1,\alpha_2}(t+\delta))\\
&\quad = q_2 U_{\alpha_1,\alpha_2}(t) + q_1^2 \operatorname{Cov}(Y_{\alpha_1,\alpha_2}(t), Y_{\alpha_1,\alpha_2}(t+\delta))\\
&\quad \sim q_2 U_{\alpha_1,\alpha_2}(t) - q_1^2 U_{\alpha_1,\alpha_2}(t) U_{\alpha_1,\alpha_2}(t+\delta)\\
&\qquad + q_1^2 \bigg(\frac{(t+\delta)^{\alpha_1+\alpha_2}}{C_1 C_2} E_{\alpha_1-\alpha_2,\alpha_1+\alpha_2+1}(-C_2 (t+\delta)^{\alpha_1-\alpha_2}/C_1)\\
&\qquad + \frac{t^{2\alpha_1}}{C_1^2} E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2 t^{\alpha_1-\alpha_2}/C_1)\bigg), \qquad (3.15)
\end{aligned}$$

where in the last step we used an asymptotic result for the covariance of the IMSS (see [17, eqs (31) and (32)]).

Substituting $\delta = 0$ in (3.15), we get an asymptotic behaviour of the variance of the MFCP. Thus

$$\begin{aligned}
&\operatorname{Var}\big(Z^\delta_{\alpha_1,\alpha_2}(t)\big)\\
&\quad = \operatorname{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(t+\delta)) + \operatorname{Var}(\mathcal{M}_{\alpha_1,\alpha_2}(t)) - 2\operatorname{Cov}(\mathcal{M}_{\alpha_1,\alpha_2}(t), \mathcal{M}_{\alpha_1,\alpha_2}(t+\delta))\\
&\quad \sim q_2(U_{\alpha_1,\alpha_2}(t+\delta) - U_{\alpha_1,\alpha_2}(t)) + q_1^2\big(2U_{\alpha_1,\alpha_2}(t)U_{\alpha_1,\alpha_2}(t+\delta) - U^2_{\alpha_1,\alpha_2}(t)\\
&\qquad - U^2_{\alpha_1,\alpha_2}(t+\delta)\big) + q_1^2 \bigg(\frac{(t+\delta)^{2\alpha_1}}{C_1^2} E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2(t+\delta)^{\alpha_1-\alpha_2}/C_1)\\
&\qquad - \frac{t^{2\alpha_1}}{C_1^2} E^2_{\alpha_1-\alpha_2,2\alpha_1+1}(-C_2 t^{\alpha_1-\alpha_2}/C_1) + \frac{t^{\alpha_1+\alpha_2}}{C_1C_2} E_{\alpha_1-\alpha_2,\alpha_1+\alpha_2+1}(-C_2 t^{\alpha_1-\alpha_2}/C_1)\\
&\qquad - \frac{(t+\delta)^{\alpha_1+\alpha_2}}{C_1C_2} E_{\alpha_1-\alpha_2,\alpha_1+\alpha_2+1}(-C_2(t+\delta)^{\alpha_1-\alpha_2}/C_1)\bigg)\\
&\quad \sim \frac{q_2\alpha_2\delta}{C_2\Gamma(\alpha_2+1)} t^{\alpha_2-1} \quad \text{(using (2.2) and (2.9))}. \qquad (3.16)
\end{aligned}$$

From (3.14) and (3.16) we get

$$\operatorname{Corr}\big(Z^\delta_{\alpha_1,\alpha_2}(s), Z^\delta_{\alpha_1,\alpha_2}(t)\big) \sim d(s) t^{-(3-\alpha_2)/2}, \quad \text{as } t \to \infty,$$

where

$$d(s) = \frac{(1-\alpha_2)\delta q_1^2 (K(s+\delta) - K(s))}{\sqrt{\operatorname{Var}(Z^\delta_{\alpha_1,\alpha_2}(s))}\sqrt{\frac{q_2\alpha_2\delta}{C_2\Gamma(\alpha_2+1)}}}.$$

As $1 < (3-\alpha_2)/2 < 1.5$, the result follows. □

3.3 An application of the MFCP to risk theory

Beghin and Macci [4] introduced the following risk process:

$$R_\alpha(t) = u + ct - \sum_{i=1}^{N^h_\alpha(t)} X_i, \quad t \geq 0,$$

where $u>0$ is the initial capital, and $c>0$ is the constant premium rate. The claim numbers

$$N_\alpha^h(t) := \sum_{n\geq 1} 1_{\{T_1+T_2+\cdots+T_n\leq t\}}, \quad 0<\alpha\leq 1,\ h>0$$

form a renewal process whose interarrival times $\{T_n\}_{n\geq 1}$ has the density

$$f_\alpha^h(t) = \lambda^h t^{\alpha h-1} E_{\alpha,\alpha h}^h(-\lambda t^\alpha), \quad \lambda>0,\ t>0,$$

and the X_i are i.i.d. positive random variables that represent the claim sizes and are independent of $\{N_\alpha^h(t)\}_{t\geq 0}$. For $h=1$, the process $\{N_\alpha^h(t)\}_{t\geq 0}$ reduces to the TFPP. Beghin and Macci [4] have obtained some asymptotic results for the ruin probabilities of $\{R_\alpha(t)\}_{t\geq 0}$. For more applications of the TFPP to risk theory, we refer the reader to Biard and Saussereau [7], Constantinescu *et al.* [8], and Kumar *et al.* [21].

