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THE REFLEXIVITY INDEX OF A LATTICE OF SETS

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Abstract

We obtain a formula for the reflexivity index of a finite lattice of sets and of various types of infinite lattices of sets.

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1. Introduction

Let 2^X denote the Boolean algebra of all subsets of a set X and let X^X denote the semigroup of all endomorphisms on X, that is, functions that map X into itself. A subset A of X is *invariant* under an endomorphism f on X if $f(A) \subseteq A$, that is, $f(x) \in A$ for all $x \in A$. For any $\mathcal{L} \subseteq 2^X$ and any $\mathcal{F} \subseteq X^X$, we define

$$\begin{aligned} \operatorname{Alg} \mathcal{L} &= \{f \in X^X : f(A) \subseteq A \text{ for all } A \in \mathcal{L}\} \quad \text{and} \\ \operatorname{Lat} \mathcal{F} &= \{A \in 2^X : f(A) \subseteq A \text{ for all } f \in \mathcal{F}\}. \end{aligned}$$

That is, $\operatorname{Alg} \mathcal{L}$ is the set of all endomorphisms on X that leave each subset in $\mathcal{L}$ invariant, and $\operatorname{Lat} \mathcal{F}$ is the set of all subsets of X that are invariant under each endomorphism in $\mathcal{F}$.

We say that a subset $\mathcal{L}$ of 2^X is *reflexive* if $\mathcal{L} = \operatorname{Lat} \operatorname{Alg} \mathcal{L}$. Since $\operatorname{Lat} \mathcal{F} = \operatorname{Lat} \operatorname{Alg} \operatorname{Lat} \mathcal{F}$ for any $\mathcal{F} \subseteq X^X$, it follows that $\mathcal{L}$ is reflexive if and only if $\mathcal{L} = \operatorname{Lat} \mathcal{F}$ for some $\mathcal{F} \subseteq X^X$. Similarly, a subset $\mathcal{F}$ of X^X is *reflexive* if $\mathcal{F} = \operatorname{Alg} \operatorname{Lat} \mathcal{F}$. Since $\operatorname{Alg} \mathcal{L} = \operatorname{Alg} \operatorname{Lat} \operatorname{Alg} \mathcal{L}$ for any $\mathcal{L} \subseteq 2^X$, $\mathcal{F}$ is reflexive if and only if $\mathcal{F} = \operatorname{Alg} \mathcal{L}$ for some $\mathcal{L} \subseteq 2^X$.

For any family $\mathcal{F}$ of endomorphisms on X, $\operatorname{Lat} \mathcal{F}$ is closed under arbitrary intersections and unions and contains the trivial subsets $\varnothing$ and X. So, a reflexive family of subsets of X is necessarily a complete sublattice of 2^X containing $\varnothing$ and X. It was shown by Zhao [13] that the converse is also true, that is, each complete sublattice of 2^X containing $\varnothing$ and X is reflexive. In the remainder of this paper, it will be implicitly assumed that any subset lattice is complete and contains the trivial subsets $\varnothing$ and X.

For any subset lattice $\mathcal{L}$, $\operatorname{Alg}\mathcal{L}$ is the largest of all families $\mathcal{F}$ of endomorphisms with the property that $\mathcal{L} = \operatorname{Lat}\mathcal{F}$. This follows from the fact that $\mathcal{F} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{F}$ for any family $\mathcal{F}$ of endomorphisms. It is of interest to consider minimal families $\mathcal{F}$ of endomorphisms with this property.

Definition 1. The reflexivity index $\kappa_X(\mathcal{L})$ of a subset lattice $\mathcal{L}$ is defined by

$$\kappa_X(\mathcal{L}) = \inf\{|\mathcal{F}| : \mathcal{L} = \operatorname{Lat}\mathcal{F}\},$$

where $|\mathcal{F}|$ denotes the cardinality of $\mathcal{F}$ and X is the underlying set.

The notion of the reflexivity index was introduced and studied by Zhao [14]. In particular, it was shown that $\kappa_X(\mathcal{L})$ is finite if $\mathcal{L}$ is finite and X is countable. The reflexivity indices of some specific examples of subset lattices were also calculated [14], and some upper bounds on $\kappa_X(\mathcal{L})$ that apply in particular cases were established.

In Section 3 we establish a formula for $\kappa_X(\mathcal{L})$, where $\mathcal{L}$ is any finite subset lattice. In Section 4 we determine the reflexive indices of various types of infinite subset lattices.

The notion of reflexivity was introduced by Halmos [2, 3] in a different context. In those papers the set X is a Hilbert space, the subsets are subspaces, that is, closed linear manifolds and the endomorphisms are operators, that is, linear and bounded. For any collection $\mathcal{F}$ of operators, $\operatorname{Lat}\mathcal{F}$ is a complete sublattice of the lattice of all subspaces, that is, it is closed under the formation of arbitrarily many intersections and closed linear spans. The lattice $\operatorname{Lat}\mathcal{F}$ is not necessarily distributive, but many of the early results relating to reflexivity of subspace lattices involve distributivity. For example, it was shown in [3] that every complete atomic Boolean algebra of subspaces is reflexive. The reflexivity of other types of distributive subspace lattices had been established earlier. For example, Ringrose [10] showed that every complete nest of subspaces is reflexive, and Johnson [7] showed that every distributive lattice of subspaces of a finite-dimensional vector space is reflexive. Johnson's result was extended by Harrison, who showed that every finite distributive lattice of subspaces of a Hilbert space of any dimension is reflexive [4]. These results were generalised by Longstaff [9], who showed that completely distributive subspace lattices are reflexive. Complete distributivity is a strong form of distributivity concerning arbitrarily many meets and joins; see [9] for the precise definition. Arveson [1] has shown that separably acting commutative subspace lattices are reflexive. A commutative subspace lattice is one in which the orthogonal projections corresponding to the subspaces commute and form a set that is closed in the strong operator topology. It is separably acting if the underlying Hilbert space is separable. Commutative subspace lattices are distributive, but not necessarily completely distributive.

The notion of the reflexivity index of a subspace lattice does not appear to have received much attention in the literature. However, Arveson [1] introduced and used the concept of a minimal algebra in his analysis of commutative subspace lattices.

In [5] we determined the reflexivity index of perhaps the simplest type of subspace lattice, that is, a finite distributive lattice of subspaces of a finite-dimensional space.

There are other contexts in which the study of reflexivity poses interesting problems and yields interesting results. For example, X could be a topological space, the subsets required to be closed and the endomorphisms required to be continuous. See [6], [11] and [12] for results concerning reflexivity of lattices of closed sets, and [13] for a discussion of reflexivity in a general context.

2. Preliminaries

Suppose that $\mathcal{L}$ is a lattice of subsets of a set X. Since a subset lattice is necessarily distributive, $\mathcal{L}$ is a distributive sublattice of the Boolean algebra 2^X. In this section we introduce some terminology associated with distributive subset lattices that will be useful for our analysis of $\kappa_X(\mathcal{L})$.

For each subset A of X, let $\overline{A}$ denote the smallest set in $\mathcal{L}$ that contains A. A nonempty set A in $\mathcal{L}$ is *(completely) join-irreducible* if $A \subseteq \bigcup\{A_\omega \in \mathcal{L} : \omega \in \Omega\}$ implies that $A \subseteq A_\omega$ for some $\omega \in \Omega$. Dually, a set A in $\mathcal{L}$ is *(completely) meet-irreducible* if $A \supseteq \bigcap\{A_\omega \in \mathcal{L} : \omega \in \Omega\}$ implies that $A \supseteq A_\omega$ for some $\omega \in \Omega$. It is easy to verify that the join-irreducible sets in $\mathcal{L}$ are the sets of the form $\overline{\{x\}}$, where $x \in X$. Clearly, $x \in \overline{\{x\}}$ and $A = \bigcup\{\overline{\{x\}} : x \in A\}$ for each nonempty set $A \in \mathcal{L}$. So, each nonempty set in $\mathcal{L}$ is a union of join-irreducibles.

