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INTEGRAL MEANS AND ZERO DISTRIBUTIONS OF BLASCHKE PRODUCTS

C. N. LINDEN

1. Preliminaries. A sequence $\{z_n\}$ in $D = \{z: |z| < 1\}$ is a Blaschke sequence if and only if

$$\sum_n (1 - |z_n|) < \infty.$$

If 0 appears m times in $\{z_n\}$ then

$$B(z, \{z_n\}) = z^m \prod_{z_n \neq 0} \frac{\bar{z}_n(z_n - z)}{|z_n|(1 - z\bar{z}_n)}$$

is the Blaschke product defined by $\{z_n\}$. The set of all Blaschke products will be denoted by $\mathscr{B}$. If $B \in \mathscr{B}$ it is well-known that B is regular in D, and $|B(z, \{z_n\})| < 1$ when $z \in D$.

For a given pair of values p in $(0, \infty)$ and q in $[0, \infty)$ we denote by $\mathscr{I}(p, q)$ the class of all Blaschke products $B(z, \{z_n\})$ such that

$$I(r, p, \{z_n\}) = \int_0^{2\pi} |\log|B(re^{i\theta}, \{z_n\})||^p d\theta = O((1 - r)^{-q})$$

as $r \to 1 - 0$. In the case $q \geqq \max(p - 1, 0)$ the classes of functions $\mathscr{B}$ and $\mathscr{I}(p, q)$ are identical: this is a particular case of an elementary theorem for functions subharmonic in a disc, the analogous theorem for functions subharmonic in a half-plane appearing in [**1**].

If $0 \leqq q < \max(p - 1, 0)$ then $\mathscr{I}(p, q)$ is a proper subset of $\mathscr{B}$, and we examine the distributions of the zeros of Blaschke products in $\mathscr{I}(p, q)$ in these cases. The theorems obtained here are generalizations of results obtained elsewhere [**2**; **3**] in the case where $p = 2$ and $q = 0$. Although the earlier methods can be readily modified to prove the results of this paper for any non-negative q when $p = 2$, such modifications are not possible for other values of p except in the case of Theorem 3. Thus a different, and more direct, approach will be made to the proofs of our other theorems.

In stating our main results we use the notation

$$A(r, \theta, \psi) = \begin{cases} \{z : |\theta - \arg z| \leqq 2\psi,\ r < |z| \leqq \frac{1}{2}(1 + r)\}, & 0 \leqq \psi < \frac{1}{2}\pi, \\ \{z : r < |z| \leqq \frac{1}{2}(1 + r)\}, & \psi \geqq \frac{1}{2}\pi, \end{cases}$$

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and let $\nu(r, \theta, \psi, \{z_n\})$ denote the number of elements of $\{z_n\}$ in $A(r, \theta, \psi)$. Then we have the following theorem.

THEOREM 1. *Let $p > 1$, $0 \leqq q < p - 1$, and suppose that $\{z_n\}$ is a Blaschke sequence such that*

$$I(\rho, p, \{z_n\}) < C(1 - \rho)^{-q}, \quad 0 < \rho < 1$$

for some constant C. Then if $\gamma \geqq 1$ we have

$$\nu(r, \varphi, (1 - r)^\gamma, \{z_n\}) < K(p, q, \gamma) C^{1/p} (1 - r)^{-(1+q)/p}, \quad 0 < r < 1, \ 0 \leqq \varphi < 2\pi, \tag{1.1}$$

and if $0 < \gamma < 1$ we have

$$\nu(r, \varphi, (1 - r)^\gamma, \{z_n\}) < K(p, q, \gamma) C^{1/p} (1 - r)^{\gamma - 1 - \gamma(1+q)/p}, \quad 0 < r < 1, \ 0 \leqq \varphi < 2\pi. \tag{1.2}$$

Here, and throughout the paper, we use the symbol K to denote a constant depending on the particular parameters under consideration. The values of K need not be the same in any two successive appearances, but at each appearance there will be some means of determining its value in terms of the relevant parameters, these being indicated either directly or by implication.

Next we show that the indices appearing in the right-hand sides of (1.1) and (1.2) are best possible. We do this by means of the following two theorems.