Here we consider a risk process in which the number of claims received by an insurance company is modelled using the MFCP. We define it as follows:

$$R_{\alpha_1,\alpha_2}(t) := u + ct - \sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X_i, \quad t\geq 0, \tag{3.17}$$

where $u>0$ is the initial capital and $c>0$ is the constant premium rate. Here, the i.i.d. positive random variables X_i are the claim sizes with mean $\mu>0$ and these are independent of the MFCP $\{\mathcal{M}_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$.

Remark 3.8. As the TFPP and MFPP are particular cases of the MFCP, the risk process defined in (3.17) generalizes the risk processes studied in Biard and Saussereau [7] and Kataria and Khandakar [18]. There are several group insurance policies that insure employees of a particular institution, families, businesses, *etc*. A single group claim received implies several claims within a group. These situations can be modelled using the MFCP where the claims arrive in groups of size less than or equal to a fixed number k. This is an advantage of the risk process defined in (3.17) compared to the risk processes studied in [7] and [18].

The mean of $\{R_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ is

$$\mathbb{E}(R_{\alpha_1,\alpha_2}(t)) = u + ct - \mu q_1 U_{\alpha_1,\alpha_2}(t),$$

where $U_{\alpha_1,\alpha_2}(t)$ is given in (2.8).

It is important to note that initially the expected number of claims is higher under the MFCP regime than that of the GFCP or GCP. From (2.9) and (3.3), since $0<C\leq 1$, $0<\alpha<1$, and t is small, we have

$$\frac{t^\alpha}{C\Gamma(\alpha+1)} \geq \frac{t^\alpha}{\Gamma(\alpha+1)} > t.$$

The covariance of $\{R_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ can be obtained along similar lines to that of Proposition 10 of [21], and it is given by

$$\begin{aligned}\operatorname{Cov}(R_{\alpha_1,\alpha_2}(s), R_{\alpha_1,\alpha_2}(t)) &= \operatorname{Cov}\left(\sum_{j=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(s)} X_j, \sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X_i\right), \quad 0<s\leq t\\ &= (\mu^2 q_2 + \operatorname{Var}(X_1) q_1) U_{\alpha_1,\alpha_2}(s) + \mu^2 q_1^2 \operatorname{Cov}(Y_{\alpha_1,\alpha_2}(s), Y_{\alpha_1,\alpha_2}(t)).\end{aligned}$$

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Its variance is given by

$$\mathrm{Var}(R_{\alpha_1,\alpha_2}(t)) = (\mu^2 q_2 + \mathrm{Var}(X_1)q_1)U_{\alpha_1,\alpha_2}(t) + \mu^2 q_1^2\,\mathrm{Var}(Y_{\alpha_1,\alpha_2}(t)).$$

It is important to observe that if the mean and variance of claim sizes are finite then the risk process $\{R_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ exhibits the LRD property.

The finite-time ruin probability for the risk process $\{R_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$ is defined as follows:

$$\psi_u(t) = \mathbb{P}\{R_{\alpha_1,\alpha_2}(s) < 0 \text{ for some } s \leq t < \infty\}.$$

Now we give an asymptotic behaviour of the finite-time ruin probability when the claim sizes are subexponentially distributed and the initial capital is arbitrarily large, i.e. $u \to \infty$.

Let the distribution function $F_{X_1}(t) = \mathbb{P}\{X_1 \leq t\}$ of the claim sizes in (3.17) be subexponential, that is,

$$\lim_{t\to\infty} \big(1 - F_{X_1}^{*2}(t)\big)/(1 - F_{X_1}(t)) = 2,$$

where $F_{X_1}^{*2}(t)$ denotes the 2-fold convolution of $F_{X_1}(t)$.

The following result related to subexponential distribution will be used (see [2]).

Lemma 3.1. *Let $\{X_i\}_{i\geq 1}$ be a sequence of i.i.d. random variables having subexponential distribution $\bar{F}_{X_1}(t) = \mathbb{P}\{X_1 > t\}$ and let N be an integer-valued random variable with $\mathbb{E}(z^N) < \infty$ for some $z > 1$. Then*

$$\mathbb{P}\left\{\sum_{i=1}^{N} X_i > t\right\} \sim \mathbb{E}(N)\bar{F}_{X_1}(t), \quad \text{as } t \to \infty,$$

where N is independent of $\{X_i\}_{i\geq 1}$.

