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Gromov–Witten invariants of $\mathbb{P}^2$-stacks

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Abstract

The Gromov–Witten theory of Deligne–Mumford stacks is a recent development, and hardly any computations have been done beyond three-point genus 0 invariants. This paper provides explicit recursions which, together with some invariants computed by hand, determine all genus 0 invariants of the stack $\mathbb{P}^2_{D,2}$. Here D is a smooth plane curve and $\mathbb{P}^2_{D,2}$ is locally isomorphic to the stack quotient $[U/(\mathbb{Z}/(2))]$, where $U \to V \subseteq \mathbb{P}^2$ is a double cover branched along $D \cap V$. The introduction discusses an enumerative application of these invariants.

1. Introduction

The Gromov–Witten invariants of a smooth projective variety X are naive counts of curves in X satisfying various conditions. Such an invariant is specified by choosing a nonnegative integer g, a collection of subvarieties $V_1, \ldots, V_n$ of X, and an algebraic class $\beta \in H_2(X)$. The invariant then provides a rough count of genus g curves of class β in X which pass through general representatives for the classes $[V_1], \ldots, [V_n] \in A^*(X)$. The Gromov–Witten invariant often fails to equal the actual count, essentially because the objects involved cannot always be deformed sufficiently. An often cited example is the case of rational curves on a quintic threefold, in which case there are 'multiple cover contributions' to the Gromov–Witten invariants. These contributions were first studied by Aspinwall and Morrison [AM93].

In general, Gromov–Witten invariants are neither integral nor positive. They fail to be integral because of the presence of automorphisms, which are an issue for multiple cover contributions. The invariants can sometimes be negative due to the virtual fundamental class. This is an element of the Chow group of the stack of stable maps which enters into the definition of the Gromov–Witten invariants. It is often considered to be a technical detail, but in some cases it becomes the central object of study.

From the point of view of enumerative geometry, Gromov–Witten invariants therefore have several drawbacks. What makes them so useful is their computability and the possibility of extracting the enumerative quantities from them. This is especially true in genus 0 if X is a homogeneous space for a semisimple Lie group. Then the Gromov–Witten invariants equal the actual enumerative count, and the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations reduce all of the invariants to a small number of easily computable invariants [FP97]. The first reconstruction theorem of Kontsevich and Manin gives a criterion for when such a reduction via WDVV is possible [KM94].

This paper computes Gromov–Witten invariants of certain Deligne–Mumford stacks. For this, a more general theory is required, and the work has been done by Chen and Ruan [CR02] and Abramovich, Graber, and Vistoli [AGV02]. The stacks we study, referred to as '$\mathbb{P}^2$-stacks' in the title, are parametrized by smooth plane curves D, and are denoted by $\mathbb{P}^2_{D,2}$. If the degree of the curve D happens to be even, then there is a two to one cover of $\mathbb{P}^2$ whose branch locus is D, and $\mathbb{P}^2_{D,2}$ is the

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stack quotient of this cover by the $\mathbb{Z}/(2)$ action. If the degree of D is odd, then this picture only holds locally, but the quotient stacks glue together. The general construction of $\mathbb{P}^2_{D,2}$ is worked out in [Cad].

For the reader who is unfamiliar with stacks, the following may be helpful. If G is a finite group acting on a quasi-projective variety Y, then there is a morphism

$$[Y/G] \to Y/G,$$

where $[Y/G]$ is the stack quotient and Y/G is the scheme quotient. This morphism is a bijection on points, but the stack quotient $[Y/G]$ 'remembers' stabilizer groups of points of Y, which are forgotten in Y/G. A morphism from a scheme S to $[Y/G]$ is the same as a principal G-bundle $P \to S$ together with a G-equivariant map $P \to Y$. So if $y \in Y$ is a point with stabilizer H, then morphisms sending S to the image of y in $[Y/G]$ are equivalent to principal H-bundles on S. This is essentially how the extra information is 'stored' in $[Y/G]$. In general, if one has a morphism $S \to Y/G$, then a lifting to a morphism $S \to [Y/G]$ requires some extra data on S, and such a lifting need not exist.

In this paper, the genus 0 Gromov–Witten invariants of $\mathbb{P}^2_{D,2}$ are reduced to a small number of invariants using the WDVV equations. These basic invariants are then computed by hand. Such a technique might be possible more generally.

The Gromov–Witten invariants of $\mathbb{P}^2_{D,2}$ take as input numerical equivalence classes on the inertia stack of $\mathbb{P}^2_{D,2}$. In this paper we instead use the coarse moduli space of the inertia stack, which is isomorphic to a disjoint union of $\mathbb{P}^2$ with D. The interesting classes are the class of a point in $\mathbb{P}^2$, the fundamental class of D, and the class of a point in D, and we denote them by T_2, T_3, and T_4 respectively.

A twisted stable map is a representable morphism of stacks $\mathfrak{C} \to \mathbb{P}^2_{D,2}$ whose induced morphism of coarse moduli spaces $C \to \mathbb{P}^2$ is an ordinary stable map such that all 'twisted' points of $\mathfrak{C}$ are either nodes or marked points. A twisted point is a point which has a nontrivial stabilizer group, and for stable maps to $\mathbb{P}^2_{D,2}$, this group will always be $\mathbb{Z}/(2)$. The stack $\mathfrak{C}$ is required to have a certain local picture at nodes and marked points [ACV03, § 2.1], and this local picture completely determines $\mathfrak{C}$ given the set of twisted points and the orders of their cyclic stabilizer groups.

Twisted stable maps to $\mathbb{P}^2_{D,2}$ are best understood in terms of tangencies. If C is a smooth curve and $f : C \to \mathbb{P}^2$ is a morphism for which $f^{-1}(D)$ is zero dimensional, then f lifts to a morphism $C \to \mathbb{P}^2_{D,2}$ if and only if $f^*(D) = 2E$ for some divisor E (see [Cad, Theorem 3.3.6]). This leads to the idea that a twisted stable map $F : \mathfrak{C} \to \mathbb{P}^2_{D,2}$ has to have an even order contact with D at every untwisted point of $\mathfrak{C}$. In general, we can characterize genus 0 twisted stable maps $F : \mathfrak{C} \to \mathbb{P}^2_{D,2}$ as follows. Let C be the coarse curve of $\mathfrak{C}$. If $f : C \to \mathbb{P}^2$ is a stable map, then to have a twisted stable map $F : \mathfrak{C} \to \mathbb{P}^2_{D,2}$ lying over f, it is necessary and sufficient that the following hold.

(i) For each component C_i of C not mapping into D, the points of C_i having an odd order of contact with D are precisely the points which are twisted in $\mathfrak{C}$.

(ii) For each component C_i of C which maps into D, the number of points of C_i which are twisted in $\mathfrak{C}$ is congruent modulo 2 to the intersection number $D \cdot C_i$.

This work was motivated by the idea that Gromov–Witten invariants of $P^2_{D,2}$ might count rational plane curves having prescribed contacts with D. While this is often true, there are plenty of counterexamples. When D is a line, the numbers of such curves can be computed from a recursion of Caporaso and Harris [CH98, § 1.4]. The first discrepancy between the Gromov–Witten invariants and the actual numbers occurs in degree 4 with the invariant $I_4(T_2^7 T_4^4)$. If this invariant were enumerative, it would count the number of rational quartics passing through seven general points

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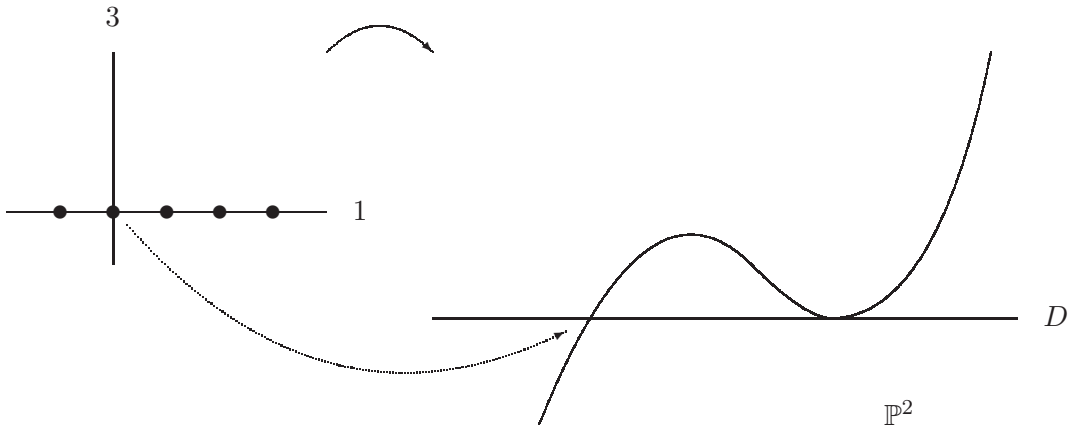
in $\mathbb{P}^2$ and four general points on D. However, the actual number is 398, while the invariant is 416. This is explained by the existence of twisted stable maps from reducible curves into $\mathbb{P}^2_{D,2}$ which are illustrated by the diagram below. The dots correspond to twisted points and the numbers indicate the degree of the component they label.



This shows that there is a contribution to the Gromov–Witten invariant coming from rational cubics which are tangent to the line and pass through the seven general points. There are 36 such curves, but each of them counts with multiplicity 1/2, because the twisted curve in the diagram has a 'ghost automorphism' which 2-commutes with the map. Ghost automorphisms are automorphisms of the twisted curve which cover the identity morphism on its coarse moduli space, and it is shown in [ACV03, Proposition 7.1.1] that they all come from twisted nodes. It is important that the component mapping onto D has an odd number of twisted points, because otherwise a representable morphism would not exist.

When D is a line, we have compared the Gromov–Witten invariants

$$I_d(T_2^{3d-1-a-b}T_3^{d-a-2b}T_4^a)$$

with the number of rational degree d curves C which have b tangencies with D at smooth points of C, meet D transversely at a general points, and meet $3d-1-a-b$ general points in $\mathbb{P}^2$. For all except one case with $d \leqslant 6$, the invariants either give the same answer as the recursion of Caporaso and Harris or otherwise the discrepancy can be accounted for in a manner similar to the above example. The remaining case, $I_6(T_2^{11}T_4^6)$, involves a virtual fundamental class computation which we have not done.

When D is a conic, there are similar contributions which prevent the invariants from being enumerative, but when $\deg(D) \geqslant 3$ no rational curve can map onto D, so such contributions are eliminated. This does not mean that they become enumerative, however. When D is a quartic, there are components consisting entirely of multiple covers which contribute to the invariants. The simplest example is $I_2(T_2)$, which if enumerative would count conics passing through a point which are tangent to D four times. The extra contribution comes from double covers of lines which are tangent to D once such that the branch points of the cover lie at the transverse intersection points. This type of contribution can occur whenever $\deg(D) \geqslant 4$.