THEOREM 2. *If $n(r, \{z_n\})$ denotes the number of elements of $\{z_n\}$ for which $|z_n| \leqq r$, and*

$$n(r, \{z_n\}) < C(1 - r)^{-\alpha}, \quad 0 < r < 1 \tag{1.3}$$

for some positive constant C and some constant α in $(0, 1)$, then for each p in $(1, \infty)$ we have

$$I(r, p, \{z_n\}) < K(\alpha, p) C^p (1 - r)^{1 - \alpha p}, \quad 0 < r < 1. \tag{1.4}$$

THEOREM 3. *Let*

$$z_n = (1 - n^{-\beta}) e^{in^{-\beta\gamma}}, \quad n = 2, 3, \ldots,$$

where $0 < \gamma \leqq 1$, $\beta > 1$, and $p > 1$. Then if

$$\beta\gamma\lambda = p(\beta - 1) - \beta\gamma(p - 1) \tag{1.5}$$

we have

$$I(r, p, \{z_n\}) < K(p, \beta, \gamma, \lambda)((1 - r)^{\lambda\gamma} + (1 - r)^\lambda + (1 - r)^\gamma), \quad 0 < r < 1, \tag{1.6}$$

while

$$\nu(r, 0, (1 - r)^\gamma, \{z_n\}) \sim K(1 - r)^{-1/\beta} \tag{1.7}$$

as $r \to 1 - 0$.

MacLane and Rubel [**2**] have proved Theorem 2 in the case where $p = 2$ and $\alpha = \frac{1}{2}$. As an immediate corollary of Theorem 1 and Theorem 2 we note the following generalization of another result proved by these two authors in the same paper.

COROLLARY. *If $p > 1$, $0 \leqq q < p - 1$, and the elements of $\{z_n\}$ are contained in a finite number of Stolz angles, then $B(z, \{z_n\}) \in \mathscr{I}(p, q)$ if and only if*

$$n(r, \{z_n\}) = O((1 - r)^{-(q+1)/p})$$

as $r \rightarrow 1 - 0$.

As well as showing that (1.1) is best possible, this Corollary demonstrates the existence of Blaschke products that do not belong to $\mathscr{I}(p, q)$ for any pair of numbers p and q for which $0 \leqq q < p - 1 < \infty$. For example

$$z_n = 1 - \frac{1}{n(\log n)^2}, n = 2, 3, 4, \ldots$$

defines a Blaschke product $\{z_n\}$ of this type.

In order to see that (1.2) is best possible, we put $\lambda = -q \leqq 0$ in the inequality (1.6) of Theorem 3. Then β can be chosen in the range $(1, \infty)$ to satisfy (1.5), so that

$$I(r, p, \{z_n\}) = O((1 - r)^{-q})$$

as $r \rightarrow 1 - 0$ since $\lambda \leqq \lambda\gamma \leqq 0$. Further the relation (1.7) becomes

$$\nu(r, 0, (1 - r)^{\gamma}, \{z_n\}) \sim K(1 - r)^{\gamma - 1 - \gamma(1+q)/p}$$

as $r \rightarrow 1 - 0$, as required.

Finally we note that the proof of Theorem 3 has already been given elsewhere [**3**] in the case $p = 2$. The proof given in this special case needs only obvious amendments to apply in the generality stated here. Thus we may forego the details of the proof of Theorem 3, and we prove only Theorems 1 and 2 in this paper.

2. The proof of Theorem 1. Without loss of generality we suppose that $\varphi = 0$ and that $z_n = r_n e^{i\theta_n} \neq 0$ for all natural numbers n. Then it is easily shown (see e.g. [**2**]) that

$$\log|B(\rho e^{i\theta}, \{z_n\})|^{-2} = \sum_{n=1}^{\infty} \log\left(1 + \frac{(1 - \rho^2)(1 - r_n^2)}{P(\rho, r_n, \theta, \theta_n)}\right), \tag{2.1}$$

where

$$P(\rho, r_n, \theta, \theta_n) = (\rho - r_n)^2 + 4\rho r_n \sin^2 \tfrac{1}{2}(\theta - \theta_n).$$

