CONCERNING THE MINIMUM FUNCTION OF A STOCHASTIC MATRIX

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A square matrix is said to be stochastic if its elements are non-negative and if each of its row sums is equal to one. Thus $\lambda = 1$ is always an eigenvalue of a stochastic matrix.

It is the intent of this paper to establish the theorem stated below. Though the result itself is well known (e.g. [1, pp. 102-104]) it is believed that the proof given is new. In any event it is self contained.

Let $A = (a_{ij})$ be an $m \times n$ matrix and let u and v be positive integers such that $1 \leq u \leq m$, $1 \leq v \leq n$. Let α denote a strictly increasing sequence of u integers $(i_1, \ldots, i_u)$ chosen from $1, \ldots, m$, and let β denote a strictly increasing sequence of v integers $(j_1, \ldots, j_v)$ chosen from $1, \ldots, n$. Then $A[\alpha|\beta]$ is that submatrix of A with rows indexed by α and columns indexed by β. $A[\alpha|\beta)$ is the submatrix of A with rows indexed by α and columns indexed by the complement of β in $\{1, 2, \ldots, n\}$. $A(\alpha|\beta]$ and $A(\alpha|\beta)$ are defined analogously.

We shall write A' for the transpose of A.

THEOREM. Let A be an $n \times n$ stochastic matrix. Then one is a simple zero of the minimum function of A.

Proof. Given y such that $(A' - I)^2 y = 0$, set $x = (A' - I)y$. It suffices to show that $x = 0$. In any event we can assume that x and y are real. Then, since $(A' - I)x = 0$, we have

$$(1) \qquad \sum_{i=1}^{n} a_{ij} x_i = x_j, \quad j = 1, \ldots, n$$

and thus $|x_j| \leq \sum_{i=1}^{n} a_{ij} |x_i|$. If strict inequality holds for

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any j, we would have

$$\sum_{j=1}^{n} |x_j| < \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}|x_i| = \sum_{i=1}^{n} |x_i|,$$

a contradiction. Thus we have

$$(2) \qquad \sum_{i=1}^{n} a_{ij}|x_i| = |x_j|, \quad j = 1, \ldots, n.$$

Let $E = \{i | x_i > 0\}$, $F = \{i | x_i < 0\}$, and $H = \{i | x_i = 0\}$. Then (2) may be written

$$(3) \qquad \sum_{i \varepsilon E} a_{ij} x_i - \sum_{i \varepsilon F} a_{ij} x_i = \begin{cases} x_j & \text{if } j \varepsilon E \\ -x_j & \text{if } j \varepsilon F \\ 0 & \text{if } j \varepsilon H \end{cases}$$

while (1) may be written as

$$(4) \qquad \sum_{i \varepsilon E} a_{ij} x_i + \sum_{i \varepsilon F} a_{ij} x_i = x_j, \quad j = 1, \ldots, n.$$

It readily follows from (3) and (4) that $\sum_{i \varepsilon F} a_{ij} x_i = 0$ if $j \varepsilon E$, $\sum_{i \varepsilon E} a_{ij} x_i = 0$ if $j \varepsilon F$, and $\sum_{i \varepsilon E} a_{ij} x_i = \sum_{i \varepsilon F} a_{ij} x_i = 0$ if $j \varepsilon H$. Thus $A[F|E] = 0$, $A[E|F] = 0$, and $A(H|H] = 0$.

The definition of x leads to the equation

$$(5) \qquad \sum_{i=1}^{n} a_{ij} y_i = y_j + x_j, \quad j = 1, \ldots, n.$$

Since $A(H|H] = 0$, this becomes

$$(6) \qquad \sum_{i \varepsilon H} a_{ij} y_i = y_j, \quad j \varepsilon H.$$

Clearly $\sum_{j \varepsilon H} a_{ij} \leq 1$ for any i. Suppose for some

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$i_o \varepsilon H$, $\sum_{j \varepsilon H} a_{i_o j} < 1$ and $y_{i_o} \neq 0$. Then we would have, using (6),

$$\sum_{j \varepsilon H} a_{i_o j} |y_{i_o}| < |y_{i_o}| \Rightarrow \sum_{i \varepsilon H} \sum_{j \varepsilon H} a_{ij} |y_i| < \sum_{i \varepsilon H} |y_i|$$

$$= \sum_{j \varepsilon H} |y_j| \leq \sum_{j \varepsilon H} \sum_{i \varepsilon H} a_{ij} |y_i| ,$$

a contradiction. Thus $i_o \varepsilon H$, $\sum_{j \varepsilon H} a_{i_o j} < 1 \Rightarrow y_{i_o} = 0$.

If $i_o \varepsilon H$, and $\sum_{j \varepsilon H} a_{i_o j} = 1$, certainly $a_{i_o j} = 0$ for $j \varepsilon E \cup F$.

It follows that $a_{ij} y_i = 0$ when $i \varepsilon H$, $j \varepsilon E \cup F$, and thus

(7) $$\sum_{i \varepsilon H} a_{ij} y_i = 0, \quad j \varepsilon E \cup F.$$

Since $A[F|E] = 0$, we have from (5) and (7)

$$\sum_{i \varepsilon E} a_{ij} y_i = y_j + x_j, \quad j \varepsilon E;$$

whence

(8) $$\sum_{j \varepsilon E} \sum_{i \varepsilon E} a_{ij} y_i = \sum_{j \varepsilon E} y_j + \sum_{j \varepsilon E} x_j .$$

But $A[E|F] = 0$ and $A(H|H] = 0$ and therefore

$$\sum_{j \varepsilon E} \sum_{i \varepsilon E} a_{ij} y_i = \sum_{j=1}^{n} \sum_{i \varepsilon E} a_{ij} y_i = \sum_{i \varepsilon E} y_i .$$

Thus (8) yields $\sum_{j \varepsilon E} x_j = 0$ and E is seen to be void.

Similarly F is empty, and thus for all i, $i \varepsilon H$. This means that $x = 0$, as was to be proved.

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REFERENCES

1. F. Gantmacher, (translated by J. Bremner), Applications of the theory of matrices, Interscience, New York - London, 1959.

2. M. Marcus and H. Minc, A survey of matrix theory and matrix inequalities, Allyn and Bacon, Boston, 1964.

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