For each join-irreducible set $\overline{\{x\}}$, let $\underline{\{x\}} = \bigcup\{A \in \mathcal{L} : A \subset \overline{\{x\}}\}$. Here $\subset$ denotes strict inclusion: $A \subset B$ means that $A \subseteq B$ and $A \neq B$. The set $\underline{\{x\}}$ is well defined, since $\varnothing \subset \overline{\{x\}}$, and is in $\mathcal{L}$, since $\mathcal{L}$ is closed under arbitrary unions. It is easy to show that if $\underline{\{x\}} \subseteq A \subseteq \overline{\{x\}}$ for some $A \in \mathcal{L}$, then $A = \underline{\{x\}}$ if $x \notin A$ and $A = \overline{\{x\}}$ if $x \in A$. We say that $\underline{\{x\}}$ is the *(immediate) predecessor* of $\overline{\{x\}}$ in the lattice $\mathcal{L}$.

Now let $\partial\{x\} = \overline{\{x\}} \setminus \underline{\{x\}}$. Since $\underline{\{x\}}$ is the immediate predecessor of $\overline{\{x\}}$ in $\mathcal{L}$, for each A in $\mathcal{L}$ either $A \cap \partial\{x\} = \varnothing$ or $\partial\{x\}$. We call sets of the form $\partial\{x\}$ *atoms* of the lattice $\mathcal{L}$. The atoms are not necessarily sets in $\mathcal{L}$. In fact, the atoms are all in $\mathcal{L}$ if and only if $\mathcal{L}$ is complemented, that is, a Boolean algebra. It is easy to verify that $x \in \partial\{x\}$ for each $x \in X$ and that $A = \bigcup\{\partial\{x\} : x \in A\}$ for each nonempty set $A \in \mathcal{L}$. Furthermore, the atoms of $\mathcal{L}$ are either equal or disjoint, that is, $\partial\{x\} = \partial\{y\}$ or $\partial\{x\} \cap \partial\{y\} = \varnothing$ for any x and y in X, and

$$\partial\{x\} = \partial\{y\} \Leftrightarrow \overline{\{x\}} = \overline{\{y\}} \Leftrightarrow x \in \partial\{y\} \Leftrightarrow y \in \partial\{x\}. \tag{1}$$

In the remainder of this paper, Γ denotes a subset of X with the property that the set of atoms $\{\partial\{x\} : x \in \Gamma\}$ is a partition of X.

We shall use two parameters associated with the join-irreducible sets in $\mathcal{L}$ in our analysis of the reflexivity index of $\mathcal{L}$. Each nonempty set A in $\mathcal{L}$ is the union of join-irreducibles and so $A = \bigcup\{\overline{\{x\}} : x \in \Gamma'\}$ for some $\Gamma' \subseteq \Gamma$. The *height* $\mathfrak{H}(A)$ of A is defined as the maximal value of $|\Gamma'|$ in such a representation. The *breadth* $\mathfrak{B}(A)$ of A is defined as the minimal value of $|\Gamma'|$ in such a representation. The height $\mathfrak{H}(A)$ is the number (finite or infinite) of join-irreducibles contained in A. Clearly, $\mathfrak{H}\overline{\{x\}} = 1$ if and only if

$\overline{\{x\}}$ is an atom, and $\mathfrak{B}(A) = 1$ if and only if A is join-irreducible. For convenience, we define $\mathfrak{H}(\varnothing) = \mathfrak{B}(\varnothing) = 0$. We shall show that the reflexivity index of $\mathcal{L}$ is determined by the sizes of the atoms $\partial\{x\}$, the heights of the corresponding join-irreducibles $\overline{\{x\}}$ and the breadths of the corresponding predecessors $\underline{\{x\}}$.

For any $\mathcal{F} \subset X^X$, let $\mathcal{F}^*$ denote the subsemigroup of X^X generated by the mappings in $\mathcal{F}$ and the identity map, where function composition is the binary operation. The following proposition lists some useful properties of sets of the form $\mathcal{F}^*x = \{f(x) : f \in \mathcal{F}^*\}$.

Proposition 2. *Suppose that $\mathcal{F} \subseteq X^X$. Then;*

(i) $x \in \mathcal{F}^*x$ *and* $y \in \mathcal{F}^*x \Rightarrow \mathcal{F}^*y \subseteq \mathcal{F}^*x$*;*
(ii) $\mathcal{F}^*x \in \operatorname{Lat}\mathcal{F}$ *for each* $x \in X$*;*
(iii) $A = \bigcup\{\mathcal{F}^*x : x \in A\}$ *for each nonempty set* A *in* $\operatorname{Lat}\mathcal{F}$.

Proof. Properties 1 and 2 follow from the fact that $\mathcal{F}^*$ is a semigroup that includes the identity map.

If $x \in A \in \operatorname{Lat}\mathcal{F}$, then $x \in \mathcal{F}^*x \subseteq A$ by properties 1 and 2. So,

$$A = \bigcup\{\{x\} : x \in A\} \subseteq \bigcup\{\mathcal{F}^*x : x \in A\} \subseteq A.$$

This proves property 3. □

The next proposition will be used repeatedly.

Proposition 3. *Suppose that $\mathcal{L}$ is a lattice of subsets of X and that $\mathcal{F} \subseteq X^X$. Then $\mathcal{L} = \operatorname{Lat}\mathcal{F}$ if and only if $\mathcal{F}^*x = \overline{\{x\}}$ for each $x \in X$.*

Proof. Suppose that $\mathcal{F}^*x = \overline{\{x\}}$ for each $x \in X$. Then $\overline{\{x\}} \in \operatorname{Lat}\mathcal{F}$ for all $x \in X$ by Proposition 2 and hence $\mathcal{L} \subseteq \operatorname{Lat}\mathcal{F}$. Similarly, $\mathcal{F}^*x \in \mathcal{L}$ for all $x \in X$ and so, by Proposition 2, $\operatorname{Lat}\mathcal{F} \subseteq \mathcal{L}$. So, $\mathcal{L} = \operatorname{Lat}\mathcal{F}$.

Now suppose that $\mathcal{L} = \operatorname{Lat}\mathcal{F}$ and that $x \in X$. Then $\mathcal{F}^*x \in \operatorname{Lat}\mathcal{F} = \mathcal{L}$ by Proposition 2. But $\mathcal{F} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{F} = \operatorname{Alg}\mathcal{L}$ and $x \in \overline{\{x\}}$. So, $\mathcal{F}^*x \subseteq \overline{\{x\}}$. Since $x \in \mathcal{F}^*x$, it follows that $\mathcal{F}^*x = \overline{\{x\}}$. □

Corollary 4. *Suppose that $\mathcal{L}$ is a lattice of subsets of a set X and that $\kappa_X(\mathcal{L})$ is finite. Then each join-irreducible $\overline{\{x\}}$ in $\mathcal{L}$ is countable.*

Proof. Suppose that $\operatorname{Lat}\mathcal{F} = \mathcal{L}$, where $\mathcal{F}$ is a finite family of endomorphisms. Then $\mathcal{F}^*$ is countable and the same is true of $\mathcal{F}^*x$ for each $x \in X$. But $\mathcal{F}^*x = \overline{\{x\}}$ by Proposition 3. □

The following corollary gives an upper bound for the reflexive index of a subset lattice. It contains a proof of the reflexivity of a subset lattice that is essentially the same as that in Zhao [13].

