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STABILITY ANALYSIS OF A k-OUT-OF-N:G REPARABLE SYSTEM

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Abstract

A k-out-of-N:G reparable system with an arbitrarily distributed repair time is studied in this paper. We translate the system into an Abstract Cauchy Problem (ACP). Analysing the spectrum of the system operator helps us to prove the well-posedness and the asymptotic stability of the system.

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1. Introduction

The k-out-of-N:G system works well when at least k of the components work. Several different aspects of related problems have been investigated, such as in [3–5, 9]. Reference [5] formulated a mathematical model of the k-out-of-N system with common-cause shock (CCS) failure and studied the system with the assumption that the repair time of the failed system was arbitrarily distributed.

To the best of our knowledge, most of the research on the k-out-of n:G system only considers system availability and other reliable indexes. Thus far, researchers have not considered whether the availability of the system exists; or, if it does, whether the availability is asymptotically stable. Obviously, it is true when the repair time of the system is exponentially distributed. However, is it still true or not if the repair time of the system is arbitrarily distributed? This is still an open question, and completing the proof of this question is meaningful both in theory and in practice.

In this paper, using C_0-semigroup theory and spectral theory, we prove that the

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k-out-of-N:G reparable system with arbitrarily distributed repair time is asymptotically stable.

The Hille-Yosida theorem in reference [1] and the stability theorem (Theorem 14) in reference [6] will be used in proving the well-posedness and asymptotic stability of the system. To be self-contained, let us recall briefly the statement of these results.

THEOREM 1.1. *Let A be an operator on a Banach space X. Then A is the generator of a C_0 semigroup $T(t)$ if and only if*:

(1) *A is closed and $D(A)$ is dense in X*;
(2) $(0, +\infty) \subset \rho(A)$ *and for all* $\lambda \geq 0$, $\|R(\lambda; A)\| \leq 1/\lambda$.

THEOREM 1.2. *Let $\mathbf{X}$ be a Banach space and $T(t)$ be a uniformly bounded C_0-semigroup. Suppose that $\alpha_p(A) \cap i\mathbb{R} = \alpha_p(A^*) \cap i\mathbb{R} = \{0\}$, and $\{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma > 0$, or $\gamma = ia$, $a \neq 0$, $a \in \mathbb{R}\}$ belongs to the resolvent set of A. If the algebraic multiplicity of 0 in $\mathbf{X}^*$ is one, then the time-dependent solution of the system strongly converges to its static solution as $t \to \infty$.*

This paper is organised as follows: Section 2 describes the system. Well-posedness and the asymptotic stability of the system are proved in Sections 3 and 4 respectively. Section 5 concludes the paper.

2. System description

2.1. The model of a k-out-of-N:G redundant system The k-out-of-N:G reparable system presented in this paper consists of $N \geq 1$ identical units, $r \geq 1$ repair facilities and it requires k, $N \geq k \geq 1$, units to make the system operational. The following assumptions are associated with the model:

(1) The units in the system can fail individually or due to CCS failures;
(2) The repair rate of a unit when the system is in operation is constant;
(3) The failure rate when i units have failed is denoted by a_i and the chance of the occurrence of such failures is c_0;
(4) The failure rate of CCS from state i to state CCS is denoted by d_i and the chance of the occurrence of such failure is c_1 with $c_0 + c_1 = 1$;
(5) All failures are statistically independent;
(6) The repair time of the system is arbitrarily distributed;
(7) The repaired unit is as good as new.

2.2. Notation We denote by i the number of failed units, where $i = 0, 1, \ldots, N-k$ and by j the failed state of the system, where $j = N - k + 1$ means failure of the system and $j = N-k+2$ means failure of the system due to CCS failure. We denote by $p_i(t)$ the probability that the system is in state i, $i = 0, 1, \dots, N-k$, at time t and by $\mu_j(x)$ the repair rate of the repair time when the system is in state j and has elapsed repair time of x. We denote by $p_j(x, t)$ the probability that the failed system is in state j and has an elapsed repaired time of x. Here X_j are random variables representing the repair time when the system is in state j, $G_j(\cdot)$ is the distribution function of X_j, $g_j(\cdot)$ is the probability density function of X_j, and $E_j(x)$ is the mean repair time when the system is in state j and has an elapsed repaired time x. Also a_i is the failure rate of i units failed where the chance of occurrence of such failures is c_0; b is the constant repair rate of a unit; b_i is the $\min(i, r)b$ and d_i is the number of constant CCS failures from state i to state $N-k+2$ where the chance of occurrence of such failures is c_1.