When D is a cubic, the moduli space of twisted stable maps still has components consisting of multiple covers, but they do not contribute to the Gromov–Witten invariants. In [Cad05], it is shown that all of the positive degree genus 0 Gromov–Witten invariants of $\mathbb{P}^2_{D,2}$ are enumerative. Stronger enumerative results have recently been obtained. In [CC07], more general 'rth root' stacks were used to compute the numbers of rational degree d curves with arbitrary contact conditions imposed relative to a smooth cubic, except for the case of a single contact of order $3d$.

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An interesting problem is to determine the actual number of rational curves satisfying tangency conditions to D when $\deg(D) \geqslant 4$. When D is a line the problem was solved by Caporaso and Harris [CH98] and, when D is a conic, it was solved by Vakil [Vak00]. The results of this paper could be used for higher degrees if the nonenumerative contributions could be worked out. In the case of a quartic, the Gromov–Witten invariants of $\mathbb{P}^2_{D,2}$ are close to being enumerative in the sense that the extra contributions come from multiple cover components which have the expected dimension. This means that the extra contributions are probably not hard to compute. When the degree is larger than 4, there will be contributions from multiple cover components having greater than the expected dimension. For example, multiple covers of bitangent lines always have this property (when they contribute to an invariant). This makes the extra contributions much more difficult to compute.

It would also be interesting to find a general formula for the nonenumerative contributions in the case of a line or conic. Here again, components having greater than the expected dimension would be difficult to handle. In any case, a new idea is probably required. The referee wondered whether there is a 'splitting axiom' which would allow one to account for the contributions from reducible source curves, as in the example illustrated above. There is a splitting axiom for twisted Gromov–Witten invariants [AGV06, Proposition 5.3.1], but it does not seem to apply to this situation. Some other kind of formula might hold in this situation, but none is known to the author.

To give the reader a sense for the role played by the virtual fundamental class, we give some invariants which are computed in this paper. Firstly, it should be mentioned that for stacks, there can exist genus 0, n-pointed degree 0 invariants with $n > 3$. This never happens for varieties due to the forgetting-a-point axiom. For the $\mathbb{P}^2$-stacks considered in this paper, there is exactly one such invariant, $I_0(T_3^3T_4)$ in the above notation, and it is equal to $-1/4$. When D is a line or a conic, there exist infinitely many invariants of degree $\delta := \deg(D)$. These come from stable maps which on the level of coarse moduli spaces have image D, which is why they only occur when D is rational. If

$$\lambda_k = \frac{(-1)^k k!}{2^{k+1}} \quad \text{for } k \geqslant 0,$$

then when D is a line, the Gromov–Witten invariant $I_1(T_3^kT_4^{k+3}) = \lambda_k$, while if D is a conic, $I_2(T_3^kT_4^{k+6}) = \lambda_k$.

The paper is organized as follows. Sections 2 and 3 contain general results about stable maps into $X_{D,r}$. Section 2 defines evaluation maps and Gromov–Witten invariants, while § 3 deals with the virtual fundamental class and works out the expected dimension of the stack of stable maps. Along the way, we investigate the tangent bundle of $X_{D,r}$ in § 3.4. We relate it to the tangent bundle of X by showing that the sheaf of logarithmic vector fields on X relative to D pulls back to the sheaf of logarithmic vector fields on $X_{D,r}$ relative to '$\frac{1}{r}D$'. Section 4 contains direct computations of some degree 0 and 1 invariants which are needed in order to use the recursions. Finally, in § 5 we define the big quantum product, work out the recursions, and provide an algorithm which can be used to compute any genus 0 invariant.

1.1 Notation and conventions

Throughout this paper, all schemes are equipped with structure morphisms to $\operatorname{Spec}\mathbb{C}$ and all morphisms of schemes respect these structure morphisms. As a consequence, the same is true for all stacks that appear. We use μ_r to denote the group of rth roots of unity in $\mathbb{C}$. Given a stack $\mathfrak{X}$, we use $\widehat{\mathfrak{X}}$ to denote the coarse moduli space of its inertia stack, assuming it exists.

In this paper, we deviate slightly from [AGV02] where Gromov–Witten invariants are defined in terms of a stack of twisted stable maps with sections of all gerbes, which is denoted by $\overline{\mathcal{M}}_{g,n}(\mathfrak{X}, \beta)$. We prefer instead to work with the stack $\mathcal{K}_{g,n}(\mathfrak{X}, \beta)$, which is the one defined in [AV02]. We implicitly assume that all twisted stable maps are balanced. In addition, we work with evaluation morphisms

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from $\mathcal{K}_{g,n}(\mathfrak{X},\beta)$ to the coarse moduli space of the inertia stack, rather than the rigidification of the inertia stack used in [AGV02].

2. Stable maps to rth root stacks

Let X be a smooth projective variety over $\mathbb{C}$, $D \subseteq X$ be a smooth divisor, and r be a positive integer. In this section, we investigate stable maps into $X_{D,r}$ and define Gromov–Witten invariants, although we leave a discussion of the virtual fundamental class for later. First we introduce the contact type of a twisted stable map and define evaluation maps in terms of this. It is straightforward to show that the definition in [AGV02] gives rise to the same evaluation maps.

2.1 The rth root construction

We now give a summary of results from [Cad, § 2] which are needed in this paper. Given a scheme X, an effective Cartier divisor $D \subseteq X$, and a positive integer r which is invertible on X, there is a Deligne–Mumford stack $X_{D,r}$ over X on which the pair $(\mathcal{O}_X(D), s_D)$ has an rth root. More precisely, if $f : S \to X$ is a morphism of schemes, then an object of $X_{D,r}$ over f consists of a triple (M, t, φ), where M is an invertible sheaf on S, t is a global section of M, and $\varphi : M^r \to f^*\mathcal{O}_X(D)$ is an isomorphism such that $\varphi(t^r) = f^*s_D$. There is a universal object on $X_{D,r}$ covering the projection $\pi : X_{D,r} \to X$. We denote the universal line bundle $\mathcal{T}$ and its section τ. The vanishing locus of τ is a substack $\mathfrak{G} \subseteq X_{D,r}$ which is mapped into D by π, and the restriction $\mathfrak{G} \to D$ is an étale gerbe with band μ_r. The projection π is an isomorphism away from D, and it is ramified over D. In fact, $\pi^{-1}(D)$ is the rth-order infinitesimal neighborhood of $\mathfrak{G}$ in $X_{D,r}$.

2.2 Twisted curves

Recall that a prestable curve over a Noetherian scheme S is a flat, proper morphism $C \to S$ whose geometric fibers are curves with at worst nodal singularities. An n-marked prestable curve has in addition n sections $s_i : S \to C$ whose images are disjoint and do not intersect the singular locus of $C \to S$. To give n such sections is equivalent to giving n disjoint effective Cartier divisors $D_i \subseteq C$ which map isomorphically to S (see [Cad, § 4]). The passage from curves to twisted curves requires the addition of stack structure at the nodes and markings [AV02, Definition 4.1.2]. The stack structure at a marking is always obtained from the coarse moduli space by applying the rth root construction to the image of the section. We do not need to consider the stack structure at the node, but for an interesting treatment see [Ols].

Given a Deligne–Mumford stack $\mathfrak{C}$, n divisors D_i, and n positive integers r_i, we use $\mathfrak{C}_{\mathbb{D},\vec{r}}$ for the result of applying the r_1th root construction along D_1, followed by the r_2th root construction along D_2, and so on.

PROPOSITION 2.2.1. *To give a twisted nodal n-pointed genus g curve over a connected Noetherian scheme S is equivalent to giving:*

(i) *a nodal n-pointed genus g curve C over S with markings $D_1, \dots, D_n$;*

(ii) *for each marking $D_i \subseteq S$, a positive integer r_i; and*

(iii) *a twisted nodal 0-pointed genus g curve $\mathfrak{C}$ over S, whose coarse moduli scheme is C.*

Then the twisted curve is isomorphic to $\mathfrak{C}_{\mathbb{D},\vec{r}}$ in such a way that the ith marking of the twisted curve is sent to the gerbe over D_i.

This can be proven in the same way as [Cad, Theorem 4.1] and also follows from [Ols]. Note that $\mathfrak{C}_{\mathbb{D},\vec{r}} \cong \mathfrak{C} \times_C C_{\mathbb{D},\vec{r}}$. Moreover, the divisors D_i do not pass through any nodes of the fibers, so the r_ith root construction along D_i is taking place away from the points of $\mathfrak{C}$ which have stack structure.

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While this statement is not very precise, it is meant to assure the reader that much of what was done in [Cad] carries over without change. In particular, the notion of contact type extends to arbitrary stable maps $\mathfrak{C}_{\mathbb{D},\vec{r}} \to X_{D,r}$.

2.3 Contact type

In [Cad, § 3.2], we derived some key results about line bundles and sections on $C_{\mathbb{D},\vec{r}}$. They extend without change to $\mathfrak{C}_{\mathbb{D},\vec{r}}$, because the results of [Cad, § 3.1], from which they follow, are stated in sufficient generality. We state the more general results here.

Let S be a *connected*, Noetherian scheme and let C, $\mathfrak{C}$, and D_i be as in Proposition 2.2.1. Let $\gamma : \mathfrak{C}_{\mathbb{D},\vec{r}} \to \mathfrak{C}$ be the projection, and let $\mathcal{T}_i$ be the tautological sheaf associated to the r_ith root construction along D_i. The important properties of these sheaves are contained in the next two corollaries.

Corollary 2.3.1. *Let $\mathcal{L}$ be an invertible sheaf on $\mathfrak{C}_{\mathbb{D},\vec{r}}$. Then there exist an invertible sheaf L on $\mathfrak{C}$ and integers k_i satisfying $0 \leqslant k_i \leqslant r_i - 1$ such that*

$$\mathcal{L} \cong \gamma^* L \otimes \prod_{i=1}^{n} \mathcal{T}_i^{k_i}.$$

Moreover, the integers k_i are unique, L is unique up to isomorphism, and $\mathcal{T}_i^{r_i} \cong \gamma^* \mathcal{O}_{\mathfrak{C}}(D_i)$.