Corollary 5. *Suppose that $\mathcal{L}$ is a lattice of subsets of a set X. Then $\mathcal{L}$ is reflexive and $\kappa_X(\mathcal{L}) \le |X|^2$.*

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Proof. For each $u, v \in X$, let $u \otimes v$ denote the endomorphism of X defined by

$$(u \otimes v)(z) = \begin{cases} u & \text{if } z = v, \\ z & \text{if } z \neq v \end{cases}$$

and let $\mathcal{F} = \{y \otimes x : x, y \in X \text{ and } y \in \overline{\{x\}}\}$. Clearly, $\overline{\{x\}} \subseteq \mathcal{F}^* x$ for each $x \in X$. If $y \in \overline{\{x\}}$, then $\overline{\{y\}} \subseteq \overline{\{x\}}$ and so $\mathcal{F}^* x \subseteq \overline{\{x\}}$. So, $\mathcal{L} = \operatorname{Lat} \mathcal{F}$ by Proposition 3 and hence $\mathcal{L}$ is reflexive. Therefore, $\kappa_X(\mathcal{L}) \leq |\mathcal{F}| \leq |X|^2$. □

3. Finite lattices

We obtain a formula for the reflexivity index of a finite subset lattice.

3.1. Uncountable sets. First, we deal with the case in which the underlying set is uncountable.

Proposition 6. *Suppose that $\mathcal{L}$ is a finite lattice of subsets of an uncountable set X. Then $\kappa_X(\mathcal{L}) = |X|$.*

Proof. Suppose that $\mathcal{L} = \operatorname{Lat} \mathcal{F}$ for some $\mathcal{F} \subseteq X^X$. Since $\mathcal{L}$ is finite and X is uncountable, $|\overline{\{x\}}| = |X|$ for some $x \in X$. Now $\mathcal{F}^* x = \overline{\{x\}}$ by Proposition 3 and so $|X| = |\mathcal{F}^* x| \leq |\mathcal{F}^*|$. Therefore, $|\mathcal{F}^*|$ is uncountable and hence $|\mathcal{F}^*| = |\mathcal{F}|$. So, $|\mathcal{F}| \geq |X|$ and hence $\kappa_X(\mathcal{L}) \geq |X|$. Now $\kappa_X(\mathcal{L}) \leq |X|^2$ by Corollary 5 and $|X| = |X|^2$, since X is uncountable. So, $\kappa_X(\mathcal{L}) = |X|$. □

3.2. Reflexivity index 1. We start by characterising those finite subset lattices whose reflexivity indices are 1. First, we establish a necessary condition.

Lemma 7. *Suppose that $\mathcal{L} = \operatorname{Lat}\{f\}$ for some $f \in X^X$. Then X is countable and, for each $x \in X$, either $\overline{\{x\}}$ is a finite atom or $\partial\{x\}$ is a singleton and $\underline{\{x\}}$ is join-irreducible.*

Proof. The countability of X follows from Proposition 6. Now let $\mathcal{F} = \{f\}$. Then $\mathcal{F}^* = \{f^n : n \geq 0\}$, where $f^0 = \mathrm{id}$, the identity map, and $f^{n+1} = f \circ f^n$ for $n \geq 0$. Also, $\mathcal{F}^* x = \{f^n(x) : n \geq 0\} = \overline{\{x\}}$ by Proposition 3.

If $\overline{\{x\}}$ is an atom, then $\mathcal{F}^* x = \overline{\{x\}} = \partial\{x\}$. Now consider $\mathcal{F}^* f(x)$. Since $f(x) \in \partial\{x\}$, $\mathcal{F}^* f(x) = \partial\{x\}$ and so $f^n f(x) = x$ for some $n \geq 0$. It follows that $\mathcal{F}^* x$, that is, $\overline{\{x\}}$, is finite.

If $\overline{\{x\}}$ is not an atom, then $\underline{\{x\}} \neq \varnothing$ and so $f^n(x) \in \underline{\{x\}}$ for some $n > 0$. Let $y = f^{n-1}(x)$ and consider $\mathcal{F}^* y = \{y\} \cup \mathcal{F}^* f(y)$. Since $y \in \partial\{x\}$, $\mathcal{F}^* y = \overline{\{y\}} = \overline{\{x\}} = \partial\{x\} \cup \underline{\{x\}}$. But $\mathcal{F}^* y \cap \partial\{x\} = \{y\}$ and so $x = y$ and $|\partial\{x\}| = 1$. Also, $\mathcal{F}^* y \cap \underline{\{x\}} = \mathcal{F}^* f(y) = \overline{\{f(y)\}}$ by Proposition 3. So, $\underline{\{x\}} = \overline{\{f(y)\}}$ and hence is join-irreducible. □

The necessary condition in Lemma 7 is also sufficient.

Theorem 8. *Suppose that $\mathcal{L}$ is a finite lattice of subsets of a countable set X. Then $\kappa_X(\mathcal{L}) = 1$ if and only if for each $x \in X$, either $\overline{\{x\}}$ is a finite atom or $\partial\{x\}$ is a singleton and $\underline{\{x\}}$ is join-irreducible.*

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Proof. Suppose that $\mathcal{L}$ satisfies the stated conditions. We need to construct an endomorphism f such that $\mathrm{Lat}\{f\} = \mathcal{L}$. We define f 'piecewise', that is, by its restriction to each of the atoms $\partial\{x\} : x \in \Gamma$. So, suppose that $x \in \Gamma$.

If $\mathfrak{H}\overline{\{x\}} = 1$, then $\overline{\{x\}} = \partial\{x\}$ and $|\partial\{x\}| = K < \infty$. Let $f|_{\partial\{x\}}$ be a *cycle* map on $\partial\{x\}$, that is, $f(e_k) = e_{k+1}$ for $1 \leq k < K$ and $f(e_K) = e_1$, where $e_1, e_2, \ldots, e_K$ is an ordering of the elements of $\partial\{x\}$. Note that $f(x) = x$ if $|\partial\{x\}| = 1$. If $\mathfrak{H}\overline{\{x\}} > 1$, then $|\partial\{x\}| = 1$ and $\underline{\{x\}} = \overline{\{y\}}$ for some $y \in \Gamma$. We define $f(x) = y$.

The endomorphism f is now defined on each of the atoms $\partial\{x\} : x \in \Gamma$ and hence on all of X. By Proposition 3, it is sufficient to show that $\mathcal{F}^*x = \overline{\{x\}}$ for all $x \in X$, where $\mathcal{F}^*$ is the subsemigroup generated by f and the identity map. For this, we use an induction argument based on height. Note that $\mathfrak{H}\overline{\{x\}}$ is finite for each $x \in X$, since $\mathcal{L}$ is finite.

If $\mathfrak{H}\overline{\{x\}} = 1$, then $\overline{\{x\}} = \partial\{x\}$ and, since $f|_{\partial\{x\}}$ is a cycle map, it is clear that $\mathcal{F}^*x = \partial\{x\} = \overline{\{x\}}$.

Now suppose that $\mathfrak{H}\overline{\{x\}} > 1$. Then $\mathcal{F}^*x = \{x\} \cup \mathcal{F}^*f(x)$, where $\underline{\{x\}} = \overline{\{f(x)\}}$ and $\mathfrak{H}\overline{\{f(x)\}} = \mathfrak{H}\overline{\{x\}} - 1$. By the inductive hypothesis, $\mathcal{F}^*f(x) = \overline{\{f(x)\}}$, and $\{x\} = \partial\{x\}$ since $|\partial\{x\}| = 1$. So, $\mathcal{F}^*x = \{x\} \cup \mathcal{F}^*f(x) = \partial\{x\} \cup \underline{\{x\}} = \overline{\{x\}}$. □

3.3. Reflexivity index 2. The following theorem gives a simple condition which ensures that the reflexivity index of a finite subset lattice is at most 2.

Theorem 9. *Suppose that $\mathcal{L}$ is a finite lattice of subsets of a countable set X and suppose that, for each $x \in X$, $\mathfrak{B}\underline{\{x\}} \leq |\partial\{x\}|$. Then $\kappa_X(\mathcal{L}) \leq 2$.*

Proof. We show that there are endomorphisms f and g with the property that $\mathrm{Lat}\{f, g\} = \mathcal{L}$. Choose any $x \in \Gamma$.