Throughout this paper, we denote

$$W = \max_j \left\{ \sup_{x \in R^+} \mu_j(x) \right\} < \infty,$$

where $\int_0^x \mu_j(\tau)\,d\tau < \infty$ for any $x < \infty$, $\int_0^\infty \mu_j(x)\,dx = \infty$.

The transition diagram of the k-out-of-N:G redundant system with r repair facilities and the presence of chance CCS failures is depicted in Figure 1 below.

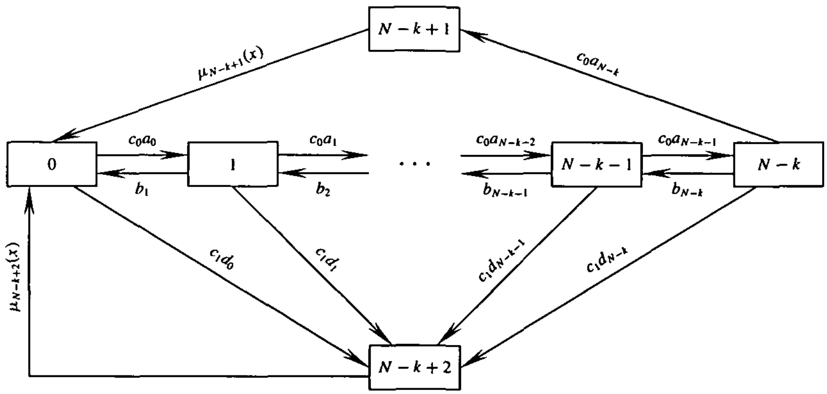

FIGURE 1. Transition diagram of the system.

2.3. System formulation The mathematical model associated with Figure 1 can be expressed as follows:

$$\frac{dp_0(t)}{dt} = -h_0p_0(t) + b_1p_1(t) + \sum_{j=N-k+1}^{N-k+2} \int_0^\infty p_j(x,t)\mu_j(x)\,dx, \tag{2.1}$$

$$\frac{dp_i(t)}{dt} = c_0 a_{i-1} p_{i-1}(t) - h_i p_i(t) + b_{i+1} p_{i+1}(t), \quad i = 1, \ldots, N-k-1, \tag{2.2}$$

$$\frac{dp_{N-k}(t)}{dt} = c_0 a_{N-k-1} p_{N-k-1}(t) - h_{N-k} p_{N-k}(t), \tag{2.3}$$

$$\frac{\partial p_j(x,t)}{\partial t} + \frac{\partial p_j(x,t)}{\partial x} = -\mu_j(x) p_j(x,t), \quad j = N-k+1, N-k+2, \tag{2.4}$$

where $h_0 = c_0 a_0 + c_1 d_0$ and $h_n = c_0 a_n + b_n + c_1 d_n$, $n = 1, \ldots N-k$.

The boundary conditions are given by

$$p_{N-k+1}(0,t) = c_0 a_{N-k} p_{N-k}(t) \quad \text{and} \tag{2.5}$$

$$P_{N-k+2}(0,t) = \sum_{i=0}^{N-k} c_1 d_i p_i(t) \tag{2.6}$$

and the initial values are given by

$$p_0(0) = 1, \quad p_i(0) = 0, \quad p_j(x,0) = 0, \tag{2.7}$$

where $i = 1, \ldots, N-k$ and $j = N-k+1, N-k+2$. We formulate this model as an Abstract Cauchy Problem (ACP) in Banach space. For simplicity, we introduce

$$A = \operatorname{diag}\left(-h_0, -h_1, \ldots, -h_{N-k}, -\frac{d}{dx} - \mu_{N-k+1}(x), -\frac{d}{dx} - \mu_{N-k+2}(x)\right)$$

and

$$E = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots & 0 & 0 & e_{N-k+1} & e_{N-k+2} \\ c_0 a_0 & 0 & b_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{N-k} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c_0 a_{N-k-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where e_j stands for $\int_0^\infty \cdot \mu_j(x)\,dx$, $j = N-k+1, N-k+2$. We take the state space $\mathbf{X}$ as follows:

$$\mathbf{X} = \left\{ \vec{y} \in \mathbb{R}^{N-k+1} \times L^1[0,\infty) \times L^1[0,\infty) \;\middle|\; \|\vec{y}\| = \sum_{i=0}^{N-k} |y_i| + \sum_{j=N-k+1}^{N-k+2} \|y_j(x)\|_{L^1[0,\infty)} \right\}.$$

It is obvious that $(\mathbf{X}, \|\cdot\|)$ is a Banach space. The domain of the operator A is $D(A) = \{\vec{p} \in \mathbf{X} \mid dp_j(x)/dx + \mu_j(x) p_j(x) \in L^1[0,\infty)$, $p_j(x)$ are absolutely continuous functions, $j = N-k+1, N-k+2$, and satisfy $p_{N-k+1}(0) = c_0 a_{N-k} p_{N-k}$, $p_{N-k+2}(0) = \sum_{i=0}^{N-k} c_1 d_i p_i\}$.

