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RECURRENCE AND TRANSIENCE OF CRITICAL BRANCHING PROCESSES IN RANDOM ENVIRONMENT WITH IMMIGRATION AND AN APPLICATION TO EXCITED RANDOM WALKS

ELISABETH BAUERNSCHUBERT,* *University of Tuebingen*

Abstract

We establish recurrence and transience criteria for critical branching processes in random environments with immigration. These results are then applied to the recurrence and transience of a recurrent random walk in a random environment on $\mathbb{Z}$ disturbed by cookies inducing a drift to the right of strength 1.

Keywords: Critical branching process in a random environment with immigration; excited random walk in a random environment; cookies of strength 1; recurrence; transience

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1. Introduction

This paper complements [4]. In [4] a random walk in a random environment on $\mathbb{Z}$ which is transient to the left and disturbed by cookies of strength 1 to the right was considered. As can be seen in [4], the study of special kinds of branching processes is essential to obtain results on recurrence and transience of these excited random walks.

We first motivate the discussion of a critical branching process in a random environment with immigration (critical BPIRE for short) within the present paper by introducing the random walk we are dealing with.

1.1. Excited random walk in a random environment

Our model is explained as follows. Consider a sequence $(p_x)_{x\in\mathbb{Z}} \in (0, 1)^{\mathbb{Z}}$ and put a random number M_x of cookies on every integer $x \in \mathbb{Z}$. Now a nearest-neighbor random walk $(S_n)_{n\geq 0}$ is started at 0 with the following transition probabilities. If a random walker reaches site x and if there is still at least one cookie on this site, he/she removes one cookie and goes to $x + 1$. Otherwise he/she jumps to the right with probability p_x and to the left with probability $1 - p_x$. For an illustration of this model, see Figure 1, which was previously presented in [4].

The cookies in our model have maximal strength and induce a drift to the right. On the other hand, we will assume a random environment $(p_x)_{x\in\mathbb{Z}}$ that makes a random walk in a random environment (RWRE for short), i.e. a random walk where $M_x = 0$ for all $x \in \mathbb{Z}$, be recurrent. So the question arises as to when the drift caused by the cookies succeeds in forcing the random walk to $+\infty$. In Theorem 1 below, criteria for transience and recurrence of the process are given.

Let us introduce the notation for the model. Set $\Omega := ([0, 1]^{\mathbb{N}})^{\mathbb{Z}}$. The elements from Ω are chosen at random according to a probability measure $\mathbb{P}$ on Ω with corresponding expectation operator $\mathbb{E}$. For a fixed environment $\omega = ((\omega(x, i))_{i\geq 1})_{x\in\mathbb{Z}} \in \Omega$ and $z \in \mathbb{Z}$, define a nearest-

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* Postal address: Mathematisches Institut, Eberhard Karls Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany. Email address: elisabeth.bauernschubert@uni-tuebingen.de

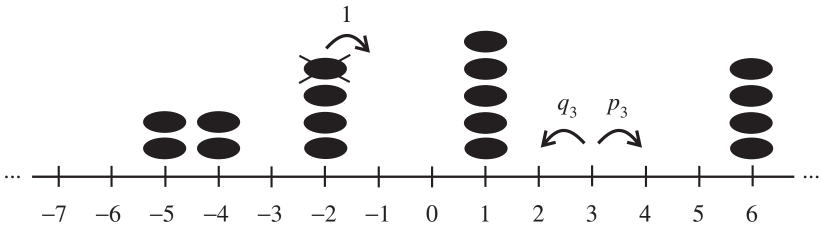

FIGURE 1: Model of the random walk (see [4]). If there are cookies at his/her current position $x \in \mathbb{Z}$, the random walker removes one and makes a step to $x + 1$. If there is no cookie he/she jumps to the right with probability p_x and to the left with probability $q_x := 1 - p_x$.

neighbor random walk $(S_n)_{n\geq 0}$ on a suitable probability space Ω' with probability measure $\mathrm{P}_{z,\omega}$, which satisfies

$$\mathrm{P}_{z,\omega}[S_0 = z] = 1,$$
$$\mathrm{P}_{z,\omega}[S_{n+1} = S_n + 1 \mid (S_m)_{1\leq m\leq n}] = \omega(S_n, \#\{m \leq n\colon S_m = S_n\}),$$
$$\mathrm{P}_{z,\omega}[S_{n+1} = S_n - 1 \mid (S_m)_{1\leq m\leq n}] = 1 - \omega(S_n, \#\{m \leq n\colon S_m = S_n\}).$$

The value of $\omega(x, i)$ serves as the transition probability from x to $x+1$ upon the ith visit at site x. Furthermore, define $\mathrm{P}_z[\cdot] := \mathbb{E}[\mathrm{P}_{z,\omega}[\cdot]]$ as the *annealed* or *averaged* probability measure with corresponding expectation operator E_z. The random walk $(S_n)_{n\geq 0}$ is called *recurrent* (*transient*) if $S_n = 0$ infinitely often ($\lim_{n\to\infty} S_n \in \{\pm\infty\}$) P_0-almost surely (P_0-a.s.).

With the convention $\sup\varnothing = 0$, the number of cookies of strength 1 at site $x \in \mathbb{Z}$ is defined by

$$M_x := \sup\{i \geq 1\colon \omega(x, j) = 1 \text{ for all } 1 \leq j \leq i\}.$$

In this paper we postulate the following for the model.

Assumption A. *There exists,* $\mathbb{P}$*-a.s.,* $(p_x)_{x\in\mathbb{Z}} \in (0, 1)^{\mathbb{Z}}$ *such that the following assumptions hold.*

(A1) *It holds* $\mathbb{P}$*-a.s. that* $\omega(x, i) = p_x$ *for all* $i > M_x$*. Furthermore,* $\mathbb{P}[p_x = \frac{1}{2}] < 1$.

(A2) $(p_x, M_x)_{x\in\mathbb{Z}}$ *is independent and identically distributed (i.i.d.).*

(A3) $\mathbb{E}[|\log\rho_0|] < \infty$ *and* $\mathbb{E}[\log\rho_0] = 0$*, where* $\rho_x := (1 - p_x)p_x^{-1}$ *for* $x \in \mathbb{Z}$.

(A4) $\mathbb{P}[M_0 = \infty] = 0$ *and* $\mathbb{P}[M_0 = 0] > 0$.

If $M_x = 0$ $\mathbb{P}$-a.s. for all $x \in \mathbb{Z}$, assumptions (A2) and (A3) imply that the RWRE is recurrent, i.e. $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$ P_0-a.s.; see, e.g. [17, Theorem 1.7].

Under Assumption A, $(S_n)_{n\geq 0}$ can be seen as a recurrent RWRE disturbed by cookies of strength 1 to the right. In accordance with an RWRE and an excited random walk (ERW), our model is called an *excited random walk in a random environment* (ERWRE for short). In Section 3 we show the following recurrence and transience criteria for the ERWRE.

Theorem 1. *Let Assumption A hold, and assume that* $\mathbb{E}[|\log\rho_0|^{\delta}] < \infty$ *for every* $0 < \delta < 6$.

(i) *If* $\mathbb{E}[(\log_+ M_0)^{2+\varepsilon}] < \infty$ *for some* $\varepsilon > 0$ *then* $S_n = 0$ *infinitely often* P_0*-a.s.*

(ii) *If* $\liminf_{t\to\infty}(t^{\lambda}\mathbb{P}[\log M_0 > t]) > 0$ *for some* $0 < \lambda < 2$ *then* $\lim_{n\to\infty} S_n = +\infty$*,* P_0*-a.s.*

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Remark 1. The tail assumption $\liminf_{t\to\infty}(t^{\lambda}\mathbb{P}[\log M_0 > t]) > 0$ implies that

$$\mathbb{E}[(\log_+ M_0)^{2-\varepsilon}] = \infty \quad \text{for all } \varepsilon \le 2-\lambda.$$

Remark 2. In the classical model of the ERW the underlying process is a simple symmetric random walk. Criteria for the recurrence and transience behavior of the classical ERW are given in Theorem 3.10 of [13]. Hence, if Assumption A holds with $\mathbb{P}[p_x = \frac{1}{2}] = 1$ in (A1), the process $(S_n)_{n\ge 0}$ is recurrent if and only if $\mathbb{E}[M_0] \le 1$. Note that this criterion is different from that given in Theorem 1.

Theorem 1 should be compared to the following result from [4], where the underlying RWRE is transient to the left, and where the following recurrence and transience criteria for the ERWRE were obtained.