If $|\partial\{x\}| < \infty$, let $f|_{\partial\{x\}}$ be a cycle map on $\partial\{x\}$. If $\underline{\{x\}} = \varnothing$ (that is, $\overline{\{x\}}$ is an atom), let $f_2(y) = y$ for each $y \in \partial\{x\} = \overline{\{x\}}$. If $\underline{\{x\}} \neq \varnothing$, let $\underline{\{x\}} = \bigcup_{j=1}^{J} \overline{\{x_j\}}$ be a minimal representation of $\underline{\{x\}}$ as a join of join-irreducibles, with $x_j \in \Gamma$ for each j. Now let $g|_{\partial\{x\}}$ be any map whose range is $\{x_1, x_2, \ldots, x_J\}$. Such a map exists since $J = \mathfrak{B}\underline{\{x\}} \leq |\partial\{x\}|$.

If $|\partial\{x\}| = \infty$, then $\partial\{x\}$ and $\overline{\{x\}}$ are both countably infinite. Let $f|_{\partial\{x\}}$ be a *successor* map on $\partial\{x\}$, that is, $f(e_k) = e_{k+1}$ for $k \geq 1$, where $e_1, e_2, e_3, \ldots$ is an ordering of the elements of $\partial\{x\}$. Let $g|_{\partial\{x\}}$ be a map whose range is $\overline{\{x\}}$. We also require that for each $z \in \overline{\{x\}}$, $g(e_k) = z$ for infinitely many k. Such a map exists since $\partial\{x\}$ and $\overline{\{x\}}$ are both countably infinite.

The endomorphisms f and g are now defined on each of the atoms $\partial\{x\} : x \in \Gamma$ and hence on all of X. Let $\mathcal{F}^*$ be the subsemigroup generated by f, g and the identity map. By Proposition 3, it is sufficient to show that $\mathcal{F}^*x = \overline{\{x\}}$ for all $x \in X$. For this, we use an induction argument based on height. First, we observe that each $\overline{\{x\}}$ is invariant under both f and g, so $\mathcal{F}^*x \subseteq \overline{\{x\}}$. It is also clear from the properties of $f|_{\partial\{x\}}$ and $g|_{\partial\{x\}}$ that $\partial\{x\} \subseteq \mathcal{F}^*x$. In particular, if $\mathfrak{H}\overline{\{x\}} = 1$, then $\overline{\{x\}} = \partial\{x\}$ and so $\mathcal{F}^*x = \overline{\{x\}}$.

Now suppose that $\mathfrak{H}\overline{\{x\}} > 1$ and that $\underline{\{x\}} = \bigcup_{j=1}^{J} \overline{\{x_j\}}$ is a minimal representation of $\underline{\{x\}}$ as a join of join-irreducibles, with $x_j \in \Gamma$ for each j. Since $\partial\{x\} \subseteq \mathcal{F}^*x$, it follows from the properties of $g|_{\partial\{x\}}$ that $x_j \in \mathcal{F}^* x$ for each x_j. Therefore, $\mathcal{F}^* x_j \subseteq \mathcal{F}^* x$ for each x_j. Now $\mathfrak{H}\overline{\{x_j\}} < \mathfrak{H}\overline{\{x\}}$. So, by the inductive hypothesis, we may assume that $\mathcal{F}^* x_j = \overline{\{x_j\}}$. Therefore,

$$\underline{\{x\}} = \bigcup_{j=1}^{J} \overline{\{x_j\}} = \bigcup_{j=1}^{J} \mathcal{F}^* x_j \subseteq \mathcal{F}^* x.$$

So, $\overline{\{x\}} = \partial\{x\} \cup \underline{\{x\}} \subseteq \mathcal{F}^* x$ and hence $\mathcal{F}^* x = \overline{\{x\}}$. □

From Theorems 8 and Theorem 9, we obtain the following result.

Corollary 10. *Suppose that $\mathcal{L}$ is a finite lattice of subsets of a countable set X and suppose that $|\partial\{x\}| = \infty$ for each $x \in X$. Then $\kappa_X(\mathcal{L}) = 2$.*

Example 11. In Zhao [14], the reflexivity indices of various types of lattices of subsets of $\mathbb{N}$, the set of natural numbers $\{1, 2, 3, \ldots\}$, are examined. These include lattices generated by finite families of subsets of the form $p\mathbb{N} = \{p, 2p, 3p, \ldots\}$, where p is a prime. It was shown that if $\mathcal{L}$ is the lattice generated by $2\mathbb{N}$, $3\mathbb{N}$ and $5\mathbb{N}$, then $\kappa(\mathcal{L}) \leq 4$, and readers are invited to determine the reflexivity index of the lattice generated by the subsets $p_1\mathbb{N}$, $p_2\mathbb{N}$, $p_3\mathbb{N}$, $\ldots$, $p_n\mathbb{N}$, where p_1, p_2, p_3, $\ldots$, p_n are distinct primes. Suppose that $\mathcal{L}$ is this lattice. It is not difficult to show that the join-irreducibles in $\mathcal{L}$ are $\mathbb{N}$ itself and each of the subsets $k\mathbb{N}$, where k is a product of any of the primes p_1, p_2, p_3, $\ldots$, p_n. Furthermore, each of the atoms in $\mathcal{L}$ is countably infinite. So, $\kappa_X(\mathcal{L}) = 2$.

3.4. A general formula. It is possible to modify the proof of Theorem 9 to obtain a precise formula for the reflexivity index of a finite lattice $\mathcal{L}$ of subsets of a countable set X. For this, it is convenient to define another two parameters associated with elements of the lattice. For each $x \in X$, we define

$$\rho(x) = \frac{\mathfrak{B}\underline{\{x\}}}{|\partial\{x\}|}$$

and

$$\kappa(x) = \begin{cases} \rho(x) & \text{if } \mathfrak{H}\overline{\{x\}} > 1 \text{ and } |\partial\{x\}| = 1, \\ 1 + \lceil \rho(x) \rceil & \text{if } \mathfrak{H}\overline{\{x\}} > 1 \text{ and } 1 < |\partial\{x\}| < \infty, \\ 1 & \text{if } \mathfrak{H}\overline{\{x\}} = 1 \text{ and } |\partial\{x\}| < \infty, \\ 2 & \text{if } |\partial\{x\}| = \infty. \end{cases}$$

We shall show that $\kappa_X(\mathcal{L}) = \sup\{\kappa(x) : x \in X\}$.

Proposition 12. *$\kappa_X(\mathcal{L}) \geq \kappa(x)$ for each $x \in X$.*

Proof. Suppose that $\mathcal{L} = \operatorname{Lat} \mathcal{F}$, where $\mathcal{F}$ is a finite family of endomorphisms such that $|\mathcal{F}| = \kappa_X(\mathcal{L})$. Suppose also that $x \in X$.

Trivially, $|\mathcal{F}| \geq 1$ and $|\mathcal{F}| \geq 2$ if $|\partial\{x\}| = \infty$ by Theorem 8. So, $\kappa(x) \leq |\mathcal{F}|$ if $\mathfrak{H}\overline{\{x\}} = 1$ or if $|\partial\{x\}| = \infty$.

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Suppose that $\mathfrak{H}\overline{\{x\}} > 1$ and $|\partial\{x\}| = 1$. Then, by Proposition 3,

$$\overline{\{x\}} = \mathcal{F}^*x = \{x\} \cup \bigcup\{\mathcal{F}^* f(x) : f \in \mathcal{F}\} = \{x\} \cup \bigcup\{\overline{f(x)} : f \in \mathcal{F}\}.$$

Therefore, $\underline{\{x\}} \subseteq \bigcup\{\overline{f(x)} : f \in \mathcal{F}\}$ and so $\kappa(x) = \mathfrak{B}\underline{\{x\}} = \rho(x) \le |\mathcal{F}|$.