Then Equations (2.1)–(2.6) can be written as an ACP in the Banach space $\mathbf{X}$ as

$$\begin{cases} \dfrac{d\vec{p}(t)}{dt} = (A+E)\vec{p}(t), \quad t \geq 0, \\ \vec{p}(0) = (1, 0, \dots, 0), \\ \vec{p}(t) = \big(p_0(t), p_1(t), \dots, p_{N-k}(t), p_{N-k+1}(x,t), p_{N-k+2}(x,t)\big). \end{cases} \tag{2.8}$$

3. Well-posedness of the system

In Section 2, we formulated the system as an ACP (see Equation (2.8)). Obviously, if we can prove that the system operator $(A+E)$ generates a C_0-semigroup, we can deduce that the ACP has a unique solution [2]. We begin with proving the following propositions.

THEOREM 3.1. *The operator $A+E$ generates a C_0-semigroup $T(t)$.*

PROOF. It can be checked that the operator E is bounded. By perturbation theory [2], we know that if the operator A generates a C_0-semigroup, then $A+E$ will generate a C_0-semigroup. Thus, by Theorem 1.1, we only need to prove: (1) $\gamma \in \rho(A)$ and $\|(\gamma I - A)^{-1}\| \leq 1/\gamma$ when $\gamma > 0$, and (2) $D(A)$ is dense in $\mathbf{X}$.

(1) $\gamma \in \rho(A)$ and $\|(\gamma I - A)^{-1}\| \leq 1/\gamma$ when $\gamma > 0$.

For any $\vec{y} = (y_0, \dots, y_{N-k}, y_{N-k+1}(x), y_{N-k+2}(x)) \in \mathbf{X}$, consider the equation $(\gamma I - A)\vec{p} = \vec{y}$, that is,

$$(\gamma + h_i)p_i = y_i, \tag{3.1}$$

$$\frac{dp_j(x)}{dx} = -(\gamma + \mu_j(x))p_j(x) + y_j(x), \tag{3.2}$$

$$p_{N-k+1}(0) = c_0 a_{N-k} p_{N-k}, \qquad p_{N-k+2}(0) = \sum_{i=0}^{N-k} c_1 d_i p_i, \tag{3.3}$$

where $i = 0, 1, \dots, N-k$ and $j = N-k+1, N-k+2$. Solving (3.1)–(3.2) with the help of (3.3), we can obtain that

$$p_i = \frac{y_i}{\gamma + h_i}, \quad p_j(x) = p_j(0)e^{-\int_0^x (\gamma+\mu_j(\xi))d\xi} + \int_0^x e^{-\int_\tau^x (\gamma+\mu_j(\xi))d\xi} y_j(\tau)\,d\tau,$$

where $i = 0, 1, \dots, N-k$ and $j = N-k+1, N-k+2$. By the Fubini theorem, we have

$$\|\vec{p}\| = \sum_{i=0}^{N-k} |p_i| + \sum_{j=N-k+1}^{N-k+2} \|p_j(x)\|_{L^1[0,\infty)}$$

$$\leq \sum_{i=0}^{N-k}|p_i| + \sum_{j=N-k+1}^{N-k+2}\left\{|p_j(0)|\int_0^\infty e^{-\gamma x}\,dx + \int_0^\infty |y_j(\tau)|\,d\tau \int_\tau^\infty e^{-\gamma(x-\tau)}\,dx\right\}$$
$$\leq \frac{1}{\gamma}\|\vec{y}\|. \tag{3.4}$$

Equation (3.4) shows that $(\gamma I - A)^{-1} : \mathbf{X} \to \mathbf{X}$ exists and $\|(\gamma I - A)^{-1}\| \leq 1/\gamma$ when $\gamma > 0$.

(2) The domain $D(A)$ is dense in $\mathbf{X}$.