The following inequalities hold for the finite-time ruin probability of $\{R_{\alpha_1,\alpha_2}(t)\}_{t\geq 0}$:

$$\begin{aligned}\mathbb{P}\left\{\sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X_i > u + ct\right\} &\leq \mathbb{P}\left\{\sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t^*)} X_i > u + ct^* \text{ for some } t^* \leq t < \infty\right\}\\ &\leq \mathbb{P}\left\{\sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X_i > u\right\}.\end{aligned} \tag{3.18}$$

The PGF of the MFCP is given in (3.8), which is finite for some $z > 1$. On dividing (3.18) by

$$\mathbb{P}\left\{\sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X_i > u\right\}$$

and taking the limit $u \to \infty$, we get

$$\mathbb{P}\left\{\sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t^*)} X_i > u + ct^* \text{ for some } t^* \leq t < \infty\right\} \sim \mathbb{P}\left\{\sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X_i > u\right\},$$

where we have used $\lim_{u\to\infty} \bar{F}_{X_1}(u+ct)/\bar{F}_{X_1}(u) = 1$. On using Lemma 3.1, we get

$$\psi_u(t) \sim \mathbb{E}(\mathcal{M}_{\alpha_1,\alpha_2}(t))\bar{F}_{X_1}(u), \quad \text{as } u \to \infty. \tag{3.19}$$

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Thus the finite-time ruin probability has the following asymptotic behaviour as the initial capital $u \to \infty$:

$$\psi_u(t) \sim q_1 U_{\alpha_1,\alpha_2}(t)\bar{F}_{X_1}(u),$$

where we used (3.3) in (3.19).

Next we extend the above result to the case of m-dimensional risk processes, which is motivated by Proposition 5 of [7].

Let us consider the following independent risk processes:

$$R^{(j)}_{\alpha_1,\alpha_2}(t) := u^{(j)} + c^{(j)}t - \sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} X^{(j)}_i, \quad j = 1, 2, \ldots, m.$$

The above collection of independent risk processes can be rewritten as follows:

$$\bar{R}_{\alpha_1,\alpha_2}(t) := \bar{u} + \bar{c}t - \sum_{i=1}^{\mathcal{M}_{\alpha_1,\alpha_2}(t)} \bar{X}_i, \quad t \geq 0, \tag{3.20}$$

where

$$\bar{R}_{\alpha_1,\alpha_2}(t) = \big(R^{(1)}_{\alpha_1,\alpha_2}(t), R^{(2)}_{\alpha_1,\alpha_2}(t), \ldots, R^{(m)}_{\alpha_1,\alpha_2}(t)\big)$$

is an m-dimensional risk process, $\bar{u} = \big(u^{(1)}, u^{(2)}, \ldots, u^{(m)}\big)$ is the initial capital vector, $\bar{c} = \big(c^{(1)}, c^{(2)}, \ldots, c^{(m)}\big)$ is the premium intensity vector, and $\bar{X}_i = \big(X^{(1)}_i, X^{(2)}_i, \ldots, X^{(m)}_i\big)$ is the ith claim size vector. Also, $\{\bar{X}_i\}_{i\geq 1}$ is a sequence of i.i.d. random vectors with the following joint distribution:

$$\begin{aligned} F(x_1, x_2, \ldots, x_m) &= \mathbb{P}\big\{X^{(1)} \leq x_1, X^{(2)} \leq x_2, \ldots, X^{(m)} \leq x_m\big\} \\ &= \prod_{j=1}^{m} \mathbb{P}\big\{X^{(j)} \leq x_j\big\} \\ &= \prod_{j=1}^{m} F^{(j)}(x_j). \end{aligned}$$

Next we give an extension of the result given in (3.19) for the risk process defined in (3.20). Its proof follows similar lines to that of the result given in (3.19).

Proposition 3.5. *For $j = 1, 2, \ldots, m$, let the distribution of the claim sizes $F^{(j)}$ be subexponentially distributed. For an initial capital vector $\bar{u}$, let*

$$\tau_{\max}(\bar{u}) = \inf\big\{s > 0\colon\ \max\big\{R^{(1)}_{\alpha_1,\alpha_2}(s), R^{(2)}_{\alpha_1,\alpha_2}(s), \ldots, R^{(m)}_{\alpha_1,\alpha_2}(s)\big\} < 0\big\}$$

be the first time all the components of $\bar{R}_{\alpha_1,\alpha_2}(t)$ are negative. Then, for any $t > 0$, we have

$$\mathbb{P}\{\tau_{\max}(\bar{u}) \leq t\} \sim \mathbb{E}(\mathcal{M}_{\alpha_1,\alpha_2}(t))^m \prod_{j=1}^{m}\big(1 - F^{(j)}(u_j)\big),$$

as $u_j \to \infty$ for all $j = 1, 2, \ldots, m$.