Corollary 2.3.2. *Given the decomposition in the previous corollary, every global section of $\mathcal{L}$ is of the form $\gamma^* s \otimes \tau_1^{k_1} \otimes \cdots \otimes \tau_n^{k_n}$ for a unique global section s of L, where τ_i is the tautological section of $\mathcal{T}_i$.*

If $F : \mathfrak{C}_{\mathbb{D},\vec{r}} \to X_{D,r}$ is any morphism, then Corollary 2.3.1 associates to $F^*\mathcal{T}$ a unique n-tuple of integers $k_1, \ldots, k_n$, where $\mathcal{T}$ is the tautological sheaf of $X_{D,r}$. Let $U \subseteq C$ be the complement of the divisors D_i and let $\mathfrak{U} \subseteq \mathfrak{C}$ be its preimage. Note that $\mathfrak{U}$ is isomorphic to its preimage in $\mathfrak{C}_{\mathbb{D},\vec{r}}$.

Proposition 2.3.3. *The morphism F is representable if and only if its restriction to $\mathfrak{U}$ is representable and for every i, r_i divides r, and k_i is relatively prime to r_i.*

The proof goes through as in [Cad, Proposition 3.3.3]. Being representable is a pointwise condition by [AV02, Lemma 4.4.3], so for F to be representable is equivalent to its restrictions to $\mathfrak{U}$ and the gerbes over $D_1, \ldots, D_n$ being representable.

Now assume that F is representable.

Definition 2.3.4. Let $\varrho_i = k_i r / r_i$ for $1 \leqslant i \leqslant n$. We define the *contact type* of F to be the n-tuple $\vec{\varrho} = (\varrho_1, \ldots, \varrho_n)$.

Multiplying the identity $\gcd(k_i, r_i) = 1$ by r/r_i yields $\gcd(\varrho_i, r) = r/r_i = \varrho_i/k_i$. Thus, $\vec{k}$ and $\vec{r}$ are determined by $\vec{\varrho}$ via the following formulas:

$$r_i = \frac{r}{\gcd(\varrho_i, r)}, \quad k_i = \frac{\varrho_i}{\gcd(\varrho_i, r)}. \tag{2.3.5}$$

For any $\beta \in N_1(X)$, the contact type determines n locally constant functions on $\mathcal{K}_{g,n}(X_{D,r}, \beta)$ with integer values from 0 to $r-1$. We define $\mathcal{K}_{g,n}(X_{D,r}, \beta, \vec{\varrho})$ to be the open and closed substack of $\mathcal{K}_{g,n}(X_{D,r}, \beta)$ consisting of stable maps with contact type $\vec{\varrho}$. In [Cad], we found that a stable map from a smooth twisted curve into $X_{D,r}$ which does not map into the gerbe has to have a contact type for which $D \cdot \beta - \sum \varrho_i$ is a multiple of r. Equation (3.5.1) implies that this holds in general.

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2.4 Evaluation maps

In [AGV02], evaluation maps are defined which go from $\mathcal{K}_{g,n}(X_{D,r},\beta)$ to a rigidification of the inertia stack of $X_{D,r}$. This allows for a richer structure than is required for our purposes. We instead define evaluation maps which go to the coarse moduli scheme of the inertia stack, which we denote by $\widehat{X_{D,r}}$. It turns out that $\widehat{X_{D,r}}$ is isomorphic to a disjoint union of X with $r-1$ copies of D. We denote these copies of D by $(\widehat{X_{D,r}})_i$ for $1 \leqslant i \leqslant r-1$, and let $(\widehat{X_{D,r}})_0$ denote the component X. A morphism from a connected scheme S to $\widehat{X_{D,r}}$ is therefore determined by a morphism $S \to X$ and an integer from 0 to $r-1$ which is positive only if $S \to X$ factors through D.

Given an object ξ of $\mathcal{K}_{g,n}(X_{D,r},\beta)$ over a connected Noetherian scheme S, let $e_i(\xi)$ be the pair (g,ϱ_i), where ϱ_i is the ith component of the contact type of ξ and $g : S \to X$ is the composition of the ith section $S \to D_i \subseteq \mathfrak{C}$ with the morphism $\mathfrak{C} \to X$.

$$\begin{array}{ccccc} & & \mathfrak{C}_{\mathbb{D},\vec{r}} & \xrightarrow{F} & X_{D,r} \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow \\ D_i & \hookrightarrow & \mathfrak{C} & \xrightarrow{f} & X \end{array}$$

In light of the following lemma, this defines a morphism $e_i : \mathcal{K}_{g,n}(X_{D,r},\beta) \to \widehat{X_{D,r}}$.

LEMMA 2.4.1. *If $\varrho_i \neq 0$, then the morphism $g : S \to X$ factors through D.*

Proof. The morphism F determines an invertible sheaf M on $\mathfrak{C}_{\mathbb{D},\vec{r}}$, which is isomorphic to $\gamma^*L \otimes \prod_{j=1}^n \mathcal{T}_j^{k_j}$ by Corollary 2.3.1. It also determines a section t of M, which corresponds to $\gamma^*s \prod \tau_j^{k_j}$ by Corollary 2.3.2. Moreover, M^r is isomorphic to $\gamma^*f^*\mathcal{O}_X(D)$ in such a way that t^r goes to $\gamma^*f^*s_D$. Taking rth powers of our expressions for M and t, the uniqueness assertions of Corollaries 2.3.1 and 2.3.2 imply that $L^r(\sum \varrho_j D_j) \cong f^*\mathcal{O}_X(D)$ by an isomorphism sending $s^r \prod s_{D_j}^{\varrho_j}$ to f^*s_D. It follows that if $\varrho_i \neq 0$, then f^*s_D vanishes on D_i, which proves the lemma. □

2.5 Gromov–Witten invariants

To define Gromov–Witten invariants, one must first define the virtual fundamental class $[\mathcal{K}_{g,n}(X_{D,r},\beta)]^v$. This is an element of the Chow group of $\mathcal{K}_{g,n}(X_{D,r},\beta)$ which has the expected dimension. For now we assume that it has already been defined.

Fix integers $g,n \geqslant 0$ and an effective class $\beta \in N_1(X)$. Let $\vec{e} = e_1 \times \cdots \times e_n : \mathcal{K}_{g,n}(X_{D,r},\beta) \to \widehat{X_{D,r}}^{\times n}$ and let $p : \widehat{X_{D,r}}^{\times n} \to \operatorname{Spec}\mathbb{C}$ be the structure morphism. Since $\mathcal{K}_{g,n}(X_{D,r},\beta)$ is proper over $\mathbb{C}$ (see [AV02, Theorem 1.4.1]), there is a proper pushforward $\vec{e}_*$ on Chow groups with rational coefficients defined by [Vis89, Definition 3.6].

Given any numerical equivalence classes $a_1,\ldots,a_n \in N^*_{\mathbb{Q}}(\widehat{X_{D,r}})$, the Gromov–Witten invariant is defined to be

$$I^g_\beta(a_1\cdots a_n) := p_*((a_1 \times \cdots \times a_n) \cap \vec{e}_*[\mathcal{K}_{g,n}(X_{D,r},\beta)]^v), \tag{2.5.1}$$

which is regarded as an element of $\mathbb{Q}$. As the definition is clearly symmetric in $a_1,\ldots,a_n$, we regard I^g_β as an element of the dual of $\operatorname{Sym}^*(N^*_{\mathbb{Q}}(\widehat{X_{D,r}}))$. The following proposition is an immediate consequence of the definitions.

PROPOSITION 2.5.2. *Suppose $a_i = [V_i]$ for some (irreducible) subvariety $V_i \subseteq \widehat{X_{D,r}}$ for $1 \leqslant i \leqslant n$. Let $\vec{\varrho}$ be such that $V_i \subseteq (\widehat{X_{D,r}})_{\varrho_i}$ for each i. If there are no stable maps $\mathfrak{C} \to X_{D,r}$ in $\mathcal{K}_{g,n}(X_{D,r},\beta,\vec{\varrho})$ such that the ith marked point is sent under $\mathfrak{C} \to X_{D,r} \to X$ into the image of $V_i \subseteq \widehat{X_{D,r}} \to X$ for all i, then $I^g_\beta(a_1\cdots a_n) = 0$.*

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3. Virtual fundamental class

3.1 Preliminaries

This section is devoted to studying the virtual fundamental class for the moduli space of stable maps into $X_{D,r}$. To begin, we consider an arbitrary smooth, proper, Deligne–Mumford stack $\mathfrak{X}$ which has a projective coarse moduli scheme. Note that $X_{D,r}$ is smooth by Proposition 3.4.1. The reader may also wish to consult [AGV02, § 4.6].

The virtual fundamental class of $\mathcal{K} := \mathcal{K}_{g,n}(\mathfrak{X}, \beta)$ is constructed in the same way as it is for varieties [Beh97, BF97, LT98]. Here we adopt the approach of Behrend and Fantechi. Let $\mathfrak{M} := \mathfrak{M}_{g,n}^{\mathrm{tw}}$ be the smooth Artin stack of twisted curves with no stability condition. This is obtained from [Ols, Theorem 1.9] by base extension to $\operatorname{Spec}\mathbb{C}$. We define an element $E^\bullet$ of the derived category of $\mathcal{K}$ which is equivalent to a two-term complex of locally free sheaves in positions -1 and 0. Then we define a morphism $\phi : E^\bullet \to L_{\mathcal{K}/\mathfrak{M}}$ which is surjective on h^{-1} and an isomorphism on h^0. After verifying that ϕ is a perfect relative obstruction theory, the construction of [BF97, § 7] produces a virtual fundamental class $[\mathcal{K}]^v \in A_*(\mathcal{K})$. Note that the mapping cone construction allows one to go from a perfect relative obstruction theory to an absolute theory which determines the same virtual fundamental class [GP99, Appendix B].

Before proceeding to define ϕ, we state some consequences of this construction. On any connected component of $\mathcal{K}$, $[\mathcal{K}]^v$ is homogeneous of degree $\operatorname{rk}(E^\bullet) + \dim(\mathfrak{M})$, which is called the expected dimension. By [BF97, § 4], a perfect obstruction theory for $\mathcal{K}$ gives rise to an obstruction theory in the classical sense at any point of $\mathcal{K}$. It follows that at every point of $\mathcal{K}$, the dimension is greater than or equal to the expected dimension with equality implying that $\mathcal{K}$ is a local complete intersection at that point [Har04, Corollary 7.4].