If we set $L = \big\{(p_0, \dots, p_{N-k}, p_{N-k+1}(x), p_{N-k+2}(x)) \mid p_j(x) \in C_0^\infty[0,\infty)$, and there exist numbers c_j such that $p_j(x) = 0, x \in [0, c_j], j = N-k+1, N-k+2\big\}$, it is obvious that L is dense in $\mathbf{X}$. So it suffices to prove that $D(A)$ is dense in L.

Take $\vec{p} \in L$. Then there are $c_j > 0$, such that $p_j(x) = 0$, $x \in [0, c_j]$, $j = N-k+1, N-k+2$. It follows that $p_j(x) = 0, x \in [0, 2s]$, where $0 < 2s < \min\{c_j\}$. Set

$$f^s(0) = \big(p_0, p_1, \dots, p_{N-k}, f^s_{N-k+1}(0), f^s_{N-k+2}(0)\big)$$
$$= \left(p_0, p_1, \dots, p_{N-k}, c_0 a_{N-k} p_{N-k}, \sum_{i=0}^{N-k} c_1 d_i p_i\right),$$
$$f^s(x) = \big(p_0, p_1, \dots, p_{N-k}, f^s_{N-k+1}(x), f^s_{N-k+2}(x)\big),$$
$$f_j^s(x) = \begin{cases} f_j^s(0)(1 - x/s)^2, & x \in [0, s), \\ -\mu_j(x-s)^2(x-2s)^2, & x \in [s, 2s), \\ p_j(x), & x \in [2s, \infty), \end{cases}$$

where $j = N-k+1, N-k+2$ and

$$\mu_j = \frac{f_j^s(0)\int_0^s \mu_j(x)(1-x/s)^2\,dx}{\int_s^{2s}\mu_j(x)(x-s)^2(x-2s)^2\,dx}.$$

Then it is easy to verify that $f^s(x) \in D(A)$, moreover

$$\|\vec{p} - f^s(x)\| = \sum_{j=N-k+1}^{N-k+2}\int_0^\infty |p_j(x) - f_j^s(x)|\,dx$$
$$= \sum_{j=N-k+1}^{N-k+2}\int_0^{2s} |p_j(x) - f_j^s(x)|\,dx$$
$$= \sum_{j=N-k+1}^{N-k+2}\left(|f_j^s(0)|\frac{s}{3} + |\mu_j|\frac{s^5}{30}\right) \xrightarrow{s\to 0} 0.$$

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This shows that $D(A)$ is dense in L. In other words, $D(A)$ is dense in $\mathbf{X}$.

So, the operator A generates a C_0-semigroup. And it is easy to check that

$$E : \mathbf{X} \to \mathbf{X}, \quad \|E\| \le \max\left\{c_0 a_0, c_0 a_i + b_i, b_{N-k}, W\right\}, \quad i = 1, \ldots N-k-1$$

is a bounded linear operator (here, $W = \sup_{x \in \mathbb{R}^+} \mu_j(x)$, $j = N-k+1, N-k+2$). Thus $A+E$ generates a C_0-semigroup $T(t)$. □

Further, in order to reflect the physical meaning of the solution of the ACP, we introduce the following theorem.

THEOREM 3.2. *$T(t)$ is a positive C_0-semigroup of contraction.*

PROOF. (1) $T(t)$ is a positive C_0-semigroup.

By the solution of Equations (3.1)–(3.3), we know that $\vec{p}$ is a nonnegative vector if $\vec{y}$ is a nonnegative vector. In other words, $(\gamma I - A)^{-1}$ is a positive operator. It is simple to show that E is a positive operator. Note that

$$(\gamma I - A - E)^{-1} = [I - (\gamma I - A)^{-1} E]^{-1} (\gamma I - A)^{-1}. \tag{3.5}$$

When $\gamma > \max\{c_0 a_0, c_0 a_i + b_i, b_{N-k}, W\}$, it follows from Equation (3.4) that $\|(\gamma I - A)^{-1} E\| < 1$. So, $[I - (\gamma I - A)^{-1} E]^{-1}$ exists and is bounded and

$$[I - (\gamma I - A)^{-1} E]^{-1} = \sum_{k=0}^{\infty} [(\gamma I - A)^{-1} E]^k. \tag{3.6}$$

Therefore $[I - (\gamma I - A)^{-1} E]^{-1}$ is a positive operator. By Equations (3.5) and (3.6) we get that $(\gamma I - A - E)^{-1}$ is a positive operator. By [2], we know that $A+E$ generates a positive C_0-semigroup.

(2) $T(t)$ is a positive C_0-semigroup of contraction.