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4. Compound generalized counting process

In this section we study the following process:

$$X(t) := \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0,$$

which we call the compound generalized counting process (CGCP). Here $\{Y_i\}_{i\geq 1}$ is a sequence of i.i.d. random variables with cumulative distribution function H and it is independent of the GCP $\{M(t)\}_{t\geq 0}$. Kataria and Khandakar [19] studied a risk process in which a version of the CGCP is used to model the total claim received by the insurance company.

For $k=1$, the CGCP reduces to the compound Poisson process as the GCP reduces to the Poisson process. For $H=\delta_1$, the Dirac measure at 1, the CGCP reduces to the GCP. Further, for $k=1$ and $H=\delta_1$, the CGCP reduces to the Poisson process. On taking $\lambda_j=\lambda$ for all $j=1, 2, \ldots, k$, the CGCP reduces to the compound Poisson process of order k (see [27]). Also, for $\lambda_j=\lambda(1-\rho)\rho^{j-1}/(1-\rho^k)$, $0\leq\rho<1$, $j=1, 2, \ldots, k$, the CGCP reduces to the compound Pólya–Aeppli process of order k.

Theorem 4.1. *Let $0=t_0\leq t_1\leq\cdots\leq t_n=t$ be a partition of [0,t]. Then the distribution function of the CGCP has the following form:*

$$\begin{aligned}&\mathbb{P}\{X(t_1)\leq y_1, X(t_2)\leq y_2, \ldots, X(t_n)\leq y_n\}\\ &\quad=\sum_{j_1,j_2,\ldots,j_n}\prod_{l=1}^{n} p(j_l, \Delta t_l)\int_{-\infty}^{y_1}\int_{-\infty}^{y_2-x_1}\\ &\qquad\cdots\int_{-\infty}^{y_n-\sum_{l=1}^{n-1}x_l} h_{Y_1}^{*j_1}(x_1)h_{Y_1}^{*j_2}(x_2)\cdots h_{Y_1}^{*j_n}(x_n)\,\mathrm{d}x_n\,\mathrm{d}x_{n-1}\cdots\mathrm{d}x_1,\end{aligned} \tag{4.1}$$

where $\Delta t_l=t_l-t_{l-1}$, $h_{Y_1}^{*j_l}(\cdot)$ denotes the j_l-fold convolution of density function h of Y_1 and the summation is taken over all non-negative integers $j_l\geq 0$, $l=1, 2, \ldots, n$.

The proof of Theorem 4.1 follows similar lines to that of Theorem 1 of [27], and thus it is omitted.

Remark 4.1. The marginal distribution of the CGCP can be obtained by taking $n=1$ in (4.1), and it is given by

$$\mathbb{P}\{X(t)\leq y\}=\sum_{j=0}^{\infty}p(j, t)\int_{-\infty}^{y}h_{Y_1}^{*j}(x)\,\mathrm{d}x. \tag{4.2}$$

If Y_1 has discrete distribution then the summations will appear in place of integrations in (4.1) and (4.2).

The mean of the CGCP can be obtained by using Wald's identity as follows:

$$\mathbb{E}(X(t))=\mathbb{E}(M(t))\mathbb{E}(Y_1)=t\mathbb{E}(Y_1)\sum_{j=1}^{k}j\lambda_j.$$

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The variance of the CGCP can obtained as follows:

$$\begin{aligned}\operatorname{Var}(X(t)) &= \mathbb{E}(M(t))\operatorname{Var}(Y_1) + (\mathbb{E}(Y_1))^2\operatorname{Var}(M(t))\\ &= t\operatorname{Var}(Y_1)\sum_{j=1}^{k} j\lambda_j + t(\mathbb{E}(Y_1))^2\sum_{j=1}^{k} j^2\lambda_j. \qquad (4.3)\end{aligned}$$

In the following result, we obtain the PGF for a particular case of the CGCP.

Proposition 4.1. *Let*

$$D(t) = \sum_{i=1}^{M(t)} Y_i, \quad t \geq 0$$

be a CGCP where $\mathbb{P}\{Y_1 = i\} = \alpha_i$*,* $i = 0, 1, 2, \ldots$ *. Then the PGF of* $\{D(t)\}_{t\geq 0}$ *is given by*

$$G_{D(t)}(u) = \exp\left(t\sum_{j=1}^{k}\lambda_j\sum_{i=1}^{\infty}\alpha_i^{*j}(u^i - 1)\right),$$

where

$$\alpha_i^{*j} = \sum_{\substack{\sum_{m=1}^{j} l_m = i\\ l_m\in\mathbb{N}_0}} \alpha_{l_1}\alpha_{l_2}\cdots\alpha_{l_j} \quad \textit{for all } j = 1, 2, \ldots, k.$$

Proof. See Appendix A.4. □

The proof of the following result follows similar lines to that of Lemma 1 of [27].