Suppose that $\mathcal{U} \subseteq \mathcal{K}_{g,n}(\mathfrak{X}, \beta)$ is an open substack which has the expected dimension. By [BF97, 5.10], the restriction of the virtual fundamental class to $\mathcal{U}$ is equal to the virtual fundamental class determined by the restriction of ϕ to $\mathcal{U}$. Since $\mathcal{U}$ has the expected dimension, the kernel of

$$h^{-1}(\phi|_{\mathcal{U}}) : h^{-1}(E^\bullet|_{\mathcal{U}}) \to h^{-1}(L_{\mathcal{U}/\mathfrak{M}})$$

has rank 0. Moreover, it is torsion free since it is a subsheaf of a locally free sheaf. Therefore, it equals 0 and $\phi|_{\mathcal{U}}$ is an isomorphism. The following proposition follows by definition of the virtual fundamental class [BF97, § 5].

Proposition 3.1.1. *If $\mathcal{U} \subseteq \mathcal{K}_{g,n}(\mathfrak{X}, \beta)$ is an open substack which has the expected dimension, then the restriction of $[\mathcal{K}_{g,n}(\mathfrak{X}, \beta)]^v$ to $\mathcal{U}$ is $[\mathcal{U}]$, the ordinary fundamental class.*

3.2 Perfect relative obstruction theory

We continue with our previous notation. Let $\mathcal{C}$ be the universal curve over $\mathcal{K}$.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathfrak{X} \\ {\scriptstyle\pi}\downarrow & & \\ \mathcal{K} & & \end{array}$$

There exist canonical log structures on $\mathcal{C}$ and $\mathcal{K}$ which make π log smooth [Ols, § 3]. Let ω be the relative sheaf of log differentials, so that ω is a dualizing sheaf for π.

Let

$$E^\bullet = (R\pi_*(f^*T_{\mathfrak{X}}))^\vee.$$

By duality, $E^\bullet$ is isomorphic to $R\pi_*(f^*L_{\mathfrak{X}} \otimes \omega)$. Let $\mathcal{C}^{\mathrm{tw}} \to \mathfrak{M}$ be the universal twisted curve. We have morphisms $f^*L_{\mathfrak{X}} \to L_{\mathcal{C}} \to L_{\mathcal{C}/\mathcal{C}^{\mathrm{tw}}}$ and an isomorphism $\pi^*L_{\mathcal{K}/\mathfrak{M}} \to L_{\mathcal{C}/\mathcal{C}^{\mathrm{tw}}}$, the latter coming

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from the following fiber square.

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathrm{tw}} \\ {\scriptstyle\pi}\downarrow & \square & \downarrow \\ \mathcal{K} & \longrightarrow & \mathfrak{M} \end{array}$$

Composing, we obtain a morphism $f^*L_{\mathfrak{X}} \to \pi^*L_{\mathcal{K}/\mathfrak{M}}$, which by duality is equivalent to a morphism $\phi : E^\bullet \to L_{\mathcal{K}/\mathfrak{M}}$.

We show that $E^\bullet$ is equivalent to a two-term complex of locally free sheaves (it is enough for this to hold locally on $\mathcal{K}$, but it is true globally). Let $\overline{\pi} : \overline{\mathcal{C}} \to \mathcal{K}$ be the relative coarse moduli space of $\mathcal{C}$ over $\mathcal{K}$, i.e. the fibers of $\overline{\pi}$ are the coarse moduli spaces of the fibers of π. Let $\gamma : \mathcal{C} \to \overline{\mathcal{C}}$ be the projection. By [OS03, Lemma 4.2], $\gamma_* f^* T_{\mathfrak{X}}$ is a coherent sheaf which is flat over $\mathcal{K}$, and $R^i\gamma_* f^* T_{\mathfrak{X}} = 0$ for $i > 0$. It follows that $E^\bullet \cong R\overline{\pi}_*(\gamma_* f^* T_{\mathfrak{X}})$. One can now apply [Beh97, Proposition 5], since the proof of that proposition requires only that E is a coherent sheaf, flat over T.

Since deformation theory for Deligne–Mumford stacks works in the same way as for schemes (all the results of [Ill71, Ill72] apply), the remainder of the proof that ϕ is a perfect relative obstruction theory is identical to the case where $\mathfrak{X}$ is a smooth projective variety.

In summary, if $\mathcal{C} = (f : \mathfrak{C} \to \mathfrak{X}, \Sigma_1, \ldots, \Sigma_n)$ is a stable map in $\mathcal{K}_{g,n}(\mathfrak{X}, \beta)$, then the expected dimension of $\mathcal{K}_{g,n}(\mathfrak{X}, \beta)$ at $\mathcal{C}$ is

$$\chi(f^* T_{\mathfrak{X}}) + \dim(\mathfrak{M}^{\mathrm{tw}}_{g,n}). \tag{3.2.1}$$

3.3 Riemann–Roch for twisted curves

Here we recall a version of the Riemann–Roch theorem for balanced nodal twisted curves which is proven in [AGV06]. It will be applied later to $f^*TX_{D,r}$, where $f : \mathfrak{C} \to X_{D,r}$ is a twisted stable map. First we recall the definition of age.

Let $\mathfrak{C}$ be a balanced twisted nodal curve, and let $\mathcal{E}$ be a locally free sheaf on $\mathfrak{C}$. For any closed point $x \in \mathfrak{C}$ there is a closed substack $\mathfrak{G}_x \subseteq \mathfrak{C}$ called the residue gerbe of x (see [LMB00, § 11]). It is isomorphic to $B\mu_r$ for some positive integer r called the *order of twisting* of x. If $r > 1$, x is called a *twisted point*, and otherwise it is called *untwisted*. The restriction of $\mathcal{E}$ to $B\mu_r$ determines a representation of μ_r, but this is not canonical since it depends on the isomorphism $B\mu_r \to \mathfrak{G}_x$. If x is a smooth point, then we can choose an isomorphism $B\mu_r \to \mathfrak{G}_x$ so that the restriction of the tangent bundle $T\mathfrak{C}$ to $B\mu_r$ corresponds to the standard representation of μ_r. Then the representation of μ_r determined by $\mathcal{E}$ is well-defined, and we denote this representation by $\mathcal{E}_x$.

At nodes, one can study $\mathcal{E}$ by pulling back to the normalization of $\mathfrak{C}$ (see [Vis89, Definition 1.18]). Locally in the étale topology, the node looks like the stack quotient of $\operatorname{Spec}\mathbb{C}[x,y]/(xy)$ by the group μ_r which acts by $\zeta \cdot (x,y) = (\zeta x, \zeta^{-1}y)$. The normalization of this stack is two copies of the affine line each with the origin twisted to order r. So a nodal point $x \in \mathfrak{C}$ has two preimages x_i in the normalization of $\mathfrak{C}$, and each determines canonically a representation $\mathcal{E}_{x_i}$ of μ_r. The fact that $\mathfrak{C}$ is balanced (encoded in the action above) implies that these are dual representations.

DEFINITION 3.3.1. The *age* of a locally free sheaf $\mathcal{E}$ at a smooth point x of a twisted curve $\mathfrak{C}$ is a rational number denoted by $\operatorname{age}(\mathcal{E}, x)$. Let V denote the standard representation of μ_r, where r is the order of twisting of x. If

$$\mathcal{E}_x \cong \bigoplus_{i=1}^{n} V^{\otimes k_i}$$

where $0 \leqslant k_i \leqslant r-1$, then

$$\operatorname{age}(\mathcal{E}, x) = \sum_{i=1}^{n} \frac{k_i}{r}.$$

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Theorem 3.3.2. *Let $\mathfrak{C}$ be a balanced twisted nodal curve of genus g over $\mathbb{C}$ and let $\mathcal{E}$ be a locally free sheaf on $\mathfrak{C}$. Then*

$$h^0(\mathfrak{C},\mathcal{E}) - h^1(\mathfrak{C},\mathcal{E}) = \deg(\mathcal{E}) + \mathrm{rk}(\mathcal{E})(1-g) - \sum_x \mathrm{age}(\mathcal{E},x),$$

where the sum is over the twisted smooth points of $\mathfrak{C}$.

3.4 Tangent bundle of $X_{D,r}$

To apply the Riemann–Roch theorem to $f^*TX_{D,r}$, we need some results about the tangent bundle to $X_{D,r}$. Let $\mathcal{T}$ be the tautological sheaf on $X_{D,r}$ (see § 2.1) and let $\mathfrak{G}$ be the gerbe of $X_{D,r}$. We use $\mathfrak{G}_k$ to denote the kth-order infinitesimal neighborhood of $\mathfrak{G}$. Let $\pi : X_{D,r} \to X$ be the projection.

Recall from [Cad, Example 2.4.1] that $X_{D,r}$ is locally a stack quotient in the following way. If $V = \operatorname{Spec} S \subseteq X$ is an affine open set on which we have a trivialization of $\mathcal{O}_X(D)$, then the tautological section of $\mathcal{O}_X(D)$ corresponds to an element $\alpha \in S$, and if $U = \operatorname{Spec} A$, where $A = S[x]/(x^r - \alpha)$, then $X_{D,r} \times_X V$ is isomorphic to $[U/\mu_r]$, where μ_r acts on A by $(\zeta, x) \mapsto \zeta^{-1}x$ and $(\zeta, s) \mapsto s$ for $\zeta \in \mu_r$ and $s \in S$. Moreover, the preimage of $\mathfrak{G}_k$ in U is defined by the vanishing of x^k. This shows that $\mathfrak{G}_r = \pi^{-1}(D)$.

We assume that X is a smooth projective variety and $D \subseteq X$ is a smooth divisor. First we show that $X_{D,r}$ is smooth, so that $TX_{D,r}$ is locally free.

Proposition 3.4.1. *We have that $X_{D,r}$ is smooth over $\mathbb{C}$.*

Proof. Using the above notation, it suffices to show that U is smooth. Since U is a closed subscheme of the smooth variety $V \times \mathbb{A}^1$ defined by $x^r - \alpha = 0$, it suffices to show that for any closed point $p \in V \times \mathbb{A}^1$, $x^r - \alpha \notin m_p^2$, where $m_p \subseteq \mathcal{O}_{V\times\mathbb{A}^1,p}$ is the maximal ideal.