For a given value r in $(\frac{3}{4}, 1)$ we suppose that

$$0 < \rho \leqq 4r - 3, \tag{2.2}$$

and

$$2(1 - r)^{\gamma} < \theta < \pi.$$

Then, if $z_n \in A(r, 0, (1-r)^\gamma) = \mathscr{A}$, we have

$$\max\{3(1-r_n), \tfrac{3}{4}(1-\rho)\} < r_n - \rho < 1 - \rho, \tag{2.3}$$

$$\sin \tfrac{1}{2}(\theta - \theta_n) < 2 \sin \tfrac{1}{4}(\theta - \theta_n) < 2 \sin \tfrac{1}{2}\theta, \tag{2.4}$$

and it follows that

$$\begin{aligned} P(\rho, r_n, \theta, \theta_n) &< (1-\rho)^2 + 16 \sin^2 \tfrac{1}{2}\theta \\ &< (1 - \rho + 4 \sin \tfrac{1}{2}\theta)^2, \end{aligned} \tag{2.5}$$

while

$$P(\rho, r_n, \theta, \theta_n) > \frac{9}{4}(1-\rho)(1-r_n) > \frac{1}{2}(1-\rho^2)(1-r_n^2). \tag{2.6}$$

Now (2.6) implies that

$$\log\left(1 + \frac{(1-\rho^2)(1-r_n^2)}{P(\rho, r_n, \theta, \theta_n)}\right) > \frac{(1-\rho^2)(1-r_n^2)}{3P(\rho, r_n, \theta, \theta_n)},$$

so that, by (2.1) and (2.5), we have

$$\begin{aligned} \log|B(\rho e^{i\theta}, \{z_n\})|^{-2} &> \sum_{z_n \in \mathscr{A}} \frac{(1-\rho)(1-r_n)}{3P(\rho, r_n, \theta, \theta_n)} \\ &> \frac{\nu(r, 0, (1-r)^\gamma)(1-\rho)(1-r)}{6(1-\rho+4\sin\frac{1}{2}\theta)^2}. \end{aligned}$$

Consequently, the hypothesis of Theorem 1 shows that

$$\begin{aligned} C(1-\rho)^{-q} &> I(\rho, p, \{z_n\}) \\ &> \frac{\nu(r, 0, (1-r)^\gamma)^p(1-\rho)^p(1-r)^p}{6^p} \int_{(1-r)^\gamma}^{\pi} \frac{d\theta}{(1-\rho+2\theta)^{2p}} \\ &> \frac{K\nu(r, 0, (1-r)^\gamma)^p(1-\rho)^p(1-r)^p}{(1-\rho+2(1-r)^\gamma)^{2p-1}}. \end{aligned}$$

In the case where $\gamma \geqq 1$ we put $\rho = 4r - 3$. Then

$$\nu(r, 0, (1-r)^\gamma)^p < KC(1-r)^{-1-q},$$

which gives (1.1). In the case where $0 < \gamma < 1$ we put $1 - \rho = (1-r)^\gamma$. Then the condition (2.2) is satisfied if $1 - r$ is sufficiently small, and we have

$$\nu(r, 0, (1-r)^\gamma)^p < KC(1-r)^{p(\gamma-1)-\gamma(q+1)},$$

which gives (1.2).

3. The proof of Theorem 2. In proving Theorem 2 we require the following elementary lemmas.

LEMMA 1. *If $a_j \geqq 0$ for $j = 1, 2, \ldots, N$, and $p > 1$, then*

$$\left[\sum_{j=1}^{N} a_j\right]^p \leqq N^{p-1} \sum_{j=1}^{N} a_j^p.$$

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LEMMA 2. *If* $\{z_n\}$ *is a Blaschke sequence and* $0 < r < 1$ *then*

$$\sum_{|z_n|>r} (1 - |z_n|) \leqq \int_r^1 n(t, \{z_n\})dt.$$

LEMMA 3. *If* $a > 0$, $b > 0$, *and* $p > 1$ *then*

$$\int_0^{2\pi} \frac{d\theta}{(a^2 + b^2 \sin^2\frac{1}{2}\theta)^p} < \frac{K(p)}{ba^{2p-1}}.$$

Without loss of generality we suppose $\frac{1}{2} < r < 1$ and $\frac{1}{2} < r_n < 1$, where again we denote z_n by $r_n e^{i\theta_n}$. Then the expression (2.1) gives immediately

$$\log|B(re^{i\theta}, \{z_n\})|^{-2} < \sum_{n=1}^{\infty} \frac{4(1-r)(1-r_n)}{(r-r_n)^2 + \sin^2\frac{1}{2}(\theta - \theta_n)}. \tag{3.1}$$