Lemma 4.1. *Let* $\{Y_i\}_{i\geq 1}$*be a sequence of non-negative i.i.d. random variables with PMF* $\mathbb{P}\{Y_1 = i\} = \alpha_i$*,* $i = 0, 1, 2, \ldots$ *. Also, let*

$$C(t) = \sum_{i=1}^{N(t)} Y_i, \quad t \geq 0,$$

where $\{N(t)\}_{t\geq 0}$ *is the Poisson process with intensity* Λ *and which is independent of* $\{Y_i\}_{i\geq 1}$*. Then*

$$C(t) \stackrel{d}{=} \sum_{i=1}^{\infty} iN_i(t), \quad t \geq 0,$$

where $\{N_i(t)\}_{t\geq 0}$ *is the Poisson process with intensity* $\Lambda\alpha_i$*.*

For $k = 1$, the process $\{D(t)\}_{t\geq 0}$ reduces to $\{C(t)\}_{t\geq 0}$.

Remark 4.2. If

$$\alpha_i = \frac{1}{\Lambda}\sum_{j=1}^{k}\lambda_j\alpha_i^{*j},$$

where $\mathbb{P}\{Y_1 = i\} = \alpha_i$ for all $i = 0, 1, \ldots$, then from the PGF of $\{C(t)\}_{t\geq 0}$, that is,

$$\mathbb{E}\big(u^{C(t)}\big) = \exp\big(\Lambda t\big(\mathbb{E}\big(u^{Y_1}\big) - 1\big)\big) = \exp\left(\Lambda t\sum_{i=1}^{\infty}(u^i - 1)\alpha_i\right),$$

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we get $C(t) \overset{d}{=} D(t)$, $t \geq 0$. So, in this case $\{D(t)\}_{t\geq 0}$ is a Lévy process and hence it is infinitely divisible. Moreover, the characteristic function of $\{D(t)\}_{t\geq 0}$ is given by

$$\mathbb{E}\big(\mathrm{e}^{\omega\xi D(t)}\big) = \exp\left(\Lambda t \sum_{i=1}^{\infty} (\mathrm{e}^{\omega i\xi} - 1)\alpha_i\right), \quad \omega = \sqrt{-1}.$$

Thus the Lévy measure of $\{D(t)\}_{t\geq 0}$ is

$$\Pi(\mathrm{d}x) = \Lambda \sum_{i=1}^{\infty} \alpha_i \delta_i \,\mathrm{d}x.$$

Let

$$Q(t) = D(t) - t\mathbb{E}(Y_1)\sum_{j=1}^{k} j\lambda_j.$$

Under the given condition on α_i, $\{D(t)\}_{t\geq 0}$ has independent increments. Thus, for $s \leq t$, we have

$$\mathbb{E}(Q(t) - Q(s) \mid \mathscr{F}_s) = \mathbb{E}(D(t) - D(s) \mid \mathscr{F}_s) - (t-s)\mathbb{E}(Y_1)\sum_{j=1}^{k} j\lambda_j = 0,$$

where $\{\mathscr{F}_t\}_{t\geq 0}$ is a natural filtration. Thus $\{Q(t)\}_{t\geq 0}$ is a martingale with respect to $\{\mathscr{F}_t\}_{t\geq 0}$.

Theorem 4.2. *Let*

$$\alpha_i = \frac{1}{\Lambda}\sum_{j=1}^{k} \lambda_j \alpha_i^{*j},$$

where $\mathbb{P}\{Y_1 = i\} = \alpha_i$ *for all* $i = 0, 1, \ldots$ *and* $\sum_{i=1}^{\infty} i^2\alpha_i < \infty$. *Then the process* $\{D(t)\}_{t\geq 0}$ *exhibits the LRD property.*

Proof. Note that $\{D(t)\}_{t\geq 0}$ is a Lévy process under the given condition on α_i (see Remark 4.2). So, its covariance can be obtained as follows:

$$\begin{aligned}
\operatorname{Cov}(D(s), D(t)) &= \mathbb{E}(D(s)(D(t) - D(s))) + \mathbb{E}\big((D(s))^2\big) - \mathbb{E}(D(s))\mathbb{E}(D(t)) \\
&= \mathbb{E}(D(s))\mathbb{E}(D(t) - D(s)) + (\mathbb{E}(D(s)))^2 + \operatorname{Var}(D(s)) - \mathbb{E}(D(s))\mathbb{E}(D(t)) \\
&= s\operatorname{Var}(Y_1)\sum_{j=1}^{k} j\lambda_j + s(\mathbb{E}(Y_1))^2\sum_{j=1}^{k} j^2\lambda_j, \quad 0 < s \leq t. \qquad (4.4)
\end{aligned}$$