Theorem 2. ([4].) *Let assumptions (A1), (A2), and (A4) hold, and assume that* $\{p_x, M_x, x \in \mathbb{Z}\}$ *is independent under* $\mathbb{P}$, $\mathbb{E}[|\log\rho_0|] < \infty$, $\mathbb{E}[\log\rho_0] > 0$, *and* $\mathbb{E}[p_0^{-1}] < \infty$.

(i) *If* $\mathbb{E}[\log_+ M_0] < \infty$ *then* $\lim_{n\to\infty} S_n = -\infty$, P_0*-a.s.*

(ii) *If* $\mathbb{E}[\log_+ M_0] = \infty$ *and if* $\limsup_{t\to\infty}(t\mathbb{P}[\log M_0 > t]) < \mathbb{E}[\log\rho_0]$, *then* $S_n = 0$ *infinitely often* P_0*-a.s.*

(iii) *If* $\liminf_{t\to\infty}(t\mathbb{P}[\log M_0 > t]) > \mathbb{E}[\log\rho_0]$ *then* $\lim_{n\to\infty} S_n = +\infty$, P_0*-a.s.*

Theorem 1 and Theorem 2 both provide recurrence and transience criteria for the ERWRE, but in the first theorem the underlying random walk is recurrent and in the latter transient to the left. Basically, if a logarithmic moment of M_0 is finite, the random environment (without cookies) determines the recurrence/transience behavior, whereas an appropriate assumption on the tail of the distribution of M_0 implies that the drift induced by the cookies wins.

Excited random walks go back to Benjamini and Wilson [6], and have been further studied and extended by, among others, Zerner [21], [22], Basdevant and Singh [2], [3], and Kosygina and Zerner [12]. A survey on ERW is given by Kosygina and Zerner in [13]. The novelties in our model are the random transition probabilities on sites without cookies and the unbounded number of cookies per site. However, we consider only cookies of maximal strength.

A useful technique to obtain results for the one-dimensional ERW is to employ the well-known relationship between branching processes and random walks. See also [2], [3], [4], and [12] for this method. Since there are only cookies of strength 1, we can concentrate on branching processes with immigration and no emigration. In order to prove Theorem 1, we have to deal with a critical BPIRE. See Section 3 for the precise connection between our model and the critical BPIRE. Roughly speaking, an excursion to the right of the random walk can be translated into a branching process by counting the number of upcrossings from n to $n+1$, $n \in \mathbb{N}$, between downcrossings from n to $n-1$. The translation from the branching process to the excursion is given by the contour process. The cookies in the ERWRE model correspond to the immigrants and the random environment gives the random offspring distributions for the branching process. As we will see in Section 3, the recurrence of the branching process implies the recurrence of the random walk and vice versa. Thus, the discussion of the BPIRE with focus on its recurrence and transience behavior is essential.

1.2. Branching processes in random environments with immigration

The literature on branching processes is extensive; see, for instance, the survey article [18]. Vatutin *et al.* [19] contains a more recent review on branching processes in random environments.

Critical branching processes in random environments with immigration are studied in, e.g. [11] and [16]. Unfortunately, a proper transience and recurrence criteria for our model could not be found or deduced.

Let us introduce the definition of the BPIRE that we study in this paper. It differs slightly from that given in [11, p. 344f]; see also Remark 3 below.

Definition 1. Consider a sequence $e = (e_n)_{n\in\mathbb{N}} = (r_n, m_n)_{n\in\mathbb{N}}$ of pairs of random variables which take values in the set of probability measures on $\mathbb{N}_0$. For $n \in \mathbb{N}$, r_n and m_n respectively give the distributions for reproduction and immigration in generation n. Assume that the so-called *random environment* $(e_n)_{n\in\mathbb{N}}$ is i.i.d. under some probability measure $\mathbb{Q}$, and denote by $\mathbb{Q}_e[\cdot] := \mathbb{Q}[\cdot \mid e]$ the conditional distribution, and by $\mathrm{E}_{\mathbb{Q}}$ and E_e the expectations with respect to $\mathbb{Q}$ and $\mathbb{Q}_e$, respectively.

Furthermore, let $\{\xi_j^{(n)}, M_k;\ j, n, k \in \mathbb{N}\}$ be a family of $\mathbb{N}_0$-valued random variables on the same probability space which is $\mathbb{Q}$-a.s. independent under $\mathbb{Q}_e$ and satisfies, $\mathbb{Q}$-a.s. for $j, n \in \mathbb{N}$,

$$\mathbb{Q}_e[M_n \in \cdot\,] = m_n, \qquad \mathbb{Q}_e[\xi_j^{(n)} \in \cdot\,] = r_n.$$

Then the process $(Z_n)_{n\geq 0}$, given by $Z_0 := 0$ and

$$Z_n := \xi_1^{(n)} + \cdots + \xi_{Z_{n-1}}^{(n)} + M_n \quad \text{for } n \in \mathbb{N}$$

(or every process with the same distribution), is called BPIRE. The random variable $\xi_j^{(n)}$ can be understood as the number of offspring of the jth individual of generation $n-1$, and M_n as the number of immigrants in the nth generation.

Another useful way to describe the BPIRE is the following. For each $j \in \mathbb{N}$, let $(Z_n(j))_{n\in\mathbb{N}_0}$ be a branching process that starts at time j with $Z_0(j) = M_j$ individuals (or immigrants) and whose reproduction distribution is given by $(r_{n+j})_{n\in\mathbb{N}}$ under $\mathbb{Q}_e$. More precisely, we consider branching processes that have the same distribution as processes realized by $Z_n(j) = \xi_{j,1}^{(n+j)} + \cdots + \xi_{j,Z_{n-1}(j)}^{(n+j)}$, where, under $\mathbb{Q}_e$, $\{\xi_{j,i}^{(k)}, M_n;\ j, i, k, n \in \mathbb{N}\}$ is independent and $\xi_{j,i}^{(k)}$ has distribution r_k, $\mathbb{Q}$-a.s. Then the sum over the offspring at the same time plus the immigrants at that time,

$$Z_n = \sum_{j=1}^{n} Z_{n-j}(j) \quad \text{for } n \in \mathbb{N},$$

gives a BPIRE. The latter definition is similar to Key's definition in [11, p. 344f]; see also Remark 3 below.

If $\mathrm{E}_{\mathbb{Q}}[\log \mathrm{E}_e[\xi_1^{(1)}]]$ exists, $(Z_n)_{n\geq 0}$ is called *critical, subcritical,* or *supercritical* according to whether $\mathrm{E}_{\mathbb{Q}}[\log \mathrm{E}_e[\xi_1^{(1)}]]$ is equal to 0, less than 0, or greater than 0, in accordance with the standard classification of branching processes in random environments. For an extended classification, see, for instance, [19, p. 222f].

Throughout the paper, the distribution r_n will be represented by its probability generating function (PGF) $\varphi_n(s) := \sum_{k\geq 0} s^k r_n(\{k\}) = \mathrm{E}_e[s^{\xi_1^{(n)}}]$, $0 \leq s \leq 1$. Apart from Lemma 1 below and its application in the proof of Theorem 3, it will be furthermore assumed that m_n $\mathbb{Q}$-a.s. takes values in the set of dirac measures on $\mathbb{N}_0$, $\{\delta_n,\ n \in \mathbb{N}_0\}$. In this case let us write n instead of δ_n as shorthand notation. Thus, for a sequence $(M_n)_{n\in\mathbb{N}}$ of $\mathbb{N}_0$-valued random variables, $(\varphi, M) := (\varphi_n, M_n)_{n\in\mathbb{N}}$ denotes an environment where the distribution for offspring in generation n of an individual in generation $n-1$ is given by the PGF φ_n and where M_n individuals immigrate in the nth generation $\mathbb{Q}_{(\varphi,M)}$-a.s.

Remark 3. In his formulation of the BPIRE model in [11], Key does not count the number of immigrants at time n as a part of generation n; he only considers their offspring as part of the next generation.

Note that $(Z_n)_{n\geq 0}$ is a time homogeneous Markov chain under $\mathbb{Q}$. In this paper it is assumed that the BPIRE is irreducible on $\mathbb{N}$ or $\mathbb{N}_0$ under $\mathbb{Q}$, i.e. every state can be reached from any other with positive probability; see, e.g. [10, p. 151]. A sufficient condition for irreducibility on $\mathbb{N}_0$ is $\mathbb{Q}[M_1 = 0] < 1$ and $\mathbb{Q}[M_1 = 0,\ \xi_1^{(1)} = k] > 0$ for every $k \in \mathbb{N}_0$. Motivated by the application to the ERWRE, we are interested in recurrence and transience criteria for a critical BPIRE.