If $x^r - \alpha \in m_p^2$, then $\partial/\partial x(x^r - \alpha) \in m_p$, so $rx^{r-1} \in m_p$, which implies $x \in m_p$. It follows that $p \in D \times \{0\}$ and $\alpha \in m_p^2$, which contradicts the smoothness of D. □

Proposition 3.4.2. *There is an exact sequence of sheaves on $X_{D,r}$:*

$$0 \to TX_{D,r} \to \pi^*TX \to \pi^*\mathcal{O}_X(D) \otimes \mathcal{O}_{\mathfrak{G}_{r-1}} \to 0.$$

Proof. Since $\pi^*\mathcal{O}_D \cong \mathcal{O}_{\mathfrak{G}_r}$, the morphisms in the above sequence clearly exist. The injectivity of the differential follows from the fact that it is an isomorphism away from $\mathfrak{G}$. Note also that

$$\pi^*TX \to \pi^*(TX \otimes \mathcal{O}_D) \to \pi^*(\mathcal{O}(D) \otimes \mathcal{O}_D) \to \pi^*\mathcal{O}(D) \otimes \mathcal{O}_{\mathfrak{G}_{r-1}}$$

is a sequence of surjective maps. It remains to show exactness in the middle, and this can be shown locally. It suffices to show this after pulling back to U since $U \to X_{D,r}$ is étale. So we need to show that the following sequence of A-modules is exact in the middle

$$0 \to \mathrm{Der}_{\mathbb{C}}(A,A) \to \mathrm{Der}_{\mathbb{C}}(S,S) \otimes_S A \to \alpha^{-1}A/x^{-1}A \to 0.$$

The second homomorphism is defined by sending a derivation $\delta : S \to S$ to $\alpha^{-1}\delta(\alpha)$. The composition is 0 because $\alpha^{-1}\delta(x^r) = rx^{-1}\delta(x)$. Note that a derivation $A \to A$ is uniquely determined by a derivation $\delta : S \to A$ and an element $\delta(x) \in A$ such that $rx^{r-1}\delta(x) = \delta(\alpha)$. If $\sum a_i\delta_i \mapsto 0$ in $\alpha^{-1}A/x^{-1}A$, then $\sum a_i\delta_i(\alpha) \in \alpha x^{-1}A = x^{r-1}A$, so $\sum a_i\delta_i$ extends to a derivation $A \to A$. □

Let $\mathcal{E}$ be the kernel of $TX \to \mathcal{O}_D(D)$, which is locally free. From the above proposition, it follows that as subsheaves of π^*TX, $\pi^*\mathcal{E}$ is a subsheaf of $TX_{D,r}$. We have the following commutative

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diagrams with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\mathcal{E} & \longrightarrow & \pi^*TX & \longrightarrow & \pi^*\mathcal{O}_D(D) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & TX_{D,r} & \longrightarrow & \pi^*TX & \longrightarrow & \pi^*\mathcal{O}_X(D)\otimes\mathcal{O}_{\mathfrak{G}_{r-1}} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X_{D,r}} & \longrightarrow & \mathcal{T}^{\otimes r} & \longrightarrow & \pi^*\mathcal{O}_D(D) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}^{\otimes r} & \longrightarrow & \pi^*\mathcal{O}_X(D)\otimes\mathcal{O}_{\mathfrak{G}_{r-1}} & \longrightarrow & 0 \end{array}$$

For the second diagram, note that $\mathcal{T}^{\otimes r} \cong \pi^*\mathcal{O}_X(D)$ and that $\mathfrak{G}_{r-1}$ is cut out by a section of $\mathcal{T}^{\otimes(r-1)}$. By applying the snake lemma to both diagrams, we see that the cokernel of $\pi^*\mathcal{E} \to TX_{D,r}$ is isomorphic to the cokernel of $\mathcal{O}_{X_{D,r}} \to \mathcal{T}$, which is $\mathcal{T}\otimes\mathcal{O}_{\mathfrak{G}}$ since this morphism is the tautological section which cuts out $\mathfrak{G}$. So we have an exact sequence

$$0 \to \pi^*\mathcal{E} \to TX_{D,r} \to \mathcal{T}\otimes\mathcal{O}_{\mathfrak{G}} \to 0, \tag{3.4.3}$$

and therefore

$$c_1(TX_{D,r}) = \pi^*\left(c_1(TX) - \frac{r-1}{r}[D]\right). \tag{3.4.4}$$

PROPOSITION 3.4.5. *There is an exact sequence of sheaves on* $\mathfrak{G}$*:*

$$0 \to \mathcal{O}_{\mathfrak{G}} \to \pi^*\mathcal{E}\otimes\mathcal{O}_{\mathfrak{G}} \to TX_{D,r}\otimes\mathcal{O}_{\mathfrak{G}} \to \mathcal{T}\otimes\mathcal{O}_{\mathfrak{G}} \to 0.$$

Note that $\pi^*\mathcal{E}\otimes\mathcal{O}_{\mathfrak{G}}/\mathcal{O}_{\mathfrak{G}}$ is locally free since it is the kernel of a surjection of locally free sheaves.

Proof. This follows from (3.4.3), given the fact that $\operatorname{Tor}_1^{\mathcal{O}_{X_{D,r}}}(\mathcal{T}\otimes\mathcal{O}_{\mathfrak{G}}, \mathcal{O}_{\mathfrak{G}}) = \mathcal{O}_{\mathfrak{G}}$, which follows by tensoring the exact sequence

$$0 \to \mathcal{O}_{X_{D,r}} \to \mathcal{T} \to \mathcal{T}\otimes\mathcal{O}_{\mathfrak{G}} \to 0$$

with $\mathcal{O}_{\mathfrak{G}}$. □

Remark 3.4.6. The above exact sequences all hold if X is a smooth twisted curve and D is an untwisted point of X. Therefore, (3.4.3) implies that

$$T\mathfrak{C} \cong TC\Big(-\sum p_i\Big)\otimes\prod\mathcal{T}_i,$$

where $\mathfrak{C}$ is a smooth twisted curve with coarse moduli scheme C, p_i are the points of C which are twisted in $\mathfrak{C}$, and $\mathcal{T}_i$ is the tautological sheaf corresponding to p_i. Recalling the notation from the beginning of § 3.1, it follows that $(\mathcal{T}_i)_{x_i}$ is the standard representation, where $x_i \in \mathfrak{C}$ is the preimage of p_i.

Let $f : \mathfrak{C} \to X_{D,r}$ be a genus g twisted stable map of class β and contact type $\vec{\varrho}$ (see Definition 2.3.4). Then (3.4.4) implies that

$$\deg(f^*TX_{D,r}) = \deg(f^*\pi^*TX) - \frac{r-1}{r}\beta\cdot D = -\beta\cdot(K_X + D) + \frac{1}{r}\beta\cdot D. \tag{3.4.7}$$

Proposition 3.4.5 implies that the age of $f^*TX_{D,r}$ at a twisted point is equal to the age of $f^*\mathcal{T}$ at the same point. From the definition of contact type, we have

$$\operatorname{age}(f^*TX_{D,r}, x_i) = \frac{\varrho_i}{r} \tag{3.4.8}$$

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where x_i is the ith marked point. Since $\mathrm{rk}(TX_{D,r}) = \mathrm{rk}(TX) = \dim(X)$, Theorem 3.3.2 implies that

$$\chi(f^*TX_{D,r}) = -\beta\cdot(K_X + D) + \frac{1}{r}\left(\beta\cdot D - \sum_{i=1}^{n}\varrho_i\right) - \dim(X)(g-1). \tag{3.4.9}$$

3.5 Expected dimension of $\mathcal{K}_{g,n}(X_{D,r},\beta,\vec{\varrho})$

Fix g, n, β, ϱ and let $\mathcal{K} = \mathcal{K}_{g,n}(X_{D,r},\beta,\vec{\varrho})$. From (3.4.9), it is clear that the restriction of the virtual fundamental class to $\mathcal{K}$ (which we denote by $[\mathcal{K}]^v$) is homogeneous in the Chow group of $\mathcal{K}$. Its degree is the expected dimension of $\mathcal{K}$ and is denoted by $\mathrm{edim}(\mathcal{K})$. This number is important in Gromov–Witten theory because it determines which Gromov–Witten invariants can be nonzero. For example, suppose that we choose for each i an element $a_i \in N^{c_i}((\widehat{X_{D,r}})_{\varrho_i}) \subseteq N^*(\widehat{X_{D,r}})$. Then the Gromov–Witten invariant $I^g_\beta(a_1\cdots a_n)$ can be nonzero only if $\mathrm{edim}(\mathcal{K}) = \sum c_i$.

Since the dimension of $\mathfrak{M}^{\mathrm{tw}}_{g,n}$ is $3g-3+n$, we can compute the expected dimension from (3.2.1) and (3.4.9):

$$\mathrm{edim}(\mathcal{K}) = -\beta\cdot(K_X + D) + \frac{1}{r}\left(\beta\cdot D - \sum_{i=1}^{n}\varrho_i\right) + n + (3-\dim(X))(g-1). \tag{3.5.1}$$

4. Some invariants of $\mathbb{P}^2_{D,2}$

4.1 Preliminaries

In this section we compute some Gromov–Witten invariants of $\mathbb{P}^2_{D,2}$, where $D \subseteq \mathbb{P}^2$ is a smooth curve of degree δ. In the next section we use these invariants together with associativity of the big quantum product to find recursions that determine all of the genus 0 invariants.

We denote the class of a curve in $N_1(\mathbb{P}^2)$ by its degree d. Since we are only interested in genus 0 invariants, we use I_d to mean I^0_d. Recall that $\widehat{\mathbb{P}^2_{D,2}} \cong \mathbb{P}^2 \sqcup D$. We fix the following basis for $N^*(\widehat{\mathbb{P}^2_{D,2}})$:

- T_0 is the unit class of $\mathbb{P}^2$;
- T_1 is the hyperplane class of $\mathbb{P}^2$;
- T_2 is the class of a point in $\mathbb{P}^2$;
- T_3 is the unit class of D;
- T_4 is the class of a point in D.

Our first computation is an immediate consequence of Proposition 2.5.2. We use $T^{\vec{n}}$ as shorthand for $\prod_{i=0}^4 T_i^{n_i}$:

$$I_0(T^{\vec{n}}) = 0 \quad \text{if } n_2 + n_4 \geqslant 2; \tag{4.1.1}$$

$$I_1(T^{\vec{n}}) = 0 \quad \text{if } n_2 + n_4 \geqslant 3. \tag{4.1.2}$$

4.2 Degree 0, three-point invariants

For any triple $\vec{\varrho}$ of integers 0 and 1, let $\mathcal{K}_{\vec{\varrho}} = \mathcal{K}_{0,3}(\mathbb{P}^2_{D,2}, 0, \vec{\varrho})$. By (3.5.1), the expected dimension of $\mathcal{K}_{\vec{\varrho}}$ is

$$2 - \frac{\varrho_1+\varrho_2+\varrho_3}{2}. \tag{4.2.1}$$

Since the expected dimension must be an integer, it follows from this that $\mathcal{K}_{\vec{\varrho}} = \emptyset$ if there are an odd number of ones in $\vec{\varrho}$. If $\vec{\sigma}$ is any permutation of $\vec{\varrho}$, then clearly $\mathcal{K}_{\vec{\varrho}} \cong \mathcal{K}_{\vec{\sigma}}$, so there are essentially two distinct cases. Note that there is a natural morphism $F_{\vec{\varrho}} : \mathcal{K}_{\vec{\varrho}} \to \overline{\mathcal{M}}_{0,3}(\mathbb{P}^2, 0) \cong \mathbb{P}^2$ which is equal to the composition of any evaluation map $\mathcal{K}_{\vec{\varrho}} \to \widehat{\mathbb{P}^2_{D,2}}$ with the projection $\widehat{\mathbb{P}^2_{D,2}} \to \mathbb{P}^2$.