On substituting $s = t$ in (4.4), we get the variance of $\{D(t)\}_{t\geq 0}$. Alternatively, it can be obtained from (4.3). Thus, for fixed s and large t, the correlation function of $\{D(t)\}_{t\geq 0}$ has the following asymptotic behaviour:

$$\operatorname{Corr}(D(s), D(t)) = \frac{\operatorname{Cov}(D(s), D(t))}{\sqrt{\operatorname{Var}(D(s))}\sqrt{\operatorname{Var}(D(t))}} \sim \sqrt{s}t^{-1/2},$$

which implies that it has the LRD property. □

5. Concluding remarks

We introduce a time-changed version of the GCP, namely the MFCP. It is defined as the GCP time-changed by an independent IMSS. We obtain the system of fractional differential equations that governs its state probabilities. Its various distributional properties are obtained, such as its one-dimensional distribution, mean, variance, covariance, and PGF. It is shown that the MFCP exhibits the LRD property whereas its increment process has the SRD property. As pointed out in Remark 3.2, the processes that exhibit the LRD property have applications in several areas, such as internet data traffic modelling, finance, econometrics, and hydrology. So, the MFCP has potential applications in these areas and other related fields. Moreover, we consider a risk process in which the MFCP is used to model the claim numbers received by an insurance company, and we obtained some results related to its ruin probability. It is shown that the introduced risk process has the LRD property. Also, we discuss some distributional properties of a compound version of the GCP.

The MFCP can serve as a stress test for insurance companies as the initial number of claims is higher. Thus it can be an important process for start-ups and re-insurers which have a dependent claim structure in their portfolio. Further, it is shown that the MFCP exhibits the overdispersion property, so it can potentially be used when the empirical count data exhibit overdispersion. The TFPP is a useful model in finance, in optics to describe light propagation through non-homogeneous media, and in the analysis of the transport of charged carriers (see [6]). As the MFCP is a generalization of the TFPP, it might have potential applications in these fields too.

Appendix A

A.1. Proof of Proposition 3.2

Proof. On taking the Z transform with respect to the state variable on both sides of (3.10), we get

$$
\begin{aligned}
C_1\frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}t^{\alpha_1}}(Zp^{\alpha_1,\alpha_2}(n,t)) &+ C_2\frac{\mathrm{d}^{\alpha_2}}{\mathrm{d}t^{\alpha_2}}(Zp^{\alpha_1,\alpha_2}(n,t)) \\
&= -\Lambda Zp^{\alpha_1,\alpha_2}(n,t) + \sum_{j=1}^{\min\{n,k\}} \lambda_j Zp^{\alpha_1,\alpha_2}(n-j,t) \\
&= -\Lambda Zp^{\alpha_1,\alpha_2}(n,t) + \sum_{j=1}^{k} \lambda_j Zp^{\alpha_1,\alpha_2}(n-j,t) \\
&= -\Lambda\big(1-G_{X_1}(z^{-1})\big)Zp^{\alpha_1,\alpha_2}(n,t),
\end{aligned}
$$

where we used (2.5) and (3.9).

Now we take the Laplace transform with respect to the time variable, and use (1.2) with $Zp^{\alpha_1,\alpha_2}(n,0)=1$ to obtain

$$
\begin{aligned}
C_1s^{\alpha_1}\mathcal{L}(Zp^{\alpha_1,\alpha_2}(n,t);s) - C_1s^{\alpha_1-1} &+ C_2s^{\alpha_2}\mathcal{L}(Zp^{\alpha_1,\alpha_2}(n,t);s) - C_2s^{\alpha_2-1} \\
&= -\Lambda\big(1-G_{X_1}(z^{-1})\big)\mathcal{L}(Zp^{\alpha_1,\alpha_2}(n,t);s).
\end{aligned}
$$

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Thus we get

$$\mathcal{L}(Zp^{\alpha_1,\alpha_2}(n, t);s) = \frac{C_1 s^{\alpha_1-1} + C_2 s^{\alpha_2-1}}{C_1 s^{\alpha_1} + C_2 s^{\alpha_2} + \Lambda\big(1 - G_{X_1}(z^{-1})\big)}.$$

On using (2.3), we get

$$\begin{aligned} Zp^{\alpha_1,\alpha_2}(n, t) = &\sum_{m=0}^{\infty} (-C_2 t^{\alpha_1-\alpha_2}/C_1)^m E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}\big(-\Lambda\big(1 - G_{X_1}(z^{-1})\big)t^{\alpha_1}/C_1\big) \\ &- \sum_{m=0}^{\infty} (-C_2 t^{\alpha_1-\alpha_2}/C_1)^{m+1} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)(m+1)+1}\big(-\Lambda\big(1 - G_{X_1}(z^{-1})\big)t^{\alpha_1}/C_1\big), \end{aligned}$$

which agrees with (3.8) for $u = z^{-1}$. The proof is complete on using (2.6). □