Let $B_1(re^{i\theta})$, $B_2(re^{i\theta})$, and $B_3(re^{i\theta})$ be respectively the subproducts of $B(re^{i\theta}, \{z_n\})$ for which

$$r_n \leqq 2r - 1,\ 2r - 1 < r_n < \tfrac{1}{2}(1 + r), \text{ and } r_n \geqq \tfrac{1}{2}(1 + r).$$

The inequality (3.1) leads to

$$\begin{aligned} |\log|B_1(re^{i\theta})||^p &< K\left(\sum_{r_n \leq 2r-1} \frac{(1-r)(1-r_n)}{\{(r-r_n)^2 + \sin^2\frac{1}{2}(\theta-\theta_n)\}}\right)^p \\ &< K(n(2r-1))^{p-1} \sum_{r_n \leq 2r-1} \frac{(1-r)^p(1-r_n)^p}{\{(1-r_n)^2 + \sin^2\frac{1}{2}(\theta-\theta_n)\}^p}, \end{aligned}$$

by application of Lemma 1 and the abbreviation $n(t) = n(t, \{z_n\})$. Hence, by Lemma 3, we have

$$\begin{aligned} \int_0^{2\pi} |\log|\, B_1(re^{i\theta})||^p d\theta &< K(n(2r-1))^{p-1} \sum_{r_n \leq 2r-1} (1-r)^p(1-r_n)^{1-p} \\ &< K(1-r)(n(2r-1))^p, \end{aligned} \tag{3.2}$$

since $0 < 2(1 - r) \leqq 1 - r_n$.

A similar consideration of $B_2(re^{i\theta})$ leads to

$$\begin{aligned} \int_0^{2\pi} |\log|B_2(re^{i\theta})||^p d\theta &< Kn(\tfrac{1}{2}(1+r))^{p-1} \\ &\quad \times \sum_{2r-1<r_n<\frac{1}{2}(1+r)} \int_0^{2\pi} \left\{\log\left(1 + \frac{2(1-r)^2}{\sin^2\frac{1}{2}(\theta-\theta_n)}\right)\right\}^p d\theta \\ &< Kn(\tfrac{1}{2}(1+r))^p(1-r). \end{aligned} \tag{3.3}$$

Finally, we note that

$$\begin{aligned} \log|B_3(re^{i\theta})| &< K \sum_{r_n \geq \frac{1}{2}(1+r)} \frac{(1-r_n)}{1-r} \\ &\leqq \frac{K}{1-r} \int_{\frac{1}{2}(1+r)}^1 n(t)dt \\ &< KC(1-r)^{-\alpha}. \end{aligned}$$

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Hence, since $p > 1$, we have

$$(3.4)\quad \int_0^{2\pi} |\log|B_3(re^{i\theta})||^p d\theta < KC^{p-1}(1-r)^{\alpha(1-p)} \int_0^{2\pi} |\log|B_3(re^{i\theta})||d\theta$$

$$< KC^{p-1}(1-r)^{\alpha(1-p)}$$

$$\times \sum_{r_n \geq \frac{1}{2}(1+r)} \int_0^{2\pi} \frac{(1-r)(1-r_n)}{(1-r)^2 + \sin^2\frac{1}{2}(\theta - \theta_n)}\, d\theta$$

$$< KC^p(1-r)^{1-\alpha p},$$

the change of order of the operations of integration and summation being possible by uniform convergence.

Substituting from (1.3) in (3.2) and (3.3), and noting that $B(z, \{z_n\}) = B_1(z)B_2(z)B_3(z)$, we obtain (1.4) by application of Minkowski's inequality.

References

1. T. M. Flett, *On the rate of growth of mean values of holomorphic and harmonic functions*, Proc. London Math. Soc. *20* (1970), 749–768.
2. G. R. MacLane and L. A. Rubel, *On the growth of Blaschke products*, Can. J. Math. *21* (1969), 595–601.
3. C. N. Linden, *On Blaschke products of restricted growth*, Pacific J. Math. *38* (1971), 501–513.

University College of Swansea,
Swansea, Wales