Theorem 3. *Let $(Z_n)_{n\geq 0}$ be an irreducible BPIRE with PGF φ_n for offspring in generation n and M_n immigrants in generation n. Assume that the following assertions hold.*

(i) $(\varphi_n, M_n)_{n\in\mathbb{N}}$ *is i.i.d. under* $\mathbb{Q}$.

(ii) $\mathrm{E}_{\mathbb{Q}}[|\log \mu_1|^2] < \infty$, $\mathrm{E}_{\mathbb{Q}}[\log \mu_1] = 0$, *and* $\mathbb{Q}[\mu_1 = 1] < 1$, *where* $\mu_n := \varphi_n'(1)$.

(iii) $\mathrm{E}_{\mathbb{Q}}[(\log_+ M_1)^{2+\varepsilon}] < \infty$ *for some* $\varepsilon > 0$.

Then $(Z_n)_{n\geq 0}$ is recurrent.

The next theorem gives a criterion for transience of a critical BPIRE. Let $\mathbb{Q}_\varphi$ denote the conditional distribution $\mathbb{Q}[\cdot \mid \varphi]$, and write var_φ for the variance according to the measure $\mathbb{Q}_\varphi$.

Theorem 4. *Consider an irreducible BPIRE $(Z_n)_{n\geq 0}$ with PGF φ_n for offspring in generation n and M_n immigrants in generation n. Assume that the following assertions hold.*

(i) $(\varphi_n, M_n)_{n\in\mathbb{N}}$ *is i.i.d. under* $\mathbb{Q}$.

(ii) 1. $\mathrm{E}_{\mathbb{Q}}[|\log \mu_1|^\delta] < \infty$ *for every* $0 < \delta < 6$ *and* $\mathrm{E}_{\mathbb{Q}}[\log \mu_1] = 0$.

2. $\mathrm{E}_{\mathbb{Q}}[(\log_+(\mathrm{var}_\varphi(\xi_1^{(1)})\mu_1^{-2}))^2] < \infty$, *where* $\mu_n := \varphi_n'(1)$.

(iii) $\liminf_{t\to\infty}(t^\lambda \mathbb{Q}[\log M_1 > t]) > 0$ *for some* $0 < \lambda < 2$.

Then $(Z_n)_{n\geq 0}$ is transient.

Theorem 4(iii) implies in particular that $\mathrm{E}_{\mathbb{Q}}[(\log_+ M_1)^{2-\varepsilon}] = \infty$ for some $\varepsilon > 0$; see also Remark 1. Note the gap in Theorems 3 and 4 concerning assumption (iii). Theorem 4(ii.2) is a technical assumption and might be weakened, but is always satisfied in our application to the ERWRE.

The recurrence criterion for the branching process was inspired by some work on random difference equations, e.g. [1], since there is some similarity between these processes. In [9, p. 1196] Goldie examined recurrence and transience of random difference equations but characterized only positive recurrence. Also, note the short review of the main known results on these processes in [15, Section 3].

The present paper is organized as follows. Section 2 is dedicated to the proofs of Theorems 3 and 4 for the critical BPIRE. In Section 3, the relation between the ERWRE and the branching process is established in order to prove Theorem 1. Examples are also given for the different cases of the theorem.

2. Branching process in a random environment with immigration

First, let us deduce for our model an analogous result about positive recurrence for an irreducible subcritical BPIRE from Theorem 3.3 of [11].

Lemma 1. *Let $(Z_n)_{n\geq 0}$ be a BPIRE with reproduction according to the sequence of PGFs $(\varphi_n)_{n\in\mathbb{N}}$ and immigration according to probability measures $(m_n)_{n\in\mathbb{N}}$. Assume that the following assertions hold.*

(i) $(\varphi_n, m_n)_{n\in\mathbb{N}}$ *is i.i.d. under* $\mathbb{Q}$.

(ii) $\mathrm{E}_{\mathbb{Q}}[\log_+ \mathrm{E}_{(\varphi_n,m_n)_{n\in\mathbb{N}}}[M_1]] < \infty$.

(iii) $\mathrm{E}_{\mathbb{Q}}[\log_+ \mu_1] < \infty$ *and* $\mathrm{E}_{\mathbb{Q}}[\log \mu_1] < 0$, *where* $\mu_n := \varphi_n'(1)$.

Then $(Z_n)_{n\geq 0}$ is positive recurrent.

Proof. It is helpful to work with the alternative description of the BPIRE given in Definition 1. As in [11], we amplify this definition in the sense that we do not only consider branching processes $(Z_n(t))_{n\in\mathbb{N}_0}$ starting at positive times, but allow $t \in \mathbb{Z}$. Therefore, the random environment is assumed to be a sequence $e = (\varphi_x, m_x)_{x\in\mathbb{Z}}$ of i.i.d. random variables.

Recall that, for $n \geq 1$, the BPIRE can be defined as $Z_n = \sum_{j=1}^{n} Z_{n-j}(j)$. Key [11] considered in a more general setting a BPIRE of the form

$$\tilde{Z}_n^{(1)} := \sum_{j=1}^{n-1} Z_{n-j}(j).$$

We shift this process and set, for $k \in \mathbb{N}_0$,

$$\tilde{Z}_0^{(-k)} := \sum_{j=1}^{k} Z_j(-j),$$

which is a BPIRE at time 0 that started in the past at time $-k$.

Since e is a sequence of i.i.d. random variables and since the branching processes $(Z_n(t))_{n\in\mathbb{N}_0}$, $t \in \mathbb{Z}$, are independent under $\mathbb{Q}_e$, we obtain for $v \in \mathbb{N}_0$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{Q}[Z_n = v] &= \mathbb{Q}[Z_0(n) + \tilde{Z}_n^{(1)} = v] \\
&= \sum_{j=0}^{v} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[Z_0(n) = v - j,\ \tilde{Z}_n^{(1)} = j]] \\
&= \sum_{j=0}^{v} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[Z_0(n) = v - j]\mathbb{Q}_e[\tilde{Z}_n^{(1)} = j]] \\
&= \sum_{j=0}^{v} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[Z_0(0) = v - j]\mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]].
\end{aligned}$$

For the last equality, note that the processes are just shifted. Since e is i.i.d., the products under $\mathrm{E}_{\mathbb{Q}}$ have the same law. According to Lemma 2.2 of [11], $\lim_{n\to\infty} \mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]$ exists $\mathbb{Q}$-a.s.

for each $j \in \mathbb{N}_0$. Hence, by the dominated convergence theorem, $\pi(v) := \lim_{n\to\infty} \mathbb{Q}[Z_n = v]$ exists for every $v \in \mathbb{N}_0$ and

$$\pi(v) = \sum_{j=0}^{v} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[Z_0(0) = v - j] \lim_{n\to\infty} \mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]]. \tag{1}$$

Let us now show that π is a probability measure on $\mathbb{N}_0$. By (1),

$$\begin{aligned}\sum_{v\in\mathbb{N}_0} \pi(v) &= \sum_{v\in\mathbb{N}_0} \sum_{j=0}^{v} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[Z_0(0) = v - j] \lim_{n\to\infty} \mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]] \\ &= \sum_{j\in\mathbb{N}_0} \sum_{v\geq j} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[Z_0(0) = v - j] \lim_{n\to\infty} \mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]] \\ &= \sum_{j\in\mathbb{N}_0} \mathrm{E}_{\mathbb{Q}}[\lim_{n\to\infty} \mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]].\end{aligned}$$

Note that, for all $j \in \mathbb{N}_0$,

$$\tilde{\pi}(j) := \mathrm{E}_{\mathbb{Q}}[\lim_{n\to\infty} \mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]] = \lim_{n\to\infty} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_e[\tilde{Z}_0^{(1-n)} = j]] = \lim_{n\to\infty} \mathbb{Q}[\tilde{Z}_n^{(1)} = j]$$

and $\tilde{\pi}$ defines a probability measure on $\mathbb{N}_0$ according to Theorem 3.3 of [11]. Thus,

$$\sum_{v\in\mathbb{N}_0} \pi(v) = 1$$

and the subcritical BPIRE is positive recurrent; see, e.g. [10, Theorem 8.18]

Now we will prove the recurrence and transience criteria for a critical BPIRE. The recurrence criteria in Theorem 3 is inspired by a similar result for an autoregressive model defined by a random difference equation in the critical case stated in [1]. Some of the ideas in [1, p. 480f] will be employed and transferred to our BPIRE model.