A.2. Proof of Theorem 3.2

Proof. Let $f_{\alpha_1,\alpha_2}(t, x)$ be the density function of the IMSS. From (1.3) and (3.1), the Laplace transform of the PMF of the MFCP can be written as

$$\begin{aligned} \mathcal{L}(p^{\alpha_1,\alpha_2}(n, t);s) &= \int_0^{\infty} \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} x^{z_k} \mathrm{e}^{-\Lambda x} \bigg(\int_0^{\infty} \mathrm{e}^{-st} f_{\alpha_1,\alpha_2}(t, x)\,\mathrm{d}t\bigg)\,\mathrm{d}x \\ &= \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \frac{\phi(s)}{s} \int_0^{\infty} \mathrm{e}^{-\Lambda x} x^{z_k} \mathrm{e}^{-x\phi(s)}\,\mathrm{d}x \quad \text{(using (2.7))} \\ &= \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \frac{\phi(s)}{s} \frac{z_k!}{(\Lambda + \phi(s))^{z_k+1}} \\ &= \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} z_k! \frac{C_1 s^{\alpha_1-1} + C_2 s^{\alpha_2-1}}{(C_1 s^{\alpha_1} + C_2 s^{\alpha_2} + \Lambda)^{z_k+1}} \\ &= \sum_{\Omega(k,n)} \prod_{j=1}^{k} \frac{\lambda_j^{x_j}}{x_j!} \frac{z_k!}{C_1^{z_k+1}} \Bigg[C_1 \bigg(\frac{s^{(\alpha_1-1)/(z_k+1)}}{s^{\alpha_1} + C_2 C_1^{-1} s^{\alpha_2} + \Lambda C_1^{-1}} \bigg)^{z_k+1} \\ &\qquad + C_2 \bigg(\frac{s^{(\alpha_2-1)/(z_k+1)}}{s^{\alpha_1} + C_2 C_1^{-1} s^{\alpha_2} + \Lambda C_1^{-1}} \bigg)^{z_k+1} \Bigg]. \end{aligned}$$

The result follows on using (2.3). □

A.3. Proof of Proposition 3.3

Proof. Let

$$\zeta(u, t) = -\frac{t^{\alpha_1}}{C_1} \sum_{j=1}^{k} (1 - u^j)\lambda_j.$$

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Then

$$\begin{aligned}
&\psi^{\alpha_1,\alpha_2}(r,t)\\
&\quad= \frac{\partial^r}{\partial u^r} G^{\alpha_1,\alpha_2}(u,t)\Big|_{u=1}\\
&\quad= \sum_{m=0}^{\infty} (-C_2 t^{\alpha_1-\alpha_2}/C_1)^m \frac{\partial^r}{\partial u^r} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}(\zeta(u,t))\Big|_{u=1}\\
&\qquad+ \frac{C_2}{C_1} t^{\alpha_1-\alpha_2} \sum_{m=0}^{\infty} (-C_2 t^{\alpha_1-\alpha_2}/C_1)^m \frac{\partial^r}{\partial u^r} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)(m+1)+1}(\zeta(u,t))\Big|_{u=1}. \qquad \text{(A.1)}
\end{aligned}$$

On using the rth derivative of composition of two functions (see [15, eq. (3.3)]), we get

$$\frac{\partial^r}{\partial u^r} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}(\zeta(u,t))\Big|_{u=1} = \sum_{n=0}^{r} \frac{1}{n!} \big(E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}(\zeta(u,t))\big)^{(n)} A_{r,n}(\zeta(u,t))\Big|_{u=1}, \qquad \text{(A.2)}$$

where

$$\begin{aligned}
A_{r,n}(\zeta(u,t))\big|_{u=1} &= \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} (-\zeta(u,t))^{n-i} \frac{\partial^r}{\partial u^r} (\zeta(u,t))^i \Big|_{u=1}\\
&= \frac{t^{n\alpha_1}}{C_1^n} \frac{\mathrm{d}^r}{\mathrm{d}u^r} \left(\sum_{j=1}^{k} (u^j-1)\lambda_j \right)^n \Big|_{u=1}.
\end{aligned}$$

Using (2.1), we get

$$\begin{aligned}
&\big(E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}(\zeta(u,t))\big)^{(n)}\big|_{u=1}\\
&\quad= (m+1)(m+2)\cdots(m+n) E^{m+n+1}_{\alpha_1,n\alpha_1+(\alpha_1-\alpha_2)m+1}(\zeta(u,t))\big|_{u=1}\\
&\quad= \frac{(m+1)(m+2)\cdots(m+n)}{\Gamma(n\alpha_1+(\alpha_1-\alpha_2)m+1)}.
\end{aligned}$$