Now suppose that $\mathfrak{H}\overline{\{x\}} > 1$ and $1 < |\partial\{x\}| < \infty$ and consider the sets

$$S = \{f(y) : y \in \partial\{x\},\ f \in \mathcal{F}\}, \quad S_1 = S \cap \partial\{x\} \quad \text{and} \quad S_2 = S \cap \underline{\{x\}}.$$

Since $\mathcal{F}^*x = \overline{\{x\}}$,

$$S_1 = \partial\{x\} \quad \text{and} \quad \underline{\{x\}} = \bigcup\{\mathcal{F}^* z : z \in S_2\} = \bigcup\{\overline{\{z\}} : z \in S_2\}.$$

Therefore,

$$\mathfrak{B}\underline{\{x\}} \le |S_2| = |S| - |S_1| \le |\mathcal{F}||\partial\{x\}| - |\partial\{x\}| = (|\mathcal{F}| - 1)|\partial\{x\}|.$$

So, $1 + \mathfrak{B}\underline{\{x\}}/|\partial\{x\}| = 1 + \rho(x) \le |\mathcal{F}|$ and hence $\kappa(x) = 1 + \lceil\rho(x)\rceil \le |\mathcal{F}|$. □

THEOREM 13. *Suppose that $\mathcal{L}$ is a finite lattice of subsets of a countable set X. Then* $\kappa_X(\mathcal{L}) = \sup\{\kappa(x) : x \in X\}$.

PROOF. Suppose that $\kappa(x) \le K < \infty$ for each $x \in X$. By Proposition 12, it is sufficient to show that there are endomorphisms $f_1, f_2, \ldots, f_K$ with the property that $\mathrm{Lat}\{f_1, f_2, \ldots, f_K\} = \mathcal{L}$.

Suppose that $x \in \Gamma$ and $\mathfrak{H}\overline{\{x\}} > 1$. Let $\underline{\{x\}} = \bigcup_{n=1}^N \overline{\{x_n\}}$ be a minimal representation of $\underline{\{x\}}$ as a join of join-irreducibles, where $x_n \in \Gamma$ for each n. If $|\partial\{x\}| = 1$, let $f_n(x) = x_n$ for $1 \le n \le N$ and let $f_n(x) = x$ for $N = \kappa(x) < n \le K$ if $\kappa(x) < K$. If $|\partial\{x\}| < \infty$, let $f_1|_{\partial\{x\}}$ be a cycle map on $\partial\{x\}$. We also require the union of the ranges of the restrictions to $\partial\{x\}$ of the remaining $K - 1$ mappings $f_2, f_3, \ldots, f_K$ to be $\{x_1, x_2, \ldots, x_N\}$. This is possible because $\kappa(x) = 1 + \lceil\rho(x)\rceil \le K$ implies that $\mathfrak{B}\underline{\{x\}} = N \le (K - 1)|\partial\{x\}|$.

If $x \in \Gamma$, $\mathfrak{H}\overline{\{x\}} = 1$ and $|\partial\{x\}| < \infty$, let $f_1|_{\partial\{x\}}$ be a cycle map on $\partial\{x\}$ and let $f_n(y) = y$ for each $y \in \partial\{x\}$ and $1 < n \le K$ if $1 = \kappa(x) < K$.

If $x \in \Gamma$ and $|\partial\{x\}| = \infty$, then $\partial\{x\}$ and $\overline{\{x\}}$ are both countably infinite. Let $f_1|_{\partial\{x\}}$ be a successor map on $\partial\{x\}$ and let $f_2|_{\partial\{x\}}$ be any map whose range is $\overline{\{x\}}$. We also require that for each $z \in \overline{\{x\}}$, $f_2(e_k) = z$ for infinitely many k. Such a map exists since $\partial\{x\}$ and $\underline{\{x\}}$ are both countably infinite. Also, let $f_n(y) = y$ for each $y \in \partial\{x\}$ and $2 < n \le K$ if $\kappa(x) = 2 < K$.

The endomorphisms $f_1, f_2, \ldots, f_K$ are now defined on each of the atoms $\partial\{x\} : x \in \Gamma$ and hence on all of X. It is easy to show using arguments similar to those in the proof of Theorem 9 that $\mathrm{Lat}\{f_1, f_2, \ldots, f_K\} = \mathcal{L}$. □

EXAMPLE 14. Let $X = \{1, 2, 3, \ldots, n, n + 1\}$ and let $\mathcal{L}$ denote the lattice of subsets of X consisting of all subsets of $\{1, 2, 3, \ldots, n\}$, together with the set X. The join-irreducibles in $\mathcal{L}$ are the singletons $\{1\}, \{2\}, \ldots, \{n\}$ and X, and the atoms are the singletons $\{1\}, \{2\}, \ldots, \{n\}, \{n + 1\}$. It is easy to check that $\kappa(j) = 1$ for $1 \le j \le n$ and that $\kappa(n + 1) = \mathfrak{B}\underline{\{n + 1\}} = n$. So, $\kappa_X(\mathcal{L}) = n$. Indeed, $\mathrm{Alg}\,\mathcal{L} = \{f_1, f_2, \ldots, f_n, f_{n+1}\}$, where, for each $1 \le i \le n + 1$, $f_i(j) = j$ for $1 \le j \le n$ and $f_i(n + 1) = i$. Furthermore, $\mathrm{Lat}\,\mathcal{F} = \mathcal{L}$ if and only if $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ or $\mathcal{F} = \{f_1, f_2, \ldots, f_n, f_{n+1}\}$.

4. Infinite lattices

In this section we determine the reflexivity index of some infinite subset lattices and obtain conditions which ensure that the reflexivity index is finite. The following simple examples are illuminating.

EXAMPLE 15. Let $\mathbb{Z}$ denote the set of all integers and let $\mathcal{L}$ denote the set of all semi-infinite intervals in $\mathbb{Z}$ of the form $(-\infty, m] = \{n \in \mathbb{Z} : n \leq m\}$, where $m \in \mathbb{Z}$, together with $\emptyset$ and $\mathbb{Z}$. It is easy to see that $\mathcal{L}$ is a complete, totally ordered lattice of subsets of $\mathbb{Z}$ and that its reflexivity index is 1. In fact, $\mathcal{L} = \mathrm{Lat}\{f\}$, where $f(n) = n - 1$ for all $n \in Z$.

EXAMPLE 16. Let $\mathbb{R}$ denote the set of all real numbers and let $\mathcal{L}$ denote the set of all semi-infinite intervals in $\mathbb{R}$ of the form $(-\infty, x] = \{y \in \mathbb{R} : y \leq x\}$ and $(-\infty, x) = \{y \in \mathbb{R} : y < x\}$, where $x \in \mathbb{R}$, together with $\emptyset$ and $\mathbb{R}$. It is easy to see that $\mathcal{L}$ is a complete, totally ordered lattice of subsets of $\mathbb{R}$. However, its reflexivity index is infinite. This follows from Corollary 4 and the fact that the join-irreducibles in $\mathcal{L}$ are the closed intervals $(-\infty, x]$, which are uncountable.

EXAMPLE 17. Let $\mathbb{Q}$ denote the set of all rational numbers and let $\mathcal{L}$ denote the set of all semi-infinite intervals in $\mathbb{Q}$ of the form $(-\infty, q] = \{y \in \mathbb{Q} : y \leq q\}$, where $q \in \mathbb{Q}$ and $(-\infty, \alpha) = \{y \in \mathbb{Q} : y < \alpha\}$, where $\alpha \in \mathbb{R}$, together with $\emptyset$ and $\mathbb{Q}$. It is easy to see that $\mathcal{L}$ is a complete, totally ordered lattice of subsets of $\mathbb{Q}$. We shall show that its reflexivity index is infinite.

EXAMPLE 18. Let $\mathbb{R}$ denote the set of all real numbers and let $\mathcal{L}$ denote the set of all subsets of $\mathbb{R}$ which are closed under 'subtraction of 1'. That is, $A \in \mathcal{L}$ if and only if $x \in A$ implies that $x - 1 \in A$. It is easy to see that $\mathcal{L}$ is a complete lattice of subsets of $\mathbb{R}$ and that its reflexivity index is 1. In fact, $\mathcal{L} = \mathrm{Lat}\{f\}$, where $f(x) = x - 1$ for all $x \in \mathbb{R}$.