For any $\vec{p} \in D(A)$, we take

$$Q_p = \left(\frac{[p_0]^+}{p_0}, \frac{[p_1]^+}{p_1}, \ldots, \frac{[p_{N-k}]^+}{p_{N-k}}, \frac{[p_{N-k+1}(x)]^+}{p_{N-k+1}(x)}, \frac{[p_{N-k+2}(x)]^+}{p_{N-k+2}(x)}\right).$$

Here

$$[p_i]^+ = \begin{cases} p_i, & p_i > 0, \\ 0, & p_0 \le 0, \end{cases} \qquad i = 0, 1, \ldots, N-k,$$

$$[p_j(x)]^+ = \begin{cases} p_j(x) & p_j(x) > 0, \\ 0, & p_j(x) \le 0, \end{cases} \qquad j = N-k+1, N-k+2.$$

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For any $\vec{p} \in D(A)$ and Q_p, we have

$$\begin{aligned}
\langle (A+E)\vec{p}, Q_p\rangle &= \left\{-h_0 p_0 + b_1 p_1 + \sum_{j=N-k+1}^{N-k+2} \int_0^\infty p_j(x)\mu_j(x)\,dx\right\} \frac{[p_0]^+}{p_0} \\
&\quad + \sum_{i=1}^{N-k-1} \{c_0 a_{i-1} p_{i-1} - h_i p_i + b_{i+1} p_{i+1}\} \frac{[p_i]^+}{p_i} \\
&\quad + \{c_0 a_{N-k-1} p_{N-k-1} - h_{N-k} p_{N-k}\} \frac{[p_{N-k}]^+}{p_{N-k}} \\
&\quad - \sum_{j=N-k+1}^{N-k+2} \int_0^\infty \left\{\frac{dp_j(x)}{dx} + \mu_j(x) p_j(x)\right\} \frac{[p_j(x)]^+}{p_j(x)}\, dx \\
&\le -h_0[p_0]^+ + b_1[p_1]^+ \\
&\quad + \sum_{i=1}^{N-k-1} \left\{c_0 a_{i-1}[p_{i-1}]^+ - h_i[p_i]^+ + b_{i+1}[p_{i+1}]^+\right\} \\
&\quad + c_0 a_{N-k-1}[p_{N-k-1}]^+ - h_{N-k}[p_{N-k}]^+ - c_0 a_{N-k}[p_{N-k}]^+ \\
&\quad - \sum_{i=0}^{N-k} c_1 d_i [p_i]^+ = 0. \qquad (3.7)
\end{aligned}$$

From the definition of a dissipative operator and (3.7), we know that $(A+E)$ is a dissipative operator. By Philips theory [2], we derive that $(A+E)$ generates a positive C_0-semigroup of contraction. Because a C_0-semigroup is unique [8], we know this positive contraction C_0-semigroup is just $T(t)$. □

THEOREM 3.3. *The system* (2.1)–(2.7) *has a unique nonnegative time-dependent solution* $\vec{p}(x,t)$, *which satisfies* $\|\vec{p}(\cdot,t)\| = 1$, $t \in [0,\infty)$.

PROOF. From Theorem 3.2 and reference [8], we know that the system (2.1)–(2.7) has a unique nonnegative solution $\vec{p}(x,t)$ and it can be expressed as

$$\vec{p}(x,t) = T(t)(1,0,\ldots,0). \qquad (3.8)$$

By Theorem 3.2 and Equation (3.8) we obtain that

$$\|\vec{p}(\cdot,t)\| = \|T(t)(1,0,\ldots,0)\| \le \|(1,0,\ldots,0)\| = 1, \quad t \in [0,\infty).$$

On the other hand, since $(1,0,\ldots,0) \in X$, so $\vec{p}(x,t) \in D(A+E)$, and $p_j(x,t)$, $j = N-k+1, N-k+2$ is a mild solution of the system and satisfies system (2.1)–(2.7). Then we have

$$\frac{d}{dt}\|\vec{p}(\cdot,t)\| = \sum_{i=0}^{N-k} \frac{dp_i(t)}{dt} + \sum_{j=N-k+1}^{N-k+2} \frac{d}{dt}\int_0^\infty p_j(x,t)dx = 0.$$

Hence $\|\vec{p}(\cdot, t)\| = \|\vec{p}(0)\| = 1$. This just reflects the physical meaning of $\vec{p}(x, t)$. $\square$

4. Stability analysis of the system

In this section, we systematically study the stability of the k-out-of-N:G reparable system. We will prove that there exists a nonnegative steady solution of the system, and the time-dependent solution converges to this solution when time t tends to infinity. Therefore the system is asymptotically stable. We begin with proving the following lemmas.