Using the result (see [15, eq. (3.6)])

$$\frac{\mathrm{d}^r}{\mathrm{d}w^r}(g(w))^n = \sum_{\substack{m_1+m_2+\cdots+m_n=r\\ m_i\in\mathbb{N}_0}} \frac{r!}{m_1!m_2!\cdots m_n!} g^{(m_1)}(w) g^{(m_2)}(w)\cdots g^{(m_n)}(w),$$

we get

$$\begin{aligned}
\frac{\mathrm{d}^r}{\mathrm{d}u^r}\left(\sum_{j=1}^{k}(u^j-1)\lambda_j\right)^n\Big|_{u=1} &= r! \sum_{\substack{\sum_{i=1}^n m_i=r\\ m_i\in\mathbb{N}_0}} \prod_{\ell=1}^{n} \frac{1}{m_\ell!} \frac{\mathrm{d}^{m_\ell}}{\mathrm{d}u^{m_\ell}} \left(\sum_{j=1}^{k}(u^j-1)\lambda_j\right)\Big|_{u=1}\\
&= r! \sum_{\substack{\sum_{i=1}^n m_i=r\\ m_i\in\mathbb{N}}} \prod_{\ell=1}^{n} \left(\frac{1}{m_\ell!} \sum_{j=1}^{k} (j)_{m_\ell} \lambda_j \right). \qquad \text{(A.3)}
\end{aligned}$$

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On substituting (A.2)–(A.3) in the first sum of (A.1), we get

$$(-C_2 t^{\alpha_1-\alpha_2}/C_1)^m \frac{\partial^r}{\partial u^r} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)m+1}(\zeta(u,t))\Big|_{u=1} = \sum_{n=1}^{r} \frac{f_{m,n}(r,t)}{\Gamma(n\alpha_1+(\alpha_1-\alpha_2)m+1)},$$

where $f_{m,n}(r, t)$ is given in (3.12). Similarly, for the second sum of (A.1), we have

$$(-C_2 t^{\alpha_1-\alpha_2}/C_1)^m \frac{\partial^r}{\partial u^r} E^{m+1}_{\alpha_1,(\alpha_1-\alpha_2)(m+1)+1}(\zeta(u,t))\Big|_{u=1} = \sum_{n=1}^{r} \frac{f_{m,n}(r,t)}{\Gamma(n\alpha_1+(\alpha_1-\alpha_2)(m+1)+1)}.$$

Finally, the result follows on substituting the above values in (A.1). □

A.4. Proof of Proposition 4.1

Proof. The PGF of the GCP is given by (see [19])

$$G_{M(t)}(u) = \exp\left(\sum_{j=1}^{k} \lambda_j (u^j - 1)t\right) = \exp\left(t\sum_{j=1}^{k} \lambda_j u^j - \Lambda t\right).$$

Let $G^{*k}_{Y_1}(u) = \mathbb{E}\left(u^{\sum_{i=1}^{k} Y_i}\right)$. Then

$$\begin{aligned}
G_{D(t)}(u) &= \exp\left(t\sum_{j=1}^{k} \lambda_j (G_{Y_1}(u))^j - \Lambda t\right)\\
&= \exp\left(t\sum_{j=1}^{k} \lambda_j \prod_{i=1}^{j} G_{Y_i}(u) - \Lambda t\right) \quad (Y_i \text{ are identical})\\
&= \exp\left(t\sum_{j=1}^{k} \lambda_j G_{Y_1+Y_2+\cdots+Y_j}(u) - \Lambda t\right) \quad (Y_i \text{ are independent})\\
&= \exp\left(t\sum_{j=1}^{k} \lambda_j \sum_{i=0}^{\infty} \mathbb{P}\{Y_1+Y_2+\cdots+Y_j = i\} u^i - \Lambda t\right)\\
&= \exp\left(t\sum_{j=1}^{k} \lambda_j \sum_{i=0}^{\infty} \sum_{\substack{\sum_{m=1}^{j} l_m = i\\ l_m\in\mathbb{N}_0}} \alpha_{l_1}\alpha_{l_2}\cdots\alpha_{l_j} u^i - \Lambda t\right)\\
&= \exp\left(t\sum_{j=1}^{k} \lambda_j \sum_{i=0}^{\infty} \alpha_i^{*j} u^i - t\sum_{j=1}^{k}\lambda_j\right)
\end{aligned}$$

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$$= \exp\left(t\sum_{j=1}^{k}\lambda_j\sum_{i=0}^{\infty}\alpha_i^{*j}(u^i-1)\right) \quad \left(\text{as } \sum_{i=0}^{\infty}\alpha_i^{*j}=1\right)$$

$$= \exp\left(t\sum_{j=1}^{k}\lambda_j\sum_{i=1}^{\infty}\alpha_i^{*j}(u^i-1)\right).$$

This completes the proof. □

Acknowledgements

The authors thank two anonymous reviewers for their insightful comments and suggestions which led to significant improvements.

Funding information

The research of the first author was supported by an Institute post-doctoral fellowship, IIT Bombay.

Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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