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It follows from [Cad, Theorem 4.2] that $F_{(0,0,0)}$ is an isomorphism over $\mathbb{P}^2 \setminus D$. Since the expected dimension of $\mathcal{K}_{(0,0,0)}$ is 2 and $\mathbb{P}^2 \setminus D$ is smooth of dimension 2, it follows from Proposition 3.1.1 that $(F_{(0,0,0)})_*[\mathcal{K}_{(0,0,0)}]^v = [\mathbb{P}^2]$. It follows that

$$I_0(T_0^2 T_2) = I_0(T_0 T_1^2) = 1 \tag{4.2.2}$$

and that all other degree 0, three-point invariants involving neither T_3 nor T_4 are zero.

Now we study $\mathcal{K}_{(1,1,0)}$. Since e_1 and e_2 map to D, $F_{(1,1,0)}$ factors through D. For any closed point $x \in D$, we now describe the unique point in the fiber of $F_{(1,1,0)}$ over x. Let $\mathfrak{C}$ be the square root of $\mathbb{P}^1$ at two distinct points, let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the tautological sheaves corresponding to the resulting twisted points, and let $\mathcal{T}$ be the tautological sheaf on $\mathbb{P}^2_{D,2}$. There is a morphism $f : \mathfrak{C} \to \mathbb{P}^2_{D,2}$ which on the level of coarse moduli spaces sends $\mathbb{P}^1$ to x and satisfies $f^*\mathcal{T} \cong \mathcal{T}_1 \otimes \mathcal{T}_2 \otimes \mathcal{O}_{\mathfrak{C}}(-1)$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ appear with exponent 1, this morphism has contact type $(1,1,0)$. It is easy to see that any stable map in the fiber over x has such an underlying morphism f, and also that there is a unique such stable map up to isomorphism. Moreover, $f : \mathfrak{C} \to \mathbb{P}^2_{D,2}$ has a unique nontrivial 2-automorphism given by multiplication by -1 on $f^*\mathcal{T}$.

We can now compute the deformation and obstruction spaces for this stable map. By Proposition 3.4.5, we have an exact sequence

$$0 \to \mathcal{O}_{\mathfrak{C}} \to f^*TX_{D,r} \to \mathcal{T}_1 \otimes \mathcal{T}_2 \otimes \mathcal{O}_{\mathfrak{C}}(-1) \to 0.$$

If $\gamma : \mathfrak{C} \to \mathbb{P}^1$ is the projection, then it follows from [Cad, Theorem 3.1.1] that γ_* of the right-hand term is $\mathcal{O}_{\mathbb{P}^1}(-1)$, so it has no cohomology. Hence, $h^0(\mathfrak{C}, f^*TX_{D,r}) = 1$ and $h^1(\mathfrak{C}, f^*TX_{D,r}) = 0$. Since this holds for every stable map in $\mathcal{K}_{(1,1,0)}$, it follows from [BF97, Proposition 7.3] that $\mathcal{K}_{(1,1,0)}$ is smooth of the expected dimension and $[\mathcal{K}_{(1,1,0)}]^v = [\mathcal{K}_{(1,1,0)}]$.

Let $U \to \mathcal{K}_{(1,1,0)}$ be an étale surjective map. Then U is smooth and one-dimensional and every component of U dominates D. It follows by [Har77, Proposition III-9.7] that $U \to D$ is flat, which by definition implies that $F_{(1,1,0)}$ is flat. Since $\mathcal{K}_{(1,1,0)}$ is proper, $F_{(1,1,0)}$ is also proper and hence finite. The degree of $F_{(1,1,0)}$ is the degree of $F^{-1}_{(1,1,0)}(x)$ for any $x \in D$. By generic smoothness, the preimage of a general $x \in D$ is reduced, so it follows by the above analysis that $F^{-1}_{(1,1,0)}(x) \cong B\mu_2$ for general $x \in D$, and hence $F_{(1,1,0)}$ has degree $1/2$. Therefore, $(F_{(1,1,0)})_*[\mathcal{K}_{(1,1,0)}]^v = (1/2)[D]$.

It now follows that the only nonzero degree 0, three-point invariants involving either T_3 or T_4 are

$$I_0(T_1 T_3^2) = \delta/2 \quad \text{and} \quad I_0(T_0 T_3 T_4) = 1/2. \tag{4.2.3}$$

4.3 Some degree 1 invariants

Now we compute the invariants

$$I_1(T_2^2 T_3^\delta) = \delta! \quad \text{and} \quad I_1(T_2 T_3^{\delta-1} T_4) = (\delta-1)!. \tag{4.3.1}$$

It follows in the same way that $I_1(T_3^{\delta-2} T_4^2) = (\delta-2)!$ if $\delta \geqslant 2$, but this invariant also follows from the associativity relations.

Let $\varrho_i = 0$ for $i = 1,2$ and $\varrho_i = 1$ for $3 \leqslant i \leqslant \delta+2$. We have a finite morphism $F : \mathcal{K}_{0,\delta+2}(\mathbb{P}^2_{D,2}, 1, \vec{\varrho}) \to \overline{\mathcal{M}}_{0,\delta+2}(\mathbb{P}^2, 1)$ by [AV02, Theorem 1.4.1]. Since this is compatible with the evaluation maps, it factors through $\bigcap_{i=3}^{\delta+2} e_i^{-1}(D) \subseteq \overline{\mathcal{M}}_{0,\delta+2}(\mathbb{P}^2, 1)$.

Let $p : \overline{\mathcal{M}}_{0,\delta+2}(\mathbb{P}^2, 1) \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 1)$ be the flat and proper morphism which forgets the last δ markings. Let $U \subseteq \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 1)$ be the dense open subscheme consisting of stable maps $f : \mathbb{P}^1 \to \mathbb{P}^2$ such that $f(\mathbb{P}^1)$ is transverse to D and the marked points do not map into D. Let G and H be as

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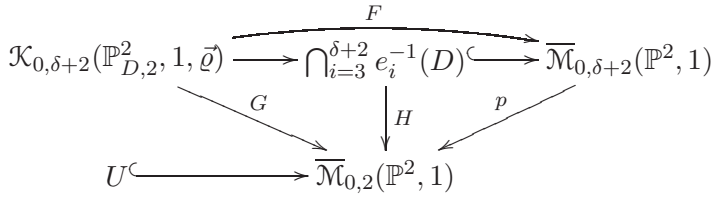
in the following diagram.



If $f : \mathfrak{C} \to \mathbb{P}^2_{D,2}$ is a stable map in $G^{-1}(U)$, then we claim that $\mathfrak{C}$ is smooth. If $\mathfrak{C}$ had a node, then there would be a component mapping with degree 0 which would have to contain at least two marked points. Both would have to be twisted by definition of U. If $\mathfrak{C}_0 \subseteq \mathfrak{C}$ is the irreducible component which maps with positive degree, then the contact type of $f|_{\mathfrak{C}_0}$ must be odd at every preimage of $\mathfrak{G}$ by [Cad, Theorem 3.3.6]. This implies that every such point must be twisted, but since there are δ points in the preimage of $\mathfrak{G}$ and only δ twisted markings, there cannot be a node.

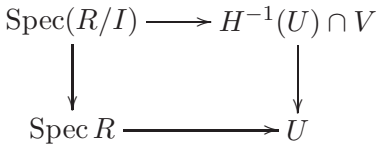
Let $V \subseteq \bigcap_{i=3}^{\delta+2} e_i^{-1}(D)$ be the open subscheme containing only maps from smooth curves into $\mathbb{P}^2$.

Proposition 4.3.2. *We have that $G^{-1}(U) \to H^{-1}(U) \cap V$ is an isomorphism.*

Proof. This follows from [Cad, Theorem 4.2] after observing that, in the notation used there, $\mathcal{U}_{0,\delta+2}(\mathbb{P}^2_{D,2},1,\vec{\varrho}) = G^{-1}(U)$ and $\mathcal{V}_{0,\delta+2}(\mathbb{P}^2,1,\vec{\varrho}) = H^{-1}(U) \cap V$. □

Proposition 4.3.3. *We have that $H^{-1}(U) \cap V \to U$ is finite and étale of degree $\delta!$.*

Proof. To show the morphism is étale, we use the following criterion. Let R be a Noetherian ring and $I \subseteq R$ a nilpotent ideal. Given a commutative diagram



it must be shown that there is a unique morphism $\operatorname{Spec} R \to H^{-1}(U) \cap V$ making the diagram commute. This means we have a smooth family of rational curves $C \to \operatorname{Spec} R$ with two disjoint sections s_1, s_2 and a morphism $f : C \to \mathbb{P}^2$ such that over $\operatorname{Spec}(R/I)$, $f^{-1}(D)$ is a disjoint union of δ sections which are disjoint from s_1 and s_2. Since I is nilpotent, $f^{-1}(D)$ must have δ connected components. Each is clearly finite over $\operatorname{Spec} R$, and each is flat by [Cad, Lemma 5.2]. Since a flat, finite, degree 1 morphism is an isomorphism, each connected component of $f^{-1}(D)$ is a section. This verifies the criterion.

The morphism is proper by Proposition 4.3.2 since G is proper. To show it is finite of degree $\delta!$, it suffices to show that each closed point has $\delta!$ preimages. The closed point is represented by a line transverse to D with two markings away from D, and a preimage is determined by an ordering of the δ intersection points. This completes the proof. □

Note that the first two evaluation maps $\mathcal{K} := \mathcal{K}_{0,\delta+2}(\mathbb{P}^2_{D,2},1,\vec{\varrho}) \to \mathbb{P}^2$ factor through G. Since the product of evaluation maps $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1) \to \mathbb{P}^2 \times \mathbb{P}^2$ is a birational morphism, it follows from the next proposition that $I_1(T_2^2 T_3^\delta) = \delta!$.