LEMMA 4.1. $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}\,dx = \int_0^\infty x g_j(x)\,dx$, *for* $j = N-k+1, N-k+2$.

PROOF. By [7], we know that

$$\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}\,dx = \int_0^\infty [1 - G_j(x)]\,dx, \quad (G(0) = 0),$$
$$\int_0^\infty x g_j(x)\,dx = \int_0^\infty [1 - G_j(x)]\,dx, \quad (G(\infty) = 1).$$

So $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}\,dx = \int_0^\infty x g_j(x)\,dx$, and we complete the proof of Lemma 4.1. $\square$

LEMMA 4.2. *There exist* $K \in \mathbb{R}$, *such that* $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}\,dx \le K$.

PROOF. Because the device is reparable when it fails, so the mean of the random variables X_j, $j = N-k+1, N-k+2$, exists and satisfies

$$E(X_j) = \int_0^\infty x g_j(x)\,dx = \int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}\,dx.$$

So there exist $K_j \in \mathbb{R}$, such that $E(X_j) \le K_j$. Let $K = \max_{j=N-k+1,N-k+2}\{K_j\}$, so $E(X_j) \le K$, that is, $\int_0^\infty e^{-\int_0^x \mu_j(\xi)d\xi}\,dx \le K$. $\square$

LEMMA 4.3. *For any* $t \ge 0$, $\int_t^\infty e^{-\int_t^x \mu_j(\xi)d\xi}\,dx \le K'$.

Let $G_j^t(x) = p\{X_j - t \le x | X_j > t\} = (G_j(x+t) - G_j(t))/(1 - G_j(t))$, $x \ge 0$. So $1 - G_j^t(x) = (1 - G_j(x+t))/(1 - G_j(t))$, then

$$\begin{aligned} E_j(t) = E\{X_j - t | X_j > t\} &= \int_0^\infty x\,dG_j^t(x) = \int_0^\infty [1 - G_j^t(x)]\,dx \\ &= \int_0^\infty \frac{1 - G_j(x+t)}{1 - G_j(t)}\,dx = \int_t^\infty \frac{1 - G_j(x)}{1 - G_j(t)}\,dx \end{aligned}$$

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$$= \int_t^\infty e^{-\int_0^x \mu_j(\xi)\,d\xi} \cdot e^{\int_0^t \mu_j(\xi)\,d\xi}\,dx = \int_t^\infty e^{-\int_t^x \mu_j(\xi)\,d\xi}\,dx.$$

Here, $E_j(t)$ means the expected time of the system to be repaired with elapsed repair time t. The system is reparable however long the elapsed repair time may be. That is, there exist $K'_j \in \mathbb{R}$, $K'_j < \infty$, such that $E_j(t) \le K'_j$. Let

$$K' = \max_{j=N-k+1,\dots,N-k+1+M}\{K'_j\},$$

then for any t, $E_j(t) \le K'$, that is, $\int_t^\infty e^{-\int_t^x \mu_j(\xi)\,d\xi}\,dx \le K'$.

It is obvious that Lemma 4.2 is a special case of Lemma 4.3.

THEOREM 4.4. *0 is the simple eigenvalue of $A + E$.*

PROOF. Consider $(A + E)\vec{p} = 0$ in terms of the following equations:

$$-h_0 p_0 + b_1 p_1 + \sum_{j=N-k+1}^{N-k+2} \int_0^\infty \mu_j(x) p_j(x)\,dx = 0, \tag{4.1}$$

$$c_0 a_{i-1} p_{i-1} - h_i p_i + b_{i+1} p_{i+1} = 0, \quad i = 1, \dots, N-k-1, \tag{4.2}$$

$$c_0 a_{N-k-1} p_{N-k-1} - h_{N-k} p_{N-k} = 0, \tag{4.3}$$

$$\frac{dp_j(x)}{dx} + \mu_j(x) p_j(x) = 0, \quad j = N-k+1, N-k+2, \tag{4.4}$$

$$p_{N-k+1}(0) = c_0 a_{N-k} p_{N-k}, \quad p_{N-k+2}(0) = \sum_{i=0}^{N-k} c_1 d_i p_i. \tag{4.5}$$

Solving (4.1)–(4.4) with the help of (4.5), we obtain that

$$p_j(x) = p_j(0) e^{-\int_0^x \mu_j(\xi)\,d\xi}, \quad j = N-k+1, N-k+2. \tag{4.6}$$