Proof of Theorem 3. Assume that $\mathbb{Q}[M_1 = 0] < 1$ since $Z_n = 0$, $\mathbb{Q}$-a.s., for all $n \in \mathbb{N}_0$ if $\mathbb{Q}[M_1 = 0] = 1$.

As in [1], let us define $Y_0 := 0$ and

$$Y_n := \log(\mu_1 \cdots \mu_n).$$

Then $(Y_n)_{n\geq 0}$ is an oscillating random walk, i.e. $\limsup_{n\to\infty}(\pm Y_n) = \infty$, $\mathbb{Q}$-a.s.; see, e.g. [10, Proposition 9.14]. The strict descending ladder epochs, see also [8, Section XII.1], are defined by $L_0 := 0$ and

$$L_n := \inf\{k > L_{n-1} \colon Y_k < Y_{L_{n-1}}\}.$$

Since $(Y_n)_{n\geq 0}$ is oscillating, L_n is $\mathbb{Q}$-a.s. finite. Let $L := L_1$, and note that $\mathrm{E}_{\mathbb{Q}}[Y_L] < 0$.

Following the strategy in [1] we consider the subprocess $(Z_{L_n})_{n\geq 0}$ and answer the following questions. Is this process a Markov chain? Is it comparable to $(Z_n)_{n\geq 0}$; more precisely, is it some kind and, if so, which kind of branching process? Is it recurrent? The third question is central for the proof of the theorem since the recurrence of the subprocess yields the recurrence of the process itself. Indeed, we show that $(Z_{L_n})_{n\geq 0}$ is a subcritical BPIRE.

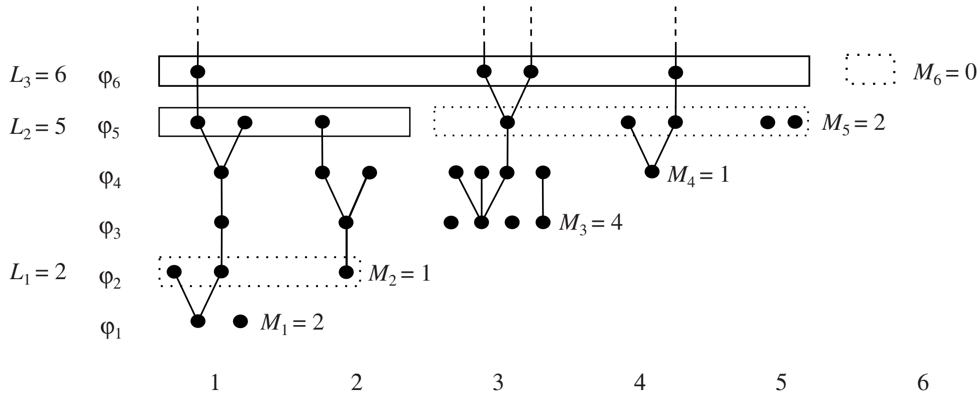

FIGURE 2: An illustration of the process and subprocess. In this figure it is assumed that $L_1 = 2$, $L_2 = 5$, and $L_3 = 6$. The boxes with solid lines represent the offspring of the previous generation, whereas boxes with dotted lines represent immigrants in generation 1, 2, and 3 of the subprocess $(Z_{L_n})_{n\geq 0}$.

Intuitively by Figure 2, $(Z_{L_n})_{n\geq 0}$ appears to be a BPIRE under $\mathbb{Q}_{(\varphi,M)}$ with reproduction distribution given by the PGF

$$\lambda_n(s) := \varphi_{L_{n-1}+1}(\varphi_{L_{n-1}+2}(\cdots \varphi_{L_n}(s))), \qquad 0 \leq s \leq 1,$$

and measure m_n for immigration in generation $n \in \mathbb{N}$, where m_n is the law of

$$\tilde{M}_n := \sum_{j=L_{n-1}+1}^{L_n} Z_{L_n-j}(j).$$

This can be made precise. Calculation and iteration yield, $\mathbb{Q}$-a.s. for $s \in [0, 1]$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{E}_{(\varphi,M)}[s^{Z_{L_n}}] &= \mathrm{E}_{(\varphi,M)}[\varphi_{L_n}(s)^{Z_{L_n-1}}]s^{M_{L_n}} \\
&= \mathrm{E}_{(\varphi,M)}[\varphi_{L_{n-1}+1}(\cdots \varphi_{L_n}(s))^{Z_{L_{n-1}}}]\varphi_{L_{n-1}+2}(\cdots \varphi_{L_n}(s))^{M_{L_{n-1}+1}} \cdots s^{M_{L_n}}.
\end{aligned}$$

The PGF for reproduction is given by the first factor conditioned on $Z_{L_{n-1}} = 1$. The remaining factors equal the PGF for immigration and coincide with the PGF of $\tilde{M}_n$ since, for $L_{n-1} + 1 \leq j \leq L_n$, $\mathbb{Q}$-a.s.,

$$\mathrm{E}_{(\varphi,M)}[s^{Z_{L_n-j}(j)}] = \varphi_{j+1}(\cdots \varphi_{L_n}(s))^{M_j}. \tag{2}$$

The sequence $(\lambda_n, m_n)_{n\in\mathbb{N}}$ is i.i.d. under $\mathbb{Q}$ since the increments of the ladder epochs $(L_n - L_{n-1})_{n\in\mathbb{N}}$ are i.i.d. under $\mathbb{Q}$; see [8, Section XII.1]. The subcriticality of $(Z_{L_n})_{n\geq 0}$ follows from $\mathrm{E}_{\mathbb{Q}}[\log \lambda_1'(1)] = \mathrm{E}_{\mathbb{Q}}[Y_L] < 0$.

Furthermore, the following arguments reveal that the subprocess is still Markovian under $\mathbb{Q}$. For $n \geq 1$ and $i_1, \ldots, i_{n+1} \in \mathbb{N}_0$, the Markov property of $(Z_n)_{n\geq 0}$ under $\mathbb{Q}_{(\varphi,M)}$ implies that

$$\begin{aligned}
&\mathbb{Q}[Z_{L_{n+1}} = i_{n+1},\ Z_{L_n} = i_n, \ldots, Z_{L_1} = i_1] \\
&\quad = \sum_{k\in\mathbb{N}} \mathrm{E}_{\mathbb{Q}}[\mathbf{1}_{\{L_n=k\}}\mathbb{Q}_{(\varphi,M)}[Z_{L_{n+1}} = i_{n+1},\ Z_{L_n} = i_n, \ldots, Z_{L_1} = i_1]] \\
&\quad = \sum_{k\in\mathbb{N}} \mathrm{E}_{\mathbb{Q}}[\mathbf{1}_{\{L_n=k\}}\mathbb{Q}_{\theta^k(\varphi,M),i_n}[Z_{L_1} = i_{n+1}]\mathbb{Q}_{(\varphi,M)}[Z_{L_n} = i_n, \ldots, Z_{L_1} = i_1]],
\end{aligned} \tag{3}$$

where θ^k denotes the shift $\theta^k(\varphi_j, M_j)_{j\in\mathbb{N}} = (\varphi_{k+j}, M_{k+j})_{j\in\mathbb{N}}$ and i_n in $\mathbb{Q}_{\theta^k(\varphi,M),i_n}$ denotes the number of ancestors. Since $(\varphi_j, M_j)_{j\in\mathbb{N}}$ is i.i.d., it follows that (3) equals

$$\sum_{k\in\mathbb{N}} \mathrm{E}_{\mathbb{Q}}[\mathbb{Q}_{\theta^k(\varphi,M),i_n}[Z_{L_1} = i_{n+1}]]\,\mathrm{E}_{\mathbb{Q}}[\mathbf{1}_{\{L_n=k\}}\mathbb{Q}_{(\varphi,M)}[Z_{L_n} = i_n, \dots, Z_{L_1} = i_1]]$$
$$= \mathbb{Q}_{i_n}[Z_{L_1} = i_{n+1}]\mathbb{Q}[Z_{L_n} = i_n, \dots, Z_{L_1} = i_1].$$