Proposition 4.3.4. *We have $G_*[\mathcal{K}]^v = \delta![\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1)]$.*

Proof. By (3.5.1), $[\mathcal{K}]^v$ is homogeneous of degree 4 in the Chow group of $\mathcal{K}$. Since U is dense in $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1)$, the only contribution to $G_*[\mathcal{K}]^v$ comes from the closure of $G^{-1}(U)$ in $\mathcal{K}$. By Propositions 4.3.2 and 4.3.3, $G^{-1}(U)$ is smooth of the expected dimension, so the result now follows from Propositions 3.1.1 and 4.3.3. □

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To compute $I_1(T_2T_3^{\delta-1}T_4)$, we work instead with the following diagram.

$$\begin{array}{ccccc}
\mathcal{K}_{0,\delta+1}(\mathbb{P}^2_{D,2},1,\vec{\sigma}) & \longrightarrow & \bigcap_{i=2}^{\delta+1} e_i^{-1}(D) & \hookrightarrow & \overline{\mathcal{M}}_{0,\delta+1}(\mathbb{P}^2,1) \\
 & \searrow^{G} & \downarrow H & & \downarrow \\
U & \hookrightarrow & e_2^{-1}(D) & \hookrightarrow & \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1)
\end{array}$$

Here $\sigma_1 = 0$ and $\sigma_i = 1$ for $2 \leqslant i \leqslant \delta+1$. The dense open set $U \subseteq e_2^{-1}(D)$ consists of lines transverse to D where the first marked point lies off of D. Let $V \subseteq \bigcap_{i=2}^{\delta+1} e_i^{-1}(D)$ be the open subscheme of maps from smooth curves into $\mathbb{P}^2$.

By the same arguments as above, one can show that $G^{-1}(U) \to H^{-1}(U) \cap V$ is an isomorphism, that $H^{-1}(U) \cap V \to U$ is finite and étale of degree $(\delta-1)!$, and that the morphism $U \to \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2,1)$ forgetting the second marking is étale. It follows that $G^{-1}(U)$ is smooth of the expected dimension and that $G_*[\mathcal{K}]^v = (\delta-1)![e_2^{-1}(D)]$. Then the result follows from the fact that the exceptional locus of $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2,1) \to \mathbb{P}^2 \times \mathbb{P}^2$ does not contain $\mathbb{P}^2 \times D$.

5. The big quantum product

5.1 Preliminaries

We continue with the notation of the last section. So $D \subseteq \mathbb{P}^2$ is a smooth curve of degree δ and $T_0, \ldots, T_4$ is the chosen basis of $N^*(\widehat{\mathbb{P}^2_{D,2}})$.

First we define the big quantum product for $\mathbb{P}^2_{D,2}$ and then we use the fact that it is associative to compute recursions. While the big quantum product exists for any smooth Deligne–Mumford stack having projective coarse moduli scheme, we only define it for $\mathbb{P}^2_{D,2}$ in order to simplify the notation. The reader may also wish to consult [AGV02, CR02]. We have adopted the notation of [FP97, § 8].

Let $(g^{ij})_{0\leqslant i,j\leqslant 4}$ be the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

This corresponds to the inverse matrix of the intersection form

$$a \otimes b \mapsto \int_{\mathfrak{I}(\mathbb{P}^2_{D,2})} a \cdot b$$

expressed in the basis $\{\pi^* T_i\}$, where $\mathfrak{I}(\mathbb{P}^2_{D,2})$ is the inertia stack of $\mathbb{P}^2_{D,2}$ and $\pi : \mathfrak{I}(\mathbb{P}^2_{D,2}) \to \widehat{\mathbb{P}^2_{D,2}}$ is the projection. The reason for the twos is that one gets a factor of $1/2$ when one integrates over $\mathfrak{G}$ instead of D. In general, an involution on the inertia stack enters into the definition of g^{ij} but for $\mathbb{P}^2_{D,2}$ this involution is the identity.

The definition of the big quantum product uses a generating function for the genus 0 Gromov–Witten invariants $I_d(T_0^{n_0} \cdots T_4^{n_4})$ called the quantum potential. It is a power series in indeterminates $y_0, \ldots, y_4$ given by

$$\Phi(y_0, \ldots, y_4) = \sum_{n_0+\cdots+n_4 \geqslant 3} \sum_{d=0}^{\infty} q^d I_d(T_0^{n_0} \cdots T_4^{n_4}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_4^{n_4}}{n_4!}.$$

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Let

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k}$$

for $0 \leqslant i, j, k \leqslant 4$.

DEFINITION 5.1.1. The big quantum product is the $R := \mathbb{Q}[[y_0, \ldots, y_4, q]]$-linear product on the free R-module with basis $T_0, \ldots, T_4$ which is given by

$$T_i * T_j = \sum_{e,f=0}^{4} \Phi_{ije} g^{ef} T_f.$$

5.2 Forgetting an untwisted point

Here we apply the forgetting a point axiom [AGV06] to the stacks $\mathbb{P}^2_{D,r}$. Let $\vec{\varrho}$ be an n-tuple and let $\vec{\sigma} = (\varrho_1, \ldots, \varrho_n, 0)$. If either $n \geqslant 3$ or $d > 0$, then it follows from [AV02, Corollary 9.1.3] that there is a morphism $F : \mathcal{K}_{0,n+1}(\mathbb{P}^2_{D,2}, d, \vec{\sigma}) \to \mathcal{K}_{0,n}(\mathbb{P}^2_{D,2}, d, \vec{\varrho})$ which forgets the last marked point. The forgetting a point axiom says firstly that $\mathcal{K}_{0,n+1}(\mathbb{P}^2_{D,2}, d, \vec{\sigma})$ is isomorphic to the universal curve over $\mathcal{K}_{0,n}(\mathbb{P}^2_{D,2}, d, \vec{\varrho})$ in such a way that F is the projection and e_{n+1} is the composition of the universal morphism with the projection $\mathbb{P}^2_{D,2} \to \mathbb{P}^2$. The second part of the axiom is that

$$F^*[\mathcal{K}_{0,n}(\mathbb{P}^2_{D,2}, d, \vec{\varrho})]^v = [\mathcal{K}_{0,n+1}(\mathbb{P}^2_{D,2}, d, \vec{\sigma})]^v.$$

From this, the following equations can be derived almost exactly as in the case of ordinary stable maps (cf. [FP97, § 7,I–III]).

(i) If $n_0 > 0$ and either $d > 0$ or $\sum n_i > 3$, then

$$I_d(T^{\vec{n}}) = 0. \tag{5.2.1}$$

(ii) If $n_1 > 0$ and either $d > 0$ or $\sum n_i > 3$, then

$$I_d(T^{\vec{n}}) = d I_d(T_0^{n_0} T_1^{n_1 - 1} T_2^{n_2} T_3^{n_3} T_4^{n_4}). \tag{5.2.2}$$

(iii) If $n_2 > 0$ and $\sum n_i > 3$, then

$$I_0(T^{\vec{n}}) = 0. \tag{5.2.3}$$

5.3 Identity and associativity

The quantum product is clearly commutative. We also need the fact that T_0 is a multiplicative identity and that the product is associative.

THEOREM 5.3.1. *For* $0 \leqslant i \leqslant 4$, $T_0 * T_i = T_i$.

Proof. By (5.2.1), the only nonzero invariants with $n_0 > 0$ are those of the form $I_0(T_0 T_i T_e)$ for any i and e. From Definition 5.1.1, we see that $T_0 * T_i = \sum I_0(T_0 T_i T_e) g^{ef} T_f$. The invariants $I_0(T_0 T_i T_e)$ are computed in (4.2.2) and (4.2.3). The theorem now follows from the definition of g^{ij}. □

For associativity of the quantum product, see [AGV06]:

$$(T_i * T_j) * T_k = T_i * (T_j * T_k) \quad \text{for all } i, j, k. \tag{5.3.2}$$

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5.4 Simplifications

Let Ψ, Ψ', and Γ be the power series defined by the following formulas:

$$\begin{aligned}\Psi &= \frac{1}{6}\sum_{i,j,k=0}^{4} I_0(T_iT_jT_k)y_iy_jy_k;\\ \Psi' &= \sum_{n_3+n_4\geqslant 4} I_0(T_3^{n_3}T_4^{n_4})\frac{y_3^{n_3}}{n_3!}\frac{y_4^{n_4}}{n_4!};\\ \Gamma &= \sum_{n_2+n_3+n_4\geqslant 0}\sum_{d=1}^{\infty}(qe^{y_1})^d I_d(T_2^{n_2}T_3^{n_3}T_4^{n_4})\frac{y_2^{n_2}}{n_2!}\frac{y_3^{n_3}}{n_3!}\frac{y_4^{n_4}}{n_4!}.\end{aligned} \tag{5.4.1}$$

By (5.2.1)–(5.2.3), $\Psi+\Psi'+\Gamma$ is congruent to Φ modulo terms of degree less than or equal to 2. Since the big quantum product only involves the third-order partial derivatives of Φ, we can use $\Psi+\Psi'+\Gamma$ in place of Φ in Definition 5.1.1.

We now introduce the stringy product, denoted by $\cdot_s$. For our purposes, it is just a convenient way to encode the effect of Ψ on the big quantum product. It is defined by

$$T_i \cdot_s T_j = \sum_{e,f}\Psi_{ije}g^{ef}T_f.$$

From (4.2.3), we compute $T_3\cdot_s T_3 = (\delta/2)T_1$, $T_3\cdot_s T_1 = \delta T_4$, $T_3\cdot_s T_4 = (1/2)T_2$, and $T_3\cdot_s T_2 = 0$. By associativity, commutativity, and the fact that T_0 is a unit, this determines the stringy product.

Once we have the stringy product, we only need to use Ψ' and Γ, and these only involve T_2, T_3, and T_4. So we use $I_d(n_2,n_3,n_4)$, or sometimes $I_d(\vec{n})$, to denote $I_d(T_2^{n_2}T_3^{n_3}T_4^{n_4})$. It is important to know which invariants can be nonzero.

PROPOSITION 5.4.2. *If $I_d(n_2,n_3,n_4)\neq 0$, then*

$$3d-1 = \frac{d\delta+n_4-n_3}{2}+n_2.$$

Proof. Let $\varrho_i = 0$ for $1\leqslant i\leqslant n_2$ and $\varrho_i = 1$ for $n_2+1\leqslant i\leqslant n_2+n_3+n_4$. Equation (3.5.1) implies that the expected dimension of $\mathcal{K}_{0,n_2+n_3+n_4}(\mathbb{P}^2_{D,2},d,\vec{\varrho})$ is $3d-d\delta/2+(n_3+n_4)/2+n_2-1$. The equality comes from setting this equal to $2n_2+n_4$. □

If we apply this to Ψ', we see that $I_0(0,n_3,n_4)$ can only be nonzero when $n_3-n_4=2$. Equation (4.1.1) imposes the additional condition $n_4\leqslant 1$. We have thus shown that $\Psi' = \lambda y_3^3y_4/6$, where $\lambda := I_0(0,3,1)$.