Substitution of (4.6) into (4.1) with the help of (4.2)–(4.5) yields that

$$(-h_0 + c_1 d_0) p_0 + (b_1 + c_1 d_1) p_1 + \sum_{i=2}^{N-k-1} c_1 d_i p_i + (a_{N-k} + c_1 d_{N-k}) p_{N-k} = 0,$$

$$c_0 a_{i-1} p_{i-1} - h_i p_i + b_{i+1} p_{i+1} = 0, \quad i = 1, \dots, N-k-1,$$

$$c_0 a_{N-k-1} p_{N-k-1} - h_{N-k} p_{N-k} = 0.$$

It is easy to check that the determinant coefficient matrix of the above equations equals 0, and, if $p_0 > 0$, then $p_i > 0$, $(i = 1, \dots, N-k)$. And

$$p_j(x) = p_j 0 e^{-\int_0^x \mu_j(\xi)\,d\xi} > 0, \quad j = N-k+1, N-k+2.$$

Using Lemma 4.2, we can deduce that $p_j(x) \in L^1[0, +\infty)$. So the vector

$$\vec{p} = (p_0, p_1, \dots, p_{N-k}, p_{N-k+1}(x), p_{N-k+2}(x)) \tag{4.7}$$

is the eigenvector corresponding to 0 for the system operator $(A + E)$. Taking $Q = (1, 1, \dots, 1)$, we have

$$\langle P, Q\rangle = \sum_{i=0}^{N-k} p_i + \sum_{j=N-k+1}^{N-k+2} \int_0^\infty p_j(x)\, dx > 0.$$

And for any $\vec{p} \in D(A + E)$, $\langle (A + E)\vec{p}, Q\rangle = 0$. So 0 is the simple eigenvalue of the system operator $(A + E)$. □

THEOREM 4.5. *The set* $\mathscr{S} = \{r \in \mathbb{C} \mid \operatorname{Re} r > 0$, *or* $r = ia$, $a \in \mathbb{R}$, $a \neq 0\}$ *belongs to the resolvent set of operator* $A + E$.

PROOF. For any $r \in \mathscr{S}$, solve $(A + E)\vec{p} = \vec{y}$ in terms of the following equations:

$$(r + h_0)p_0 - b_1 p_1 - \sum_{j=N-k+1}^{N-k+2} \int_0^\infty \mu_j(x) p_j(x) dx = y_0, \tag{4.8}$$

$$- c_0 a_{i-1} p_{i-1} + (r + h_i)p_i - b_{i+1} p_{i+1} = y_i, \quad i = 1, \dots, N - k - 1, \tag{4.9}$$

$$- c_0 a_{N-k-1} p_{N-k-1} + (r + h_{N-k}) p_{N-k} = y_{N-k}, \tag{4.10}$$

$$\frac{dp_j(x)}{dx} + (r + \mu_j(x))p_j(x) = y_j(x), \quad j = N - k + 1, N - k + 2, \tag{4.11}$$

$$p_{N-k+1}(0) = c_0 a_{N-k} p_{N-k}, \quad p_{N-k+2}(0) = \sum_{i=0}^{N-k} c_1 d_i p_i. \tag{4.12}$$

Solving Equations (4.8)–(4.11), with the help of Equations (4.12), we can obtain that

$$p_j(x) = p_j(0)e^{-\int_0^x (r+\mu_j(\xi))\, d\xi} + \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))\, d\xi} y_j(\tau)\, d\tau.$$

For $y_j(x) \in L^1[0, \infty)$, combining with Lemma 4.3, we can derive that

$$\begin{aligned}
\int_0^\infty \left| \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))\, d\xi} y_j(\tau)\, d\tau \right| dx &\le \int_0^\infty dx \int_0^x e^{-\int_\tau^x \mu_j(\xi)\, d\xi} |y_j(\tau)|\, d\tau \\
&= \int_0^\infty |y_j(\tau)|\, d(\tau) \int_\tau^\infty e^{-\int_\tau^x \mu_j(\xi)\, d\xi}\, dx \\
&\le \|y_j\|_{L^1[0,\infty)} \cdot K'.
\end{aligned}$$