To apply Lemma 1, we have to check that $\mathrm{E}_{\mathbb{Q}}[\log_+ \mathrm{E}_{(\varphi,M)}[\tilde{M}_1]] < \infty$ is satisfied. The integrability of $\log(1 + \mathrm{E}_{(\varphi,M)}[\tilde{M}_1])$ implies the integrability of $\log_+ \mathrm{E}_{(\varphi,M)}[\tilde{M}_1]$ and vice versa. We follow the strategy of the proof of Lemma 5.49 of [7]. According to [7, Lemma 5.23], $\log(1 + \mathrm{E}_{(\varphi,M)}[\tilde{M}_1])$ is integrable if and only if

$$\limsup_{n\to\infty} \mathrm{E}_{(\varphi,M)}[\tilde{M}_n]^{1/n} < \infty \quad \mathbb{Q}\text{-a.s.} \tag{4}$$

Since $\mathrm{E}_{\mathbb{Q}}[|\log \mu_1|^2] < \infty$, Theorem 1a of [8, Section XII.7, p. 414f] can be applied and yields $\mathbb{Q}[L > n] \sim c/\sqrt{n}$ for some constant $c > 0$. This implies that $\mathrm{E}_{\mathbb{Q}}[L^\beta] < \infty$ for every $0 < \beta < \frac{1}{2}$. Let $\beta = 1/(2 + \varepsilon)$. Then by, e.g. [10, Theorem 4.23], we obtain, $\mathbb{Q}$-a.s.,

$$\limsup_{n\to\infty} \frac{L_n^\beta}{n} = 0. \tag{5}$$

For the proof of (4), note that, by (2) and since $\mu_j \cdots \mu_{L_n} < 1$ for each $j < L_n$, we obtain, $\mathbb{Q}$-a.s.,

$$\mathrm{E}_{(\varphi,M)}[\tilde{M}_n] = \sum_{j=L_{n-1}+1}^{L_n} \mathrm{E}_{(\varphi,M)}[Z_{L_n-j}(j)] = \sum_{j=L_{n-1}+1}^{L_n} M_j\mu_{j+1}\cdots\mu_{L_n} \le \sum_{j=1}^{L_n} M_j. \tag{6}$$

Now, an analogous calculation to that given in [7, p. 336] yields (4). For completeness, let us give the full argument. Inequality (6) gives

$$\limsup_{n\to\infty} \mathrm{E}_{(\varphi,M)}[\tilde{M}_n]^{1/n} \le \limsup_{n\to\infty} \exp\biggl(\frac{1}{n}\log\biggl(\sum_{j=1}^{L_n} M_j\biggr)\biggr)$$
$$\le \limsup_{n\to\infty} \exp\biggl(\frac{1}{L_n^\beta}\log\biggl(1 + \sum_{j=1}^{L_n} M_j\biggr)\frac{L_n^\beta}{n}\biggr). \tag{7}$$

Since

$$\log\biggl(1 + \sum_{j=1}^{L_n} M_j\biggr) \le \sup_{1\le j\le L_n} \log(1 + M_j) + \log L_n$$

and $\log L_n/L_n^\beta \to 0$ for $n \to \infty$, we obtain

$$\limsup_{n\to\infty} \frac{1}{L_n^\beta}\log\biggl(1 + \sum_{j=1}^{L_n} M_j\biggr) \le \limsup_{n\to\infty} \frac{1}{L_n^\beta}\Bigl(\sup_{1\le j\le L_n} (\log(1 + M_j))^{1/\beta}\Bigr)^\beta$$
$$\le \limsup_{n\to\infty}\biggl(\frac{1}{L_n}\sum_{j=1}^{L_n} (\log(1 + M_j))^{1/\beta}\biggr)^\beta. \tag{8}$$

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Recall that $\mathrm{E}_{\mathbb{Q}}[(\log_+ M_1)^{1/\beta}] = \mathrm{E}_{\mathbb{Q}}[(\log_+ M_1)^{2+\varepsilon}] < \infty$. Hence, the right-hand side of (8) is finite $\mathbb{Q}$-a.s. by the law of large numbers, and (4) follows from (5), (7), and (8).

Summing up, the subprocess $(Z_{L_n})_{n\geq 0}$ is a BPIRE with reproduction according to $(\lambda_n)_{n\in\mathbb{N}}$ and immigration distribution $(m_n)_{n\in\mathbb{N}}$, where $(\lambda_n, m_n)_{n\in\mathbb{N}}$ is i.i.d. and $\mathrm{E}[\log \mathrm{E}_{(\varphi,M)}[\tilde{M}_1]] < \infty$. Applying Lemma 1 reveals that $(Z_{L_n})_{n\geq 0}$ is positive recurrent, in particular recurrent. Hence, $(Z_n)_{n\geq 0}$ is recurrent.

Before proving Theorem 4, let us deduce a useful result from Theorem 2 of [20].

Lemma 2. *Let $d \in \mathbb{N}$, $c > 0$, and $0 < a < 1$. Assume that $\{V, V_{i,n}; i, n \in \mathbb{N}_0\}$ is a family of i.i.d., a.s. nonnegative random variables. Then $\mathrm{E}_{\mathbb{Q}}[(\log_+ V)^d] < \infty$ if and only if $\sum_{n\in\mathbb{N}_0} a^n \sum_{i=0}^{\lfloor cn^{d-1}\rfloor} V_{i,n} < \infty$, $\mathbb{Q}$-a.s.*

Proof. The proof follows by induction over d. Since $\log(1+\sum_{i=1}^k x_i) \leq \sum_{i=1}^k \log(1+x_i)$ for any $k \geq 1$ and $x_i \geq 0$, $1 \leq i \leq k$, we obtain $\mathrm{E}_{\mathbb{Q}}[\log_+ V] < \infty$ if and only if $\mathrm{E}_{\mathbb{Q}}[\log_+(\sum_{i=0}^{\lfloor c\rfloor} V_{i,0})] < \infty$. Furthermore, the latter is equivalent to the almost-sure convergence of $\sum_{n\in\mathbb{N}_0} a^n \sum_{i=0}^{\lfloor c\rfloor} V_{i,n}$; see, for instance, [14, Theorem 5.4.1]. Thus, the result for $d = 1$ follows.

Let the assertion hold for some $d \geq 1$. It will be useful to consider random variables with three indices. Therefore, let $\{V, V_{j,i,n};\ j, i, n \in \mathbb{N}_0\}$ be i.i.d. It follows from Theorem 2 of [20] that $\mathrm{E}_{\mathbb{Q}}[(\log_+ V)^{d+1}]$ is finite if and only if $\mathrm{E}_{\mathbb{Q}}[(\log_+ V_0)^d]$ is finite, where $V_0 = \sum_{n\in\mathbb{N}_0} a^n V_{0,0,n}$. By the induction hypothesis, $\mathrm{E}_{\mathbb{Q}}[(\log_+ V_0)^d] < \infty$ is equivalent to the almost-sure convergence of

$$\sum_{j\in\mathbb{N}_0} a^j \sum_{i=0}^{\lfloor cj^{d-1}\rfloor} \sum_{n\in\mathbb{N}_0} a^n V_{j,i,n} = \sum_{j\in\mathbb{N}_0} \sum_{n\in\mathbb{N}_0} a^{j+n} \sum_{i=0}^{\lfloor cj^{d-1}\rfloor} V_{j,i,n} = \sum_{k\in\mathbb{N}_0} a^k \sum_{j=0}^{k} \sum_{i=0}^{\lfloor cj^{d-1}\rfloor} V_{j,i,k-j}.$$

The number of summands in $\sum_{j=0}^{k} \sum_{i=0}^{\lfloor cj^{d-1}\rfloor} V_{j,i,k-j}$ is asymptotically equal to $k^d c/d$ for $k \to \infty$. For $d \geq 1$, note that the almost-sure convergence of $\sum_{n\in\mathbb{N}_0} a^n \sum_{i=0}^{\lfloor \tilde{c}n^d\rfloor} V_{i,n}$ for some $\tilde{c} > 0$ and all $0 < a < 1$ implies the almost-sure convergence for all $\tilde{c} > 0$. Hence, the lemma follows.

Let us now prove the transience criterion for an irreducible critical BPIRE.

Proof of Theorem 4. The strategy of the proof is similar to that of Theorem 2.2 of [4]. First we discuss an autoregressive model defined by the critical random difference equation $X_n := \mu_n X_{n-1} + M_n$ for $n \in \mathbb{N}$ and $X_0 = 0$. We will show that, $\mathbb{Q}$-a.s.,

$$X_n > \mathrm{e}^{\sqrt{n}} \quad \text{for large } n. \tag{9}$$

Thereafter, $(X_n)_{n\geq 0}$ is coupled with the critical BPIRE $(Z_n)_{n\geq 0}$ to obtain the transience.