Example 18 shows that there are lattices of subsets of uncountable sets with small reflexivity indices. Note that the join-irreducibles in the lattice in Example 18 are the subsets of the form $\{x, x - 1, x - 2, \ldots\}$, and these are countable as required by Corollary 4.

4.1. Direct sums. Example 18 fits within the wider framework of lattice sums. Let Ω be an index set and let $\{X(\omega) : \omega \in \Omega\}$ be a family of mutually disjoint sets. For each $\omega \in \Omega$, let $\mathcal{L}(\omega)$ be a lattice of subsets of a set $X(\omega)$. The *direct sum* of the lattices $\mathcal{L}(\omega) : \omega \in \Omega$ is a lattice $\mathcal{L}$ of subsets of the set $X = \bigcup\{X(\omega) : \omega \in \Omega\}$. It consists of all subsets of the form $\bigcup\{A(\omega) : A(\omega) \in \mathcal{L}(\omega)\}$ and is denoted by $\biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}$.

PROPOSITION 19. $\kappa(\biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}) = \sup\{\kappa(\mathcal{L}(\omega)) : \omega \in \Omega\}$.

PROOF. Suppose that $\mathrm{Lat}\,\mathcal{F} = \biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}$. Since each of the sets $X(\omega)$ is in $\biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}$, it is easy to see that for each $\omega \in \Omega$, $\mathrm{Lat}\,\mathcal{F}_\omega = \mathcal{L}(\omega)$, where $\mathcal{F}_\omega = \{f|_{X(\omega)} : f \in \mathcal{F}\}$. Since $|\mathcal{F}_\omega| \leq |\mathcal{F}|$, it follows that

$$\kappa\Big(\biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}\Big) \geq \kappa(\mathcal{L}(\omega)) \quad \text{for each } \omega \in \Omega. \tag{2}$$

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Now suppose that $\kappa(\mathcal{L}(\omega)) \le K < \infty$ for each $\omega \in \Omega$. By (2), it is sufficient to show that $\kappa(\biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}) \le K$. For each $\omega \in \Omega$, choose K functions $f_k(\omega) : 1 \le k \le K$, defined on $X(\omega)$, such that $\mathrm{Lat}\{f_k(\omega) : 1 \le k \le K\} = \mathcal{L}(\omega)$. For each k, let f_k denote the function defined on $X = \bigcup\{X(\omega) : \omega \in \Omega\}$ whose restriction to each $X(\omega)$ is $f_k(\omega)$. It is easy to check that $\mathrm{Lat}\{f_k : 1 \le k \le K\} = \biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}$. So, $\kappa(\biguplus\{\mathcal{L}(\omega) : \omega \in \Omega\}) \le K$, as required. □

Proposition 19 provides a method of constructing subset lattices with small reflexivity index, where the underlying set is arbitrarily large.

Corollary 20. *The reflexivity index of a Boolean algebra of sets is* 1, 2 *or* ∞. *It is* 1 *if each atom is finite, it is* 2 *if each atom is countable and at least one atom is countably infinite, and it is* ∞ *if at least one atom is uncountable.*

Proof. Suppose that $\mathcal{L}$ is a Boolean algebra of subsets of a set X. Then $\mathcal{L} = \biguplus\{\mathcal{L}(x) : x \in \Gamma\}$, where $\mathcal{L}(x) = \{\emptyset, \partial\{x\}\}$ for each $x \in \Gamma$. It is easy to see that $\kappa(\mathcal{L}(x)) = 1$ if $\partial\{x\}$ is finite, 2 if $\partial\{x\}$ is countably infinite and ∞ if $\partial\{x\}$ is uncountable. An application of Proposition 19 completes the proof. □

Example 21. For each $\omega \in [0, 1)$, let $\mathbb{Z}_\omega = \mathbb{Z} + \omega = \{n + \omega : n \in \mathbb{Z}\}$ and let $\mathcal{L}(\omega)$ denote the set of all subsets of $\mathbb{Z}_\omega$ of the form $\{\omega + n - m : m \in \mathbb{N}\}$, where $n \in \mathbb{Z}$, together with $\emptyset$ and $\mathbb{Z}_\omega$. Then $\mathcal{L}(\omega) = \mathrm{Lat}\{f_\omega\}$, where $f_\omega(x) = x - 1$ for all $x \in \mathbb{Z}_\omega$ and so $\kappa(\mathcal{L}(\omega)) = 1$. So, $\kappa(\biguplus\{\mathcal{L}(\omega) : \omega \in [0, 1\}) = 1$, by Proposition 19. Note that $\biguplus\{\mathcal{L}(\omega) : \omega \in [0, 1)\}$ is the lattice $\mathcal{L}$ in Example 18.

4.2. Nests. We conclude this section by examining the reflexivity indices of nests, that is, totally ordered subset lattices. First, we establish some properties of a nest whose reflexivity index is finite.

Lemma 22. *Suppose that* $\mathcal{L}$ *is a nest of subsets of a set* X *and that* $\kappa_X(\mathcal{L})$ *is finite. Then, for each* $x \in X$, $\overline{\{x\}}$ *is countable and* $\overline{\{x\}}$ *is an atom,* $\underline{\{x\}}$ *is join-irreducible or* $\partial\{x\}$ *is countably infinite.*

Proof. Suppose that $\mathcal{L} = \mathrm{Lat}\,\mathcal{F}$, where $\mathcal{F}$ is a finite family of endomorphisms. Then $\overline{\{x\}}$ is countable by Corollary 4.

Suppose that $\partial\{x\}$ is finite and $\overline{\{x\}}$ is not an atom. By Proposition 3, $\mathcal{F}^*x = \overline{\{x\}} = \partial\{x\} \cup \underline{\{x\}}$. Let $S = \{f(y) : f \in \mathcal{F}, y \in \partial\{x\}$ and $f(y) \in \underline{\{x\}}\}$. Then

$$\underline{\{x\}} = \mathcal{F}^*x \cap \underline{\{x\}} \subseteq \bigcup\{\mathcal{F}^*z : z \in S\} = \bigcup\{\overline{\{z\}} : z \in S\} \subseteq \underline{\{x\}}.$$

Since $\mathcal{L}$ is a nest and S is finite, $\bigcup\{\overline{\{z\}} : z \in S\} = \overline{\{z_0\}}$ for some $z_0 \in S$. So, $\underline{\{x\}} = \overline{\{z_0\}}$, that is, $\underline{\{x\}}$ is join-irreducible. □

We shall show that the necessary conditions in Lemma 22 for the finiteness of $\kappa_X(\mathcal{L})$ are also sufficient. But first we need to examine the fine structure of the nest $\mathcal{L}$. An *interval* in $\mathcal{L}$ is a subset $\mathcal{I}$ of $\mathcal{L}$ with the property that if $A \in \mathcal{I}$, $B \in \mathcal{I}$ and $A \subseteq B$, then $[A, B] \subseteq \mathcal{I}$, where $[A, B] = \{C : C \in \mathcal{L}$ and $A \subseteq C \subseteq B\}$. The set $[A, B]$ is itself an interval in $\mathcal{L}$. We say that the interval $\mathcal{I}$ is *discrete* if each subinterval in $\mathcal{I}$ of the form $[A, B]$ is finite.

