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DIVISIBLE PROPERTIES AND THE STONE-ČECH COMPACTIFICATION

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Let T be an abelian infinite countable group. We say that a property $\mathscr{P}$ of subsets of T is *divisible* if it satisfies the following requirements. (We identify $\mathscr{P}$ with the set of all subsets of T which satisfy $\mathscr{P}$.)

(i) $\emptyset \notin \mathscr{P}$ and $T \in \mathscr{P}$

(ii) $A \in \mathscr{P}$ and $B \supset A$ implies $B \in \mathscr{P}$

(iii) $A \in \mathscr{P}$ and $A = B_1 \cup B_2$ implies that either B_1 or B_2 is in $\mathscr{P}$.

As we shall see there are 2^c divisible properties. We mention some properties which are obviously divisible. (1) Being infinite. (2) $\mathscr{P}$ is an ultrafilter on T. (3) Let $T = \mathbf{Z}$ (the group of integers) and let $A \in \mathscr{P}$ if and only if $\sum_{n \in A} 1/|n| = \infty$. Less obvious is the divisibility of the following properties. (4) $T = \mathbf{Z}$ and $A \in \mathscr{P}$ if and only if A contains arbitrarily long arithmatical progressions. (5) $T = \mathbf{Z}$ and $A \in \mathscr{P}$ if and only if there exists an infinite sequence $\{n_i\} \subset \mathbf{Z}$ such that for every $i \neq j$ either $n_i - n_j$ or $n_j - n_i$ is in A. In fact it is not hard to see that the divisibility of the properties (4) and (5) is equivalent to v.d. Warden's and Ramsey's theorems respectively. (6) Another divisible property for a general group T is "not being a Sidon set." This statement is Drury's theorem [**1**].

Our aim in this note is to establish a one-to-one correspondence between divisible properties of subsets of T and closed subsets of βT, the Stone–Čech compactification of the discrete group, and to use this correspondence in studying both divisible properties and some aspects of topological dynamics.

This work is closely related to H. Furstenberg's and B. Weiss' paper [**6**]. I am indebted to them and also to J. Hirshfeld for many interesting and helpful conversations.

1. Closed subsets of βT and divisible properties. Although we assume that our group T is abelian we use a multiplicative notation for the group operation. The group product induces in a natural way a semigroup structure on βT. This product considered as a function from $\beta T \times \beta T$ into βT is continuous only if T is finite. However, for a fixed $p \in \beta T$ the map $q \to qp$ of βT into itself is continuous, and for all $t \in T$

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the maps $q \to tq$ of βT into itself are continuous. The latter makes βT a T-flow and in fact the flow $(\beta T, T)$ is a universal point transitive flow. The minimal subsets of the flow $(\beta T, T)$ are all isomorphic as T-flows and they coincide with the minimal left ideals of the semigroup βT. If M is such a minimal ideal then the set of idempotents, J, of M is not empty. Each such an idempotent, v, is a right unit on M and vM is actually a group. M is the disjoint union of the groups $\{vM\}$ where $v \in J$. An idempotent of βT is called *minimal* if it belongs to some minimal ideal of βT. For more details about the flow $(\beta T, T)$ and its close relation to topological dynamics we refer the reader to [**3**] and [**7**].

As a topological space βT is of course compact, Hausdorff, and non-metric. If A is a subset of T then $\bar{A} \subset \beta T$ is an open and closed subset of βT and the collection $\{\bar{A} : A \subset T\}$ forms a basis for the topology of βT. We shall identify βT with the set of all ultrafilters on T. From this point of view $\bar{A}$ is the set of all ultrafilters containing A.

1.1. PROPOSITION. *Let Z be a non-empty closed proper subset of βT; then the requirement $\bar{A} \cap Z \neq \emptyset$ on subsets A of T defines a divisible property. Conversely for every divisible property $\mathscr{P}$ there exists a non-empty closed subset Z of βT called the kernel of $\mathscr{P}$ such that a subset A of T is in $\mathscr{P}$ if and only if $\bar{A} \cap Z \neq \emptyset$.*

Proof. The first statement is obvious. Let $\mathscr{P}$ be a divisible property. We say that a point $p \in \beta T$ is a $\mathscr{P}$-point if every set in p (where p is considered as an ultrafilter) has the property $\mathscr{P}$. Let Z be the set of all $\mathscr{P}$-points in βT. Let $A \in \mathscr{P}$ and set

$$\mathscr{F} = \{B : B \subset T \text{ and } A \backslash B \notin \mathscr{P}\}.$$

Since $\mathscr{P}$ is divisible it follows by (i) that $A \in \mathscr{F}$, $\emptyset \notin \mathscr{F}$ and by (ii) it follows that if $B \in \mathscr{F}$ and $D \supset B$ then also $D \in \mathscr{F}$. Let $B_1, B_2 \in \mathscr{F}$, then

$$A \backslash (B_1 \cap B_2) = (A \backslash B_1) \cup (A \backslash B_2).$$

Now (iii) implies that $A \backslash (B_1 \cap B_2) \notin \mathscr{P}$ i.e., $B_1 \cap B_2 \in \mathscr{F}$. Thus $\mathscr{F}$ is a filter on T. Let p be an ultrafilter containing $\mathscr{F}$. We claim that p is a $\mathscr{P}$-point. In fact let $B \in p$ and suppose $B \notin \mathscr{P}$; then $A \cap B \notin \mathscr{P}$ and hence $A \backslash (T \backslash B) = A \cap B \notin \mathscr{P}$ which means $T \backslash B \in \mathscr{F} \subset p$, a contradiction. Since $p \in \bar{A}$ this shows that $\bar{A} \cap Z \neq \emptyset$. Conversely if $A \subset T$ is such that $\bar{A} \cap Z \neq \emptyset$ then there exists a $\mathscr{P}$-point in $\bar{A}$ and $A \in \mathscr{P}$.