It follows that the quantum product is given by

$$\begin{aligned}T_i * T_j &= T_i\cdot_s T_j + \sum_{ef}(\Psi'_{ije}+\Gamma_{ije})g^{ef}T_f\\ &= T_i\cdot_s T_j + \Gamma_{ij1}T_1 + \Gamma_{ij2}T_0 + 2\Gamma_{ij3}T_4 + 2\Gamma_{ij4}T_3 + \lambda\frac{\partial^2 y_3^2y_4}{\partial y_i\partial y_j}T_4 + \frac{\lambda}{3}\frac{\partial^2 y_3^3}{\partial y_i\partial y_j}T_3.\end{aligned} \tag{5.4.3}$$

5.5 Recursions

Using (5.4.3) together with associativity, it is a simple but tedious computation to compute the recursions. The only difficulty is in knowing which products to apply associativity to, and we do not claim to have found the most efficient algorithm. The following four relations are obtained by comparing, respectively, the coefficients of T_0 in $(T_1*T_1)*T_2$ and $T_1*(T_1*T_2)$, those of T_3 in $(T_3*T_3)*T_4$ and $T_3*(T_3*T_4)$, those of T_3 in $(T_3*T_1)*T_4$ and $T_3*(T_1*T_4)$, and those of T_1

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in $(T_3 * T_3) * T_1$ and $T_3 * (T_3 * T_1)$

$$\Gamma_{222} = \Gamma_{112}^2 - \Gamma_{111}\Gamma_{122} + 2(2\Gamma_{123}\Gamma_{124} - \Gamma_{113}\Gamma_{224} - \Gamma_{114}\Gamma_{223}) \tag{5.5.1}$$

$$\delta\Gamma_{144} + 4\lambda(y_4\Gamma_{444} - y_3\Gamma_{344}) - 2\Gamma_{234} = 2(\Gamma_{134}^2 - \Gamma_{133}\Gamma_{144}) + 4(\Gamma_{334}\Gamma_{344} - \Gamma_{333}\Gamma_{444}) \tag{5.5.2}$$

$$2\delta\Gamma_{444} - \Gamma_{124} - 4\lambda y_3\Gamma_{144} = 2(\Gamma_{114}\Gamma_{134} - \Gamma_{113}\Gamma_{144}) + 4(\Gamma_{144}\Gamma_{334} - \Gamma_{133}\Gamma_{444}) \tag{5.5.3}$$

$$\begin{aligned}&\Gamma_{233} + \delta(\tfrac{1}{2}\Gamma_{111} - 2\Gamma_{134}) + 2\lambda(y_3\Gamma_{113} + y_4\Gamma_{114})\\ &\quad = \Gamma_{113}^2 - \Gamma_{111}\Gamma_{133} + 2(2\Gamma_{133}\Gamma_{134} - \Gamma_{113}\Gamma_{334} - \Gamma_{114}\Gamma_{333}).\end{aligned} \tag{5.5.4}$$

It is now possible to compute λ using (5.5.4). Comparing the coefficients of $qe^{y_1}y_2^2y_3^\delta/2\delta!$ yields

$$\delta(\tfrac{1}{2} + 2\lambda)I_1(2,\delta,0) = 2\delta I_1(2,\delta+1,1) - I_1(3,\delta+2,0).$$

The right-hand side is zero by (4.1.2) and $I_1(2,\delta,0)$ is nonzero by (4.3.1). Therefore, $\lambda = -1/4$.

By comparing coefficients in (5.5.1)–(5.5.4), we obtain recursions (5.5.5)–(5.5.8) in that order. In each recursion, d_1 and d_2 vary over positive integers and $\vec{p} := (p_2, p_3, p_4)$ and $\vec{q} := (q_2, q_3, q_4)$ vary over triples of nonnegative integers. In addition to the condition in parentheses, we also assume $d > 0$:

$(n_2 \geqslant 3)$

$$\begin{aligned}&I_d(n_2,n_3,n_4)\\ &\quad = \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}-(1,0,0)}} I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_3}{p_3}\binom{n_4}{p_4}\left[d_1^2d_2^2\binom{n_2-3}{p_2-1} - d_1^3d_2\binom{n_2-3}{p_2}\right]\\ &\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(-1,1,1)}} 2I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_3}{p_3-1}\binom{n_4}{p_4}\left[2d_1d_2\binom{n_2-3}{p_2-1} - d_1^2\binom{n_2-3}{p_2} - d_2^2\binom{n_2-3}{p_2-2}\right]\end{aligned} \tag{5.5.5}$$

$(n_4 \geqslant 2)$

$$\begin{aligned}&(d\delta + n_3 - n_4 + 2)I_d(n_2,n_3,n_4)\\ &\quad = 2I_d(n_2+1,n_3+1,n_4-1)\\ &\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(0,2,0)}} 2d_1d_2I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_2}{p_2}\left[\binom{n_3}{p_3-1}\binom{n_4-2}{p_4-1} - \binom{n_3}{p_3-2}\binom{n_4-2}{p_4}\right]\\ &\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(0,3,1)}} 4I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_2}{p_2}\left[\binom{n_3}{p_3-2}\binom{n_4-2}{p_4-1} - \binom{n_3}{p_3-3}\binom{n_4-2}{p_4}\right]\end{aligned} \tag{5.5.6}$$

$(n_4 \geqslant 3)$

$$\begin{aligned}&2\delta I_d(n_2,n_3,n_4)\\ &\quad = dI_d(n_2+1,n_3,n_4-2) - n_3dI_d(n_2,n_3-1,n_4-1)\\ &\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(0,1,-1)}} 2I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_2}{p_2}\binom{n_3}{p_3-1}\left[d_1d_2^2\binom{n_4-3}{p_4-1} - d_1^2d_2\binom{n_4-3}{p_4}\right]\\ &\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(0,2,0)}} 4I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_2}{p_2}\binom{n_3}{p_3-2}\left[d_2\binom{n_4-3}{p_4-1} - d_1\binom{n_4-3}{p_4}\right]\end{aligned} \tag{5.5.7}$$

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$$
\begin{aligned}
&(\forall \vec{n})\\
&\frac{1}{2}d^2(d\delta - n_3 - n_4)I_d(n_2, n_3, n_4)\\
&\quad = 2d\delta I_d(n_2, n_3+1, n_4+1) - I_d(n_2+1, n_3+2, n_4)\\
&\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(0,2,0)}} I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_2}{p_2}\binom{n_4}{p_4}\left[d_1^2d_2^2\binom{n_3}{p_3-1} - d_1^3d_2\binom{n_3}{p_3}\right]\\
&\qquad + \sum_{\substack{d_1+d_2=d\\ \vec{p}+\vec{q}=\vec{n}+(0,3,1)}} 2I_{d_1}(\vec{p})I_{d_2}(\vec{q})\binom{n_2}{p_2}\binom{n_4}{p_4}\left[2d_1d_2\binom{n_3}{p_3-2} - d_1^2\binom{n_3}{p_3-1} - d_2^2\binom{n_3}{p_3-3}\right].
\end{aligned}
\tag{5.5.8}
$$

Combining these recursions with (4.3.1) and Proposition 5.4.2, the genus 0 Gromov–Witten invariants of $\mathbb{P}^2_{D,2}$ can be computed by the following algorithm.

Algorithm 5.5.9. Let $d \geqslant 1$ and $n_2, n_3, n_4 \geqslant 0$ be integers. Compute $I := I_d(n_2, n_3, n_4)$ as follows.

(i) If $3d - 1 \neq (d\delta + n_4 - n_3)/2 + n_2$, then $I = 0$.
(ii) Otherwise, if $(d, n_2, n_3, n_4) = (1, 2, \delta, 0)$, then $I = \delta!$.
(iii) Otherwise, if $(d, n_2, n_3, n_4) = (1, 1, \delta - 1, 1)$, then $I = (\delta - 1)!$.
(iv) Otherwise, if $n_2 \geqslant 3$, apply recursion (5.5.5).
(v) Otherwise, if $n_4 \geqslant 2$ and $n_4 - n_3 \neq d\delta + 2$, apply recursion (5.5.6).
(vi) Otherwise, if $n_4 - n_3 = d\delta + 2$, apply recursion (5.5.7).
(vii) Otherwise, apply recursion (5.5.8).

Justification. To justify the algorithm, we first show that if $n_4 - n_3 = d\delta + 2$ then $n_4 \geqslant 3$, and if step (vii) is reached then $n_3 + n_4 \neq d\delta$. The first statement follows from $d\delta \geqslant 1$ and $n_3 \geqslant 0$. To reach the last step, we must have $n_2 \leqslant 2$, $n_4 \leqslant 1$, and $3d-1 = (d\delta+n_4-n_3)/2+n_2$. If $n_3+n_4 = d\delta$, then $n_3 \geqslant d\delta - 1$, so $3d - 1 \leqslant 3$. This implies $d = 1$, and the only two possibilities for (n_2, n_3, n_4) are handled in steps (ii) and (iii).

Since the recursions do not clearly reduce degree d invariants to degree $d-1$ invariants, it must also be shown that the algorithm terminates. To do this, fix a degree d and define a sequence of triples $(n_2^{(i)}, n_3^{(i)}, n_4^{(i)})$ to be *admissible* if in order to compute $I_d(n_2^{(i)}, n_3^{(i)}, n_4^{(i)})$, the algorithm requires one to compute $I_d(n_2^{(i+1)}, n_3^{(i+1)}, n_4^{(i+1)})$. It suffices to show that any admissible sequence starting at a given point has bounded length.

One can see this geometrically by working in the (n_3, n_4) plane, noting that n_2 is determined by n_3 and n_4. From a given point, one is only allowed to move by five vectors: $(1, -1)$, $(0, -2)$, $(-1, -1)$, $(1, 1)$, and $(2, 0)$. So $n_4 - n_3$ is nonincreasing in an admissible sequence. Moreover, a move by $(-1, -1)$ is only allowed on the line $n_4 - n_3 = d\delta + 2$, a move by $(1, 1)$ is only allowed in the range $n_4 \leqslant 1$, and these two sets are disjoint. Otherwise, $n_4 - n_3$ decreases by 2.

Once an admissible path reaches the line $n_4 - n_3 = (6 - \delta)d - 8$, it must terminate since then $n_2 = 3$. From this it is easy to get a bound on the length of an admissible path starting at (n_3, n_4). For example $\frac{1}{2}[n_4 - n_3 + 8 - (6 - \delta)d] + n_4 + \max(0, 10 - (6 - \delta)d)$ works. □

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