So, $p_j(x) \in L^1[0,\infty)$, $j = N-k+1, N-k+2$. Substitution into Equation (4.8) with the help of Equatons (4.9)–(4.10), yields that

$$(r + h_0 - c_1 d_0 g_{N-k+2}) p_0 - (b_1 + c_1 d_1 g_{N-k+2}) p_1 - \sum_{i=2}^{N-k-1} c_1 d_i g_{N-k+2} p_i$$
$$- (c_0 a_{N-k} g_{N-k+1} + c_1 d_{N-k} g_{N-k+2}) p_{N-k} = y_0 + \sum_{j=N-k+1}^{N-k+2} G_j, \tag{4.13}$$
$$- c_0 a_{i-1} p_{i-1} + (r + h_i) p_i - b_{i+1} p_{i+1} = y_i, \quad i = 1, \dots, N-k-1, \tag{4.14}$$
$$- c_0 a_{N-k-1} p_{N-k-1} + (r + h_{N-k}) p_{N-k} = y_{N-k}, \tag{4.15}$$

where

$$g_j = \int_0^\infty \mu_j(x) e^{-\int_0^x (r+\mu_j(\xi))\,d\xi}\, dx,$$
$$G_j = \int_0^\infty \mu_j(x)\, dx \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))\,d\xi} y_j(\tau)\, d\tau.$$

When $\operatorname{Re} r > 0$, or $r = ia$, $a \in \mathbb{R}$, $a \neq 0$, we have $|g_j| \geq 1$. It then follows that the coefficient matrix of Equations (4.13)–(4.15) is a strictly diagonally dominant matrix. So Equations (4.13)–(4.15) have a unique solution. Assuming that $\{\hat{p}_0, \hat{p}_1, \dots, \hat{p}_{N-k}\}$ is the unique solution of (4.13)–(4.15), then $\{\hat{p}_0, \hat{p}_1, \dots, \hat{p}_{N-k}\}$ and

$$\hat{p}_j(x) = \hat{p}_j(0) e^{-\int_0^x (r+\mu_j(\xi))\,d\xi} + \int_0^x e^{-\int_\tau^x (r+\mu_j(\xi))\,d\xi} y_j(\tau)\, d\tau, \tag{4.16}$$

$j = N-k+1, N-k+2$, is the unique solution of Equations (4.8)–(4.12). So $R(rI - A - E) = \mathbf{X}$. And because $(rI - A - E)$ is a closed operator, we can deduce that $(rI - A - E)^{-1}$ exists and is bounded. In other words, $\mathscr{S}$ belongs to the resolvent set of the system operator $A + E$. This completes the proof of Theorem 4.5. □

COROLLARY 4.6. *System* (2.1)–(2.6) *has a nonnegative stable solution.*

In Theorem 4.5, we proved that all the spectrum of $A + E$ lies in the left half-plane and there is no spectrum on the imaginary axis except 0. We observe that $\vec{p}$ in (4.7) is the eigenvector corresponding to 0 of $A + E$. It is obvious that $\vec{p}$ is nonnegative. Hence $\vec{p}$ is the nonnegative steady solution of the system.

In the same manner as that used above, we can easily prove that $\alpha_p((A+E)^*) \cap i\mathbb{R} = \{0\}$ and the algebraic multiplicity of 0 in $\mathbf{X}^*$ is one. Due to space limitations, we do not give the proofs.

Theorem 4.7. *Let $\hat{p}$ be the nonnegative eigenvector corresponding to* 0 *which satisfies* $\|\hat{p}\| = 1$. *Let* $Q = (1, \dots, 1)$, *then the time-dependent solution* $\hat{p}(\cdot, t)$ *of the system tends to the steady solution* $\hat{p}$:

$$\lim_{t\to\infty} \hat{p}(\cdot, t) = \langle \vec{p}_0, Q\rangle \hat{p} = \hat{p},$$

where $\vec{p}_0$ *is the initial value of the system.*

By Theorem 1.2, we know that Theorem 4.7 holds. Thus we proved that $\hat{p}$, the eigenvector corresponding to 0 of the system operator $A+E$, is the unique nonnegative steady solution of this $k - out - of - N{:}G$ system, and $\lim_{t\to\infty} \hat{p}(\cdot, t) = \hat{p}$.

5. Concluding remarks

This paper introduced and analysed the well-posedness and the asymptotic stability of a k-out-of-N:G reparable system with CCS failure. We used a C_0-semigroup to prove the main results. This paper provides a strict mathematic proof for reliability of a k-out-of-N:G reparable system.

In this paper, we also point out that the system has a unique non-negative stable solution, which is just the eigenvector corresponding to 0 of the system operator $(A + E)$. Furthermore, the normalisation of this eigenvector is just the steady-state availability of the system.

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