By Zorn's lemma, each discrete interval in $\mathcal{L}$ is a subset of a maximal discrete interval. Any two maximal discrete intervals in $\mathcal{L}$ are either identical or disjoint, and each set in $\mathcal{L}$ is in a maximal discrete interval. For each $x \in X$, let $\mathcal{I}(x)$ denote the maximal discrete interval in $\mathcal{L}$ that contains $\overline{\{x\}}$. Note that $\mathcal{I}(x)$ also contains $\underline{\{x\}}$. For each $x \in X$, let

$$\mathcal{A}(x) = \{y : y \in X \text{ and } \overline{\{y\}} \in \mathcal{I}(x)\}.$$

Note that (1) implies that

$$\mathcal{A}(x) = \bigcup \{\partial\{y\} : \overline{\{y\}} \in \mathcal{I}(x)\}.$$

Since $\mathcal{I}(x) = \mathcal{I}(y) \Leftrightarrow \overline{\{x\}} \in \mathcal{I}(y) \Leftrightarrow \overline{\{y\}} \in \mathcal{I}(x)$,

$$\mathcal{A}(x) = \mathcal{A}(y) \Leftrightarrow y \in \mathcal{A}(x) \Leftrightarrow x \in \mathcal{A}(y).$$

So, any two of the sets $\mathcal{A}(x) : x \in X$ are identical or disjoint.

Theorem 23. *Suppose that $\mathcal{L}$ is a nest of subsets of a set X. Then $\kappa_X(\mathcal{L})$ is finite if and only if for each $x \in X$, $\overline{\{x\}}$ is countable, and $\overline{\{x\}}$ is an atom, $\underline{\{x\}}$ is join-irreducible or $\partial\{x\}$ is countably infinite.*

Proof. Suppose that $\mathcal{L}$ satisfies the stated conditions. We shall construct endomorphisms f and g such that $\mathrm{Lat}\{f, g\} = \mathcal{L}$. Choose subsets Ψ and Γ of X such that $\{\mathcal{A}(x) : x \in \Psi\}$ and $\{\partial\{y\} : y \in \Gamma\}$ both partition X. We may assume that $\Psi \subseteq \Gamma$. For each $x \in X$, let $\Gamma(x) = \Gamma \cap \mathcal{A}(x)$. We define f and g piecewise, that is, by their restrictions to the sets $\mathcal{A}(x) : x \in \Psi$. So, suppose that $x \in \Psi$.

The maps $f|_{\mathcal{A}(x)}$ and $g|_{\mathcal{A}(x)}$ are also defined piecewise, that is, by their restrictions to the sets $\partial\{y\}$, where $y \in \Gamma(x)$. For any such y, let $f|_{\partial\{y\}}$ be a cycle map on $\partial\{y\}$ if $1 < |\partial\{y\}| < \infty$ and let $f|_{\partial\{y\}}$ be a successor map on $\partial\{y\}$ if $|\partial\{y\}| = \infty$. If $|\partial\{y\}| = 1$ and $\overline{\{y\}}$ is an atom, let $f_1(y) = y$. If $|\partial\{y\}| = 1$ and $\overline{\{y\}}$ is not an atom, let $f(y) = z$, where $z \in \Gamma(x)$ and $\underline{\{y\}} = \overline{\{z\}}$.

The definition of $g|_{\mathcal{A}(x)}$ depends on the nature of $\mathcal{I}(x)$. Let

$$\inf \mathcal{I}(x) = \bigcap \{M : M \in \mathcal{I}(x)\}.$$

First, suppose that $\varnothing = \inf \mathcal{I}(x)$ or $\inf \mathcal{I}(x) \in \mathcal{I}(x)$, and that $y \in \Gamma(x)$. If $|\partial\{y\}| < \infty$ and $\overline{\{y\}}$ is an atom, let $g(u) = y$ for each $u \in \partial\{y\}$. If $|\partial\{y\}| < \infty$ and $\overline{\{y\}}$ is not an atom, then, by the stated conditions, $\underline{\{y\}}$ is join-irreducible. Let $g(u) = z$ for each $u \in \partial\{y\}$, where $z \in \Gamma$ and $\underline{\{y\}} = \overline{\{z\}}$. If $|\partial\{y\}| = \infty$, let $g|_{\partial\{y\}}$ be any map whose range is $\overline{\{y\}}$. We also require that for each $v \in \overline{\{y\}}$, $g(u) = v$ for infinitely many $u \in \partial\{y\}$. Such a map exists since $\partial\{y\}$ and $\overline{\{y\}}$ are both countably infinite.

Now suppose that $\varnothing \subset \inf \mathcal{I}(x) \notin \mathcal{I}(x)$. Then $\mathcal{I}(x)$, $\mathcal{A}(x)$ and $\Gamma(x)$ are all countably infinite. We require that $g|_{\Gamma(x)}$ be a map whose range is $\inf \mathcal{I}(x)$. We also require that,

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for each $v \in \inf \mathcal{I}(x)$ and each $y \in \Gamma(x)$, there exists $u \in \Gamma(x)$ such that $g(u) = v$ and $\overline{\{u\}} \subseteq \overline{\{y\}}$. Such a map exists because $\Gamma(x) \cap \overline{\{y\}}$ is countably infinite for each $y \in \Gamma(x)$ and $\inf \mathcal{I}(x)$ is countable.

We now need to define g on $\mathcal{A}(x) \setminus \Gamma(x)$. Suppose that $y \in \Gamma(x)$. Note that $\overline{\{y\}}$ is not an atom, since $\varnothing \subset \inf \mathcal{I}(x) \subset \overline{\{y\}}$. If $1 < |\partial\{y\}| < \infty$, then $\underline{\{y\}}$ is join-irreducible by the stated conditions. Write $\underline{\{y\}} = \overline{\{z\}}$, where $z \in \Gamma(x)$, and let $g(u) = z$ for each $u \in \partial\{y\} \setminus \{y\}$. If $|\partial\{y\}| = \infty$, let $g|_{\partial\{y\} \setminus \{y\}}$ be a map whose range is $\overline{\{y\}}$. We also require that for each $v \in \overline{\{y\}}$, $g(u) = v$ for infinitely many $u \in \partial\{y\} \setminus \{y\}$. Such a map exists because $\partial\{y\} \setminus \{y\}$ and $\overline{\{y\}}$ are countably infinite.

It remains to be shown that $\mathrm{Lat}\{f, g\} = \mathcal{L}$. Let $\mathcal{F}^*$ be the subsemigroup generated by f, g and the identity map. Choose $x \in X$. By Proposition 3, it is sufficient to show that $\mathcal{F}^* x = \overline{\{x\}}$.

It is easy to see that $f(x) \in \overline{\{x\}}$ and $g(x) \in \overline{\{x\}}$. So, $\mathcal{F}^* x \subseteq \overline{\{x\}}$. It is also clear that $\overline{\{x\}} \subseteq \mathcal{F}^* x$ if $|\partial\{x\}| = \infty$. Also, $\partial\{x\} \subseteq \mathcal{F}^* x$ if $|\partial\{x\}| < \infty$. So, if $\overline{\{x\}}$ is an atom or if $|\partial\{x\}| = \infty$, $\mathcal{F}^* x = \overline{\{x\}}$. Assume that $\overline{\{x\}}$ is not an atom and that $|\partial\{x\}| < \infty$. Then, by the stated conditions, $\underline{\{x\}} = \overline{\{y\}}$ for some $y \in \Gamma(x)$. Furthermore, $y \in \mathcal{F}^* x$. So,

$$\mathcal{F}^* x = \partial\{x\} \cup \mathcal{F}^* y \quad \text{where } \underline{\{x\}} = \overline{\{y\}}. \tag{3}$$

Repeated application of (3) and Proposition 2 shows that $\mathcal{A}(x) \cap \overline{\{x\}} \subseteq \mathcal{F}^* x$.