As in the proof of Theorem 3, we define $Y_n = \log\mu_1 + \cdots + \log\mu_n$. Let $\frac{1}{2} < \kappa < 1/\lambda$, and let

$$T(\kappa) := \inf\{k \in \mathbb{N}\colon Y_n \geq -n^\kappa \text{ for all } n \geq k\}.$$

By the law of the iterated logarithm, $T(\kappa)$ is finite $\mathbb{Q}$-a.s.; see, e.g. [10, Corollary 14.8, p. 275].

Choose $0 < \delta < 6$ such that $\gamma := \delta\kappa - 2 > 1$. Then

$$\begin{aligned}\mathrm{E}_{\mathbb{Q}}[T(\kappa)^{\gamma}] &= \sum_{n\in\mathbb{N}} n^{\gamma}\,\mathbb{Q}[T(\kappa) = n] \\ &\le 1 + \sum_{n\ge 2} n^{\gamma}\,\mathbb{Q}[Y_{n-1} < -(n-1)^{\kappa}] \\ &\le 1 + \sum_{n\ge 2} n^{\gamma}\,\mathbb{Q}[|Y_{n-1}| > (n-1)^{\kappa}].\end{aligned}$$

This is finite due to the complete convergence theorem of Baum and Katz [5, Theorem 3, (a) $\Rightarrow$ (b)], and the assumptions $\mathrm{E}_{\mathbb{Q}}[|\log \mu_1|^{\delta}] < \infty$ and $\mathrm{E}_{\mathbb{Q}}[\log \mu_1] = 0$. Therefore,

$$\mathrm{E}_{\mathbb{Q}}[T(\kappa)^{\gamma}] < \infty. \tag{10}$$

Consider now the autoregressive model. The recursion of X_n yields

$$X_n = M_n + \mu_n M_{n-1} + \mu_n\mu_{n-1}M_{n-2} + \cdots + \mu_2\cdots\mu_n M_1.$$

Set

$$W_n := M_1 + \mu_1 M_2 + \mu_1\mu_2 M_3 + \cdots + \mu_1\cdots\mu_{n-1}M_n$$

for $n \in \mathbb{N}$. Then, exchangeability implies that, for all $n \in \mathbb{N}$,

$$\mathbb{Q}[X_n > \mathrm{e}^{\sqrt{n}}] = \mathbb{Q}[W_n > \mathrm{e}^{\sqrt{n}}]. \tag{11}$$

Recall that $\frac{1}{2} < \kappa < 1/\lambda$ and $\gamma > 1$. By assumption (iii) of the theorem, there is some constant $c_1 > 0$ such that $\mathbb{Q}[M_1 > \mathrm{e}^{2n^{\kappa}}] > c_1 n^{-\kappa\lambda}$ for large n. Thus, for a suitable constant $c_2 > 0$, we obtain the following bound from above for large $n \in \mathbb{N}$:

$$\begin{aligned}\mathbb{Q}\Big[W_n \le \mathrm{e}^{\sqrt{n}}, T(\kappa) < n^{1/\gamma}\Big] &\le \mathbb{Q}\Big[\bigcap_{n^{1/\gamma}\le i<n} \{\mu_1\cdots\mu_i M_{i+1} \le \mathrm{e}^{\sqrt{n}}\},\ T(\kappa) < n^{1/\gamma}\Big] \\ &= \mathbb{Q}\Big[\bigcap_{n^{1/\gamma}\le i<n} \{M_{i+1} \le \mathrm{e}^{\sqrt{n}-Y_i}\},\ T(\kappa) < n^{1/\gamma}\Big] \\ &\le \mathbb{Q}\Big[\bigcap_{n^{1/\gamma}\le i<n} \{M_{i+1} \le \mathrm{e}^{2n^{\kappa}}\}\Big] \\ &\le (1 - \mathbb{Q}[M_1 > \mathrm{e}^{2n^{\kappa}}])^{n-n^{1/\gamma}-1} \\ &\le \mathrm{e}^{-c_2 n^{1-\kappa\lambda}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{Q}[W_n \le \mathrm{e}^{\sqrt{n}}] &\le \mathbb{Q}[W_n \le \mathrm{e}^{\sqrt{n}},\ T(\kappa) < n^{1/\gamma}] + \mathbb{Q}[T(\kappa) \ge n^{1/\gamma}] \\ &\le \mathrm{e}^{-c_2 n^{1-\kappa\lambda}} + \mathbb{Q}[T(\kappa) \ge n^{1/\gamma}]\end{aligned}$$

for large n. By (10) and (11), and since $\kappa\lambda < 1$, we obtain

$$\sum_{n\ge 1} \mathbb{Q}[X_n \le \mathrm{e}^{\sqrt{n}}] < \infty,$$

and (9) follows by the Borel–Cantelli lemma. Furthermore, we can now choose $N \in \mathbb{N}$ and $x_N > \mathrm{e}^{\sqrt{N}}$ such that

$$\mathbb{Q}\Bigg[\bigcap_{n\geq N+1}\{X_n > \mathrm{e}^{\sqrt{n}}\} \cap \{X_N \leq x_N\}\Bigg]\mathbb{Q}\Bigg[\bigcap_{n=1}^{N-1}\{X_n > \mathrm{e}^{\sqrt{n}}\} \cap \{X_N \geq x_N\}\Bigg] > 0. \tag{12}$$

Define $X_N^{(x_N)} := x_N$ and $X_n^{(x_N)} := \mu_n X_{n-1}^{(x_N)} + M_n$ for $n > N$, and note that $\{X_N \leq x_N, X_n > \mathrm{e}^{\sqrt{n}} \text{ for all } n > N\} \subseteq \{X_n^{(x_N)} > \mathrm{e}^{\sqrt{n}} \text{ for all } n > N\}$. Thus, (12) and the independence of $(\varphi_j, M_j)_{1\leq j\leq N}$ and $(\varphi_j, M_j)_{j\geq N+1}$ under $\mathbb{Q}$ yield

$$\begin{aligned}
\mathbb{Q}\Bigg[\bigcap_{n\geq 1}\{X_n > \mathrm{e}^{\sqrt{n}}\}\Bigg] &\geq \mathbb{Q}\Bigg[\bigcap_{n=1}^{N-1}\{X_n > \mathrm{e}^{\sqrt{n}}\} \cap \{X_N \geq x_N\} \bigcap_{n\geq N+1}\{X_n^{(x_N)} > \mathrm{e}^{\sqrt{n}}\}\Bigg] \\
&= \mathbb{Q}\Bigg[\bigcap_{n=1}^{N-1}\{X_n > \mathrm{e}^{\sqrt{n}}\} \cap \{X_N \geq x_N\}\Bigg]\mathbb{Q}\Bigg[\bigcap_{n\geq N+1}\{X_n^{(x_N)} > \mathrm{e}^{\sqrt{n}}\}\Bigg] \\
&> 0.
\end{aligned}$$

Hence, we have $\mathbb{Q}_\varphi[\bigcap_{n\geq 1}\{X_n > \mathrm{e}^{\sqrt{n}}\}] > 0$ on some $\mathbb{Q}$-nonnull set D.

The next step is to couple $(X_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$. As can be seen from the notation, the increments of the difference equation correspond to the number of immigrants in the BPIRE and the multiplication factor μ_n to the expected number of offspring of an individual in generation $n-1$.

Fix $\mathrm{e}^{-1} < \beta < 1$. We will show that

$$\mathbb{Q}_\varphi\Bigg[\bigcap_{n\in\mathbb{N}}\{Z_n \geq \beta^{\sqrt{n}} X_n\} \,\Bigg|\, \bigcap_{k\in\mathbb{N}}\{X_k > \mathrm{e}^{\sqrt{k}}\}\Bigg] > 0 \tag{13}$$

on D. Thus, $\mathbb{Q}[\lim_{n\to\infty} Z_n = \infty] > 0$ since $\mathrm{e}\beta > 1$, and the transience follows.