Since $\overline{\{x\}} = (\mathcal{A}(x) \cap \overline{\{x\}}) \cup \inf \mathcal{I}(x)$, it remains to be shown that $\inf \mathcal{I}(x) \subseteq \mathcal{F}^* x$. We may assume that $\varnothing \subset \inf \mathcal{I}(x)$. If $\inf \mathcal{I}(x) \in \mathcal{I}(x)$, then

$$\inf \mathcal{I}(x) = \underline{\{y^*\}} \subset \overline{\{y^*\}} \subseteq \overline{\{x\}} \quad \text{for some } y^* \in \Gamma(x).$$

Now $\overline{\{y^*\}}$ is not an atom and $\underline{\{y^*\}} = \inf \mathcal{I}(x)$ is not join-irreducible (since $\mathcal{I}(x)$ is a maximal discrete interval). So, by the stated conditions, $|\partial\{y^*\}| = \infty$ and therefore

$$\inf \mathcal{I}(x) \subseteq \overline{\{y^*\}} = \mathcal{F}^* y \subseteq \mathcal{F}^* x.$$

If $\inf \mathcal{I}(x) \notin \mathcal{I}(x)$, then the range of g restricted to $\Gamma(x) \cap \overline{\{x\}}$ is $\inf \mathcal{I}(x)$. Since $\Gamma(x) \cap \overline{\{x\}} \subseteq \mathcal{A}(x) \cap \overline{\{x\}} \subseteq \mathcal{F}^* x$, it follows that $\inf \mathcal{I}(x) \subseteq \mathcal{F}^* x$. An application of Lemma 22 completes the proof. □

The following corollary follows immediately from the proof of Theorem 23.

Corollary 24. *Suppose that $\mathcal{L}$ is a nest of subsets of a set X and that $\kappa_X(\mathcal{L})$ is finite. Then $\kappa_X(\mathcal{L}) \le 2$.*

The proof of the next corollary is a modification of the proof of Theorem 8.

Corollary 25. *The reflexivity index of a nest $\mathcal{L}$ of subsets of a set X is* 1 *if and only if the nontrivial sets in $\mathcal{L}$ form a discrete interval in $\mathcal{L}$, $\partial\{x\}$ is finite if $\overline{\{x\}}$ is an atom and $\partial\{x\}$ is a singleton if $\overline{\{x\}}$ is not an atom.*

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Proof. Suppose that $\mathcal{L} = \mathrm{Lat}\{f\}$, where $f \in X^X$, and that $x \in X$. Then $\mathcal{F}^* x = \{f^n(x) : n \geq 0\} = \overline{\{x\}}$, by Proposition 3. Also, Lemma 7 shows that either $\overline{\{x\}}$ is a finite atom or $\partial\{x\}$ is a singleton and $\underline{\{x\}}$ is join-irreducible.

Suppose that $x, y \in X$. Since $\mathcal{L}$ is a nest, we may assume that $\overline{\{y\}} \subseteq \overline{\{x\}}$. Since $\mathcal{F}^* x = \overline{\{x\}}$, it follows that $y = f^n(x)$ for some $n \geq 0$. Furthermore, $[\overline{\{y\}}, \overline{\{x\}}] = \bigcup\{\overline{\{f^{(m)}(x)\}} : 0 \leq m \leq n\}$. So, $[\overline{\{y\}}, \overline{\{x\}}]$ is a discrete interval in $\mathcal{L}$ and it follows that $\mathcal{I}(x) = \mathcal{I}(y)$. Therefore, there is a unique maximal discrete interval in $\mathcal{L}$, which we denote by $\mathcal{I}$. Since $\overline{\{x\}} \in \mathcal{I}$ for each $x \in X$, $\sup \mathcal{I} = X$. Furthermore, $\underline{\{x\}} \in \mathcal{I}$ for each x. Since $x \notin \underline{\{x\}}$, $\inf \mathcal{I} = \varnothing$ and so $(\varnothing, X) \subseteq \mathcal{I}$.

Now suppose that $\mathcal{L}$ satisfies the stated conditions. We construct $f \in X^X$ such that $\mathrm{Lat}\{f\} = \mathcal{L}$. The endomorphism f is defined piecewise, that is, by its restriction to each of the atoms $\partial\{x\} : x \in \Gamma$. So, suppose that $x \in \Gamma$.

If $\overline{\{x\}}$ is an atom, then $\overline{\{x\}} = \partial\{x\}$ and $|\partial\{x\}| = K < \infty$. Let $f|_{\partial\{x\}}$ be a cycle map on $\partial\{x\}$. If $\overline{\{x\}}$ is not an atom, then $|\partial\{x\}| = 1$ and $\underline{\{x\}} = \overline{\{y\}}$ for some $y \in \Gamma$. We define $f(x) = y$. The endomorphism f is now defined on each of the atoms $\partial\{x\} : x \in \Gamma$ and hence on all of X.

Suppose that $x \in X$ and that $y \in \overline{\{x\}}$. By Proposition 3, we need to show that $y \in \mathcal{F}^* x = \{f^n(x) : n \geq 0\}$. Since $y \in \overline{\{x\}}$, $\overline{\{y\}} \subseteq \overline{\{x\}}$ and so, by the stated conditions, $[\overline{\{y\}}, \overline{\{x\}}]$ is a finite interval in $\mathcal{L}$. (Note that if $\overline{\{y\}} \subset \overline{\{x\}} = X$, then $[\overline{\{y\}}, \underline{\{x\}}]$ is a finite interval of nontrivial sets in $\mathcal{L}$ and hence $[\overline{\{y\}}, \overline{\{x\}}]$ is finite.) Since $[\overline{\{y\}}, \overline{\{x\}}]$ is finite, $f^n(x) \in \partial\{y\}$ for some $n \geq 0$. If $\overline{\{y\}}$ is not an atom, $\partial\{y\}$ is a singleton and so $f^n(x) = y$. If $\overline{\{y\}}$ is an atom, $y = f^m(x)$ for some $m \geq n$, since f is a cycle map on $\overline{\{y\}}$. □

Example 26. Let ω_1 denote the first uncountable ordinal. Let $\mathcal{L}(\omega_1)$ denote the set consisting of all ordinals not greater than ω_1. Since each ordinal is the set of all smaller ordinals [8], $\mathcal{L}(\omega_1)$ is a complete nest of subsets of ω_1. It contains the trivial subsets, since $\varnothing$, the smallest ordinal, is less than ω_1 and $\omega_1 \in \mathcal{L}(\omega_1)$.

The set ω_1 is uncountable, but each of the other sets in $\mathcal{L}(\omega_1)$ is countable. A nonempty ordinal α is a *successor* if the set of all ordinals smaller than α has a maximum. The join-irreducible elements of $\mathcal{L}(\omega_1)$ are the sets α, where α is a successor ordinal. Let ω_0 denote the first infinite ordinal. Then $\omega_0 + 1 = \omega_0 \cup \{\omega_0\}$ is join-irreducible in $\mathcal{L}$, $\partial\{\omega_0 + 1) = \{\omega_0\}$ and $\underline{\{\omega_0 + 1)} = \omega_0$. So, the nest $\mathcal{L}(\omega_1)$ does not satisfy the conditions of Theorem 23 and hence $\kappa(\mathcal{L}(\omega_1)) = \infty$.

Example 27. Let $X = \omega_1 \otimes \mathbb{N}$, where ω_1 is the first uncountable ordinal and $\mathbb{N}$ is the set of natural numbers. Let $\mathcal{L}' = \mathcal{L}(\omega_1) \otimes \mathbb{N}$ denote the set consisting of all subsets of X of the form $\alpha \otimes \mathbb{N}$, where α is an ordinal not greater than ω_1. Each $\alpha \otimes N$ consists of all ordered pairs (β, n), where β is an ordinal less than α and n is a natural number.

The set $\mathcal{L}'$ is a complete nest of subsets of X that contains the trivial subsets. The set X is uncountable, but each of the other sets in $\mathcal{L}'$ is countable. The join-irreducible elements of $\mathcal{L}$ are the sets of the form $\alpha \otimes \mathbb{N}$, where α is a successor ordinal. The join-irreducibles of $\mathcal{L}'$ are all countable, and the atoms of $\mathcal{L}'$ are all countably infinite. So, the nest $\mathcal{L}$ satisfies the conditions of Theorem 23 and hence $\kappa(\mathcal{L}')$ is finite. It is clear from Corollaries 24 and 25 that $\kappa(\mathcal{L}') = 2$.

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