Let $B_0 := \bigcap_{k\geq 1}\{X_k > \mathrm{e}^{\sqrt{k}}\}$ and, for $n \in \mathbb{N}$,

$$B_n := \bigcap_{j=1}^{n}\{Z_j \geq \beta^{\sqrt{j}} X_j\} \cap \bigcap_{k\geq 1}\{X_k > \mathrm{e}^{\sqrt{k}}\}.$$

By the definition of $(Z_n)_{n\geq 0}$ in Definition 1, on D, we obtain

$$\begin{aligned}
&\mathbb{Q}_\varphi[Z_n < \beta^{\sqrt{n}} X_n, B_{n-1}] \\
&\quad = \sum_{k\in\mathbb{N}} \mathbb{Q}_\varphi[Z_n < \beta^{\sqrt{n}} X_n,\ Z_{n-1} = k, B_{n-1}] \\
&\quad = \sum_{k\in\mathbb{N}} \mathbb{Q}_\varphi\Bigg[\mu_n k - \sum_{i=1}^{k} \xi_i^{(n)} > \mu_n(k - \beta^{\sqrt{n}} X_{n-1}) + M_n(1-\beta^{\sqrt{n}}),\ Z_{n-1} = k, B_{n-1}\Bigg] \\
&\quad \leq \sum_{k>(\mathrm{e}\beta)^{\sqrt{n-1}}} \mathbb{Q}_\varphi\Bigg[\mu_n k - \sum_{i=1}^{k} \xi_i^{(n)} > (1-\beta^{\sqrt{n}-\sqrt{n-1}})\mu_n k\Bigg]\mathbb{Q}_\varphi[Z_{n-1} = k, B_{n-1}].
\end{aligned}$$

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For the inequality in the last line, note that the summation over $1 \le k \le (\mathrm{e}\beta)^{\sqrt{n-1}}$ yields 0. Now, Chebyshev's inequality implies that

$$\mathbb{Q}_\varphi\left[\mu_n k - \sum_{i=1}^{k} \xi_i^{(n)} > (1-\beta^{\sqrt{n}-\sqrt{n-1}})\mu_n k\right] \le \frac{\operatorname{var}_\varphi(\xi_1^{(n)})}{(1-\beta^{\sqrt{n}-\sqrt{n-1}})^2 \mu_n^2 k}.$$

Note that, for large n, $1-\beta^{\sqrt{n}-\sqrt{n-1}} \ge -\frac{1}{2}(\sqrt{n}-\sqrt{n-1})\log\beta$. Hence,

$$\mathbb{Q}_\varphi[Z_n < \beta^{\sqrt{n}} X_n, B_{n-1}] \le \frac{4\operatorname{var}_\varphi(\xi_1^{(n)})}{\mu_n^2(\log\beta)^2(\sqrt{n}-\sqrt{n-1})^2(\mathrm{e}\beta)^{\sqrt{n-1}}}\mathbb{Q}_\varphi[B_{n-1}]$$

for large n. Thus, it holds for some $0 < \alpha < 1$ that

$$\mathbb{Q}_\varphi[Z_n < \beta^{\sqrt{n}} X_n \mid B_{n-1}] \le \alpha^{\sqrt{n-1}} \operatorname{var}_\varphi(\xi_1^{(n)})\mu_n^{-2}$$

for large n and

$$\begin{aligned}\sum_{n\in\mathbb{N}_0} \alpha^{\sqrt{n}} \operatorname{var}_\varphi(\xi_1^{(n+1)})\mu_{n+1}^{-2} &= \sum_{k\in\mathbb{N}_0}\sum_{n=k^2}^{(k+1)^2-1} \alpha^{\sqrt{n}} \operatorname{var}_\varphi(\xi_1^{(n+1)})\mu_{n+1}^{-2} \\ &\le \sum_{k\in\mathbb{N}_0} \alpha^k \sum_{n=k^2}^{(k+1)^2-1} \operatorname{var}_\varphi(\xi_1^{(n+1)})\mu_{n+1}^{-2}.\end{aligned}$$

By the assumption that $\mathrm{E}_{\mathbb{Q}}[(\log_+(\operatorname{var}_\varphi(\xi_1^{(1)})\mu_1^{-2}))^2] < \infty$, Lemma 2 yields, on D, $\mathbb{Q}$-a.s.,

$$\sum_{n\in\mathbb{N}} \mathbb{Q}_\varphi[Z_n < \beta^{\sqrt{n}} X_n \mid B_{n-1}] < \infty. \tag{14}$$

Furthermore, we have, on D, $\mathbb{Q}$-a.s.,

$$\mathbb{Q}_\varphi[Z_n \ge \beta^{\sqrt{n}} X_n \mid B_{n-1}] \ge \mathbb{Q}_\varphi\left[\sum_{i=1}^{Z_{n-1}} \xi_i^{(n)} \ge \mu_n \beta^{\sqrt{n}-\sqrt{n-1}} Z_{n-1} \mid B_{n-1}\right] > 0 \tag{15}$$

for all $n \ge 1$. The strict inequality in (15) holds since

$$\beta^{\sqrt{n}-\sqrt{n-1}} \le 1 \quad \text{and} \quad \mathbb{Q}_\varphi\left[\sum_{i=1}^{k} \xi_i^{(n)} \ge \mu_n k\right] > 0 \quad \text{for any } k \in \mathbb{N}.$$

The left-hand side of (13) equals $\prod_{n\in\mathbb{N}} \mathbb{Q}_\varphi[Z_n \ge \beta^{\sqrt{n}} X_n \mid B_{n-1}]$ and, thus, (13) follows from (14) and (15).

Remark 4. The recurrence and transience criteria in Theorems 3 and 4 hold in the same way for a BPIRE with one ancestor. Starting with $Z_0 = 0$ in Definition 1 is only due to the proof of Lemma 1. To make sure that the BPIRE dies out infinitely often $\mathbb{Q}$-a.s. in Theorem 3, we have to additionally assume—if the process starts with one ancestor—that $\mathbb{Q}[\varphi_1(0) > 0,\ M_1 = 0] > 0$.

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3. Excited random walk in a random environment

The aim of this section is to transfer the recurrence and transience criteria from the BPIRE to the ERWRE. Therefore, note the well-known connection between branching processes with migration and excited random walks. For a simple symmetric random walk disturbed by cookies, this idea was employed, e.g. in [2], [3], and [12]. In [4] the author explained the connection between a left-transient RWRE disturbed by cookies of maximal strength and a subcritical BPIRE. In this section we establish an analogous relation between a critical BPIRE and a recurrent RWRE disturbed by cookies of maximal strength. This connection is used to prove Theorem 1. For a detailed explanation of the relation, we refer the reader to [4] or [12], but try to give the main ideas here.

Let $X_i^{(j)}$, $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, be a family of ± 1-valued random variables on Ω' such that they are independent under $\mathrm{P}_{z,\omega}$ for all $z \in \mathbb{Z}$ and $\omega \in \Omega$ and

$$\mathrm{P}_{z,\omega}[X_i^{(j)} = 1] = \omega(j, i) = 1 - \mathrm{P}_{z,\omega}[X_i^{(j)} = -1].$$

The events $\{X_i^{(j)} = 1\}$ and $\{X_i^{(j)} = -1\}$ are respectively called successes and failures. Let

$$\xi_j^{(k)} := \#\{\text{successes in } (X_i^{(k)})_{i>M_k} \text{ between the } (j-1)\text{th and the } j\text{th failure}\}.$$

Now, set $V_0 := 1$ and

$$V_k := \xi_1^{(k)} + \cdots + \xi_{V_{k-1}}^{(k)} + M_k.$$

Under Assumption A, $(V_k)_{k\ge 0}$ is a BPIRE under P_1, with one ancestor, immigrants $(M_k)_{k\ge 1}$, and offspring given by $(\xi_j^{(k)})_{j,k\in\mathbb{N}}$. Note that $\xi_j^{(k)}$ has geometric distribution with parameter $1 - p_k$, $(\mathrm{geo}_{\mathbb{N}_0}(1 - p_k)$ for short), i.e. $\mathrm{P}_{1,\omega}[\xi_j^{(k)} = n] = p_k(\omega)^n(1 - p_k(\omega))$ for $n \in \mathbb{N}_0$ and $\mathbb{P}$-almost every $\omega \in \Omega$.

To give an idea of the role of $(V_k)_{k\ge 0}$, note the following. The ERWRE can be realized recursively by

$$S_{n+1} = S_n + X^{(S_n)}_{\#\{m\le n\colon S_m = S_n\}} \quad \text{for } n \ge 0.$$

We denote the time when the ERWRE first hits $k \in \mathbb{Z}$ by

$$T_k := \inf\{n \in \mathbb{N}\colon S_n = k\}.$$

Now, consider the first excursion to the right of $(S_n)_{n\ge 0}$ and count for $k \in \mathbb{N}$ the number of upcrossings from k to $k + 1$ during this excursion:

$$U_k := \#\{n \ge 0\colon n < T_0,\ S_n = k,\ S_{n+1} = k + 1\}.$$

For the moment, assume that $S_1 = 1$. If the first excursion is finite then, up to time T_0, the random walker ate all cookies on site 1 to $\max_{1\le n\le T_0} S_n$. Let $1 \le k < \max_{1\le n\le T_0} S_n$. Before his/her return to 0, the walker stepped from k to $k + 1$ M_k times plus an additional number of times, or, more precisely, plus the number of successes in $(X_i^{(k)})_{i>M_k}$ prior to the U_{k-1}th failure, where $U_0 = 1$. Thus, on $\{T_0 < \infty\} \cap \{S_1 = 1\}$, we obtain

$$U_k = V_k \tag{16}$$

for all $1 \le k \le \max_{1\le n\le T_0} S_n$; see also [4, Lemma 3.3] or [12, Equation (14)].

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Analogously to [12, p. 1962] or [4] we call the ERWRE *recurrent from the right* if the first excursion to the right of 0, if there is any, is P_0-a.s. finite, i.e. $\mathrm{P}_1[T_0 < \infty] = 1$.

In the next step a connection between right recurrence of $(S_n)_{n\geq 0}$ and recurrence of $(V_n)_{n\geq 0}$ is established. Roughly speaking, by (16), as soon as $(U_k)_{k\geq 1}$ becomes 0—which means that the first excursion of the ERWRE is finite—$(V_k)_{k\geq 1}$ becomes 0 or extinct, and vice versa.

Lemma 3. *Let Assumption A hold. The ERWRE $(S_n)_{n\geq 0}$ is recurrent from the right if and only if $(V_k)_{k\geq 0}$ is recurrent in* 0*, i.e.* P_1[*there exists* $k \in \mathbb{N}$: $V_k = 0$] = 1.

- *If $(S_n)_{n\geq 0}$ is recurrent from the right then all excursions are* P_0*-a.s. finite.*
- *If $(S_n)_{n\geq 0}$ is not recurrent from the right then* $\mathrm{P}_0[\lim_{n\to\infty} S_n = +\infty] > 0$.

This lemma can be proven analogously to Lemmas 3.5, 3.6, and 3.7 of [4]. Instead of Lemma 3.2 of [4] we use the facts that $\mathrm{P}_0[\liminf_{n\to\infty} S_n \in \{\pm\infty\}] = 1$ and

$$\mathrm{P}_0\Big[\limsup_{n\to\infty} S_n = +\infty\Big] = 1. \tag{17}$$

Intuitively, assertion (17) is expected since the underlying random walk is recurrent and the cookies induce a drift to the right. In fact, by Lemma 15 of [21], which also holds in the setting $\Omega = ([0, 1]^{\mathbb{N}})^{\mathbb{Z}}$, $\mathrm{P}_{0,\omega}[T_k \leq t]$ is monotone with respect to the environment ω for any $t > 0$ and $k \in \mathbb{N}$. Then $\mathrm{P}_{0,\omega}[\limsup_{n\to\infty} S_n = +\infty] = P_{0,\omega}[\bigcap_{k\in\mathbb{N}} \bigcup_{t\in\mathbb{N}} \{T_k \leq t\}]$ is monotone and (17) holds.

Remark 5. In the case of right recurrence note that, due to monotonicity, there are, contrary to the model in [4], a.s. infinitely many finite excursions to the right since the underlying random environment induces a recurrent random walk. Hence, $\mathrm{P}_0[S_n = 0 \text{ infinitely often}] = 1$ if $(S_n)_{n\geq 0}$ is recurrent from the right.

Proof of Theorem 1. If $\mathbb{P}[M_1 = 0] = 1$, the statement follows from [17, Theorem 1.7] about the RWRE. Thus, assume that $\mathbb{P}[M_1 = 0] < 1$.

The process $(V_k)_{k\geq 0}$ as described above is a BPIRE with immigrants $(M_n)_{n\geq 1}$ and offspring distribution $\mathrm{geo}_{\mathbb{N}_0}(1 - p_j)$, $j \in \mathbb{N}$. It is irreducible on $\mathbb{N}_0$ since, by Assumption A, $\mathbb{P}[M_1 = 0] > 0$ and $0 < p_1 < 1$, $\mathbb{P}$-a.s. and, thus, $\mathbb{P}[M_1 = 0,\ \xi_1^{(1)} = k] > 0$ for every $k \in \mathbb{N}$. Given an environment $\omega \in \Omega$, the expected number of offspring produced by a single particle in the $(j-1)$th generation and its variance are

$$\mu_j(\omega) := \mathrm{E}_{0,\omega}[\xi_1^{(j)}] = \frac{p_j(\omega)}{1 - p_j(\omega)} = \rho_j^{-1}(\omega)$$

and

$$\mathrm{var}_{0,\omega}(\xi_1^{(j)}) = \frac{p_j(\omega)}{(1 - p_j(\omega))^2},$$

respectively. Hence, since we suppose that Assumption A holds for Theorem 1, $(V_k)_{k\geq 0}$ is a critical BPIRE according to Definition 1. Furthermore, $\mathbb{P}[\mu_1 = 1] < 1$ holds by Assumption A. Supposing that $\mathbb{E}[|\log \mu_1|^{\delta}] < \infty$ for every $0 < \delta < 6$ also includes, in particular, that $\mathbb{E}[(\log p_1)^2] < \infty$. To see this, note that

$$\mathbb{E}\bigg[\bigg(\log \frac{p_1}{1 - p_1}\bigg)^2\bigg] \geq \mathbb{E}\bigg[\bigg(\log \frac{p_1}{1 - p_1}\bigg)^2 \mathbf{1}_{\{p_1 < \kappa\}}\bigg] \quad \text{for any } \kappa > 0.$$

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Now, for small enough κ, a constant $c > 0$ can be found such that

$$\left(\log \frac{p_1}{1-p_1}\right)^2 = (\log p_1 - \log(1-p_1))^2 \geq c(\log p_1)^2$$

holds on $\{0 < p_1 < \kappa\}$. Therefore, $\mathbb{E}[(\log_+(\mathrm{var}_{0,\omega}(\xi_1^{(1)})\mu_1^{-2}))^2] = \mathbb{E}[(\log p_1)^2] < \infty$ is fulfilled, and, finally, the assumptions of Theorems 3 and 4 are satisfied.

If $\mathbb{E}[(\log_+ M_0)^{2+\varepsilon}] < \infty$ holds for some $\varepsilon > 0$ then the BPIRE $(V_k)_{k\geq 0}$ is recurrent by Theorem 3. Lemma 3 gives $\mathrm{P}_0[S_n = 0 \text{ infinitely often}] = 1$ since the underlying RWRE is recurrent and the first statement of Theorem 1 follows.

Let $\liminf_{t\to\infty}(t^{\lambda}\mathbb{P}[\log M_1 > t]) > 0$ for some $0 < \lambda < 2$. Then $(V_k)_{k\geq 0}$ is transient by Theorem 4. Thus, by Lemma 3, $\mathrm{P}_0[\lim_{n\to\infty} S_n = +\infty] > 0$. Now, following the same strategy as in the proof of Theorem 1.1(iii) of [4], we obtain $\mathrm{P}_0[\lim_{n\to\infty} S_n = +\infty] = 1$. This completes the proof.

Example 1. Suppose that the assumptions of Theorem 1 are fulfilled, and let $\lambda > 0$. Let M_0 satisfy

$$\mathbb{P}[M_0 = 0] = 1 - \frac{1}{(1+\log 2)^{\lambda}},$$
$$\mathbb{P}[M_0 \geq k] = \frac{1}{(1+\log k)^{\lambda}} \quad \text{for } k \geq 2, k \in \mathbb{N}.$$

Theorem 1 makes no statement for $\lambda = 2$, but, for $\lambda < 2$, we obtain $\lim_{n\to\infty} S_n = +\infty$ P_0-a.s. due to $\lim_{t\to\infty} t^{\lambda}\mathbb{P}[\log M_0 \geq t] = 1 > 0$. If $\lambda > 2$ then $S_n = 0$ infinitely often P_0-a.s. since we can choose $\varepsilon > 0$ such that $2 + \varepsilon < \lambda$ and get $\mathbb{E}[(\log_+ M_0)^{2+\varepsilon}] < \infty$.

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