Let $\mathscr{P}$ be a property of subsets of T (i.e., a collection of subsets of T). We say that a subset A of T has the dual property, $\mathscr{P}^*$, if and only if A has a non-empty intersection with each member of $\mathscr{P}$.

1.2. PROPOSITION. *A property $\mathscr{P}$ is divisible if and only if $\mathscr{P}^*$ is a filter. In this case $\mathscr{P} \supset \mathscr{P}^*$, $\mathscr{P}^{**} = \mathscr{P}$ and $Z = \bigcap \{\bar{A} : A \in \mathscr{P}^*\}$ is the kernel of $\mathscr{P}$. Conversely if $\mathscr{F}$ is a filter on T then $\mathscr{F}^*$ is divisible and $\mathscr{F}^{**} = \mathscr{F}$.*

Proof. Suppose $\mathscr{P}$ is divisible and let Z be its kernel. If $A \in \mathscr{P}^*$ then A intersects every neighborhood of each point in Z. Hence $\mathscr{P}^* = \{A : \bar{A} \supset Z\}$ which is a filter. Moreover it is now clear that $\mathscr{P}^* \subset \mathscr{P}$, that $Z = \bigcap \{\bar{A} : A \in \mathscr{P}^*\}$, and that $\mathscr{P}^{**} = \mathscr{P}$.

Suppose that $\mathscr{F}$ is a filter on T and let $Z = \bigcap\{\bar{A} : A \in \mathscr{F}\}$. It is easy to see that $\mathscr{F}^* = \{B \subset T : \bar{B} \cap Z \neq \emptyset\}$. Thus $\mathscr{F}^*$ is divisible and by the above $\mathscr{F}^{**}$ is a filter with $Z = \bigcap\{\bar{B} : B \in \mathscr{F}^{**}\}$. By definition $\mathscr{F} \subset \mathscr{F}^{**}$. Suppose $B \in \mathscr{F}^{**}$ but $B \notin \mathscr{F}$; let $\mathscr{G} = \{(T\backslash B) \cap A : A \in \mathscr{F}\}$. $\mathscr{G}$ is a filter base, since $(T\backslash B) \cap A = \emptyset$ for some $A \in \mathscr{F}$ would imply $B \in \mathscr{F}$. Let p be an ultrafilter containing $\mathscr{G}$; $p \notin \bar{B}$ implies $p \notin Z$. But $p \supset \mathscr{F}$ implies $p \in Z$, a contradiction. Thus $\mathscr{F} = \mathscr{F}^{**}$ and the proof is completed.

Remark. $\mathscr{F}$ is a filter if and only if $\mathscr{G} = \{A : T\backslash A \in \mathscr{F}\}$ is an ideal. Whence a property $\mathscr{P}$ is divisible if and only if the collection of subsets $\mathscr{G} = \{A : A \text{ is not in } \mathscr{P}\}$ is an ideal. If $\mathscr{P}$ is divisible and Z is its kernel then $A \in \mathscr{G}$ if and only if $\bar{A} \cap Z = \emptyset$.

2. IP **sets and Hindman's theorem.** Let $\{t_i\}_{i \in I}$ be a subset of T, where I is some index set. If $\alpha = \{i_1, i_2, \ldots, i_n\}$ is a finite subset of I we set

$$t_\alpha = \prod_{i=1}^{n} t_{i_j}.$$

We let

$$\mathrm{IP}\{t_i\}_{i \in I} = \{t_\alpha : \alpha \text{ is a finite subset of } I\}.$$

Notice that we do not allow repetitions in α. We say that $\mathrm{IP}\{t_i\}_{i \in I}$ is an *IP-system*. A subset A of T is called an *IP-set* if it contains an infinite IP-system.

2.1. THEOREM. *The property of being an* IP-*set is divisible. Let U be the set of idempotents in $\beta T\backslash T$; then $Z = \bar{U}$ is its kernel.*

Proof. We will show that a subset $A \subset T$ is an IP-set if and only if $\bar{A} \cap Z \neq \emptyset$, thereby proving the divisibility of this property.

Suppose first that $A \subset T$ is such that $\bar{U} \cap \bar{A} \neq \emptyset$. Since $\bar{A}$ is open this implies that there exists an idempotent $u \neq e$, $u \in \bar{A}$ (e is the identity element of T). Since $u^2 = u$ and since right multiplication by u is continuous we conclude that there exists $t_1 \in A$ with $t_1 u \in \bar{A}$. Thus $u \in \overline{t_1^{-1}A \cap A}$ and we can choose $t_2 \in t_1^{-1}A \cap A$ with $t_2 \neq t_1$ and $t_2 u \in \overline{t_1^{-1}A \cap A}$. Clearly $\mathrm{IP}\{t_1, t_2\} \subset A$ and $tu \in \bar{A}$ for every $t \in \mathrm{IP}\{t_1, t_2\}$. Suppose we already find $t_1, t_2, \ldots, t_n \in A$ such that $\mathrm{IP}\{t_1, \ldots, t_n\} \subset A$ and $tu \in \bar{A}$ for every $t \in \mathrm{IP}\{t_1, \ldots, t_n\}$. Then

$$V = \operatorname{cls} A \cap \{t^{-1}A : t \in \mathrm{IP}\{t_1, \ldots, t_n\}\}$$

is a neighborhood of u and we can find $t_{n+1} \in T \cap V$ with $t_{n+1} \neq t_j$, $j \leqq n$ and $t_{n+1}u \in V$. Thus $\mathrm{IP}\{t_1, \ldots, t_{n+1}\} \subset A$ and by induction we conclude that A is an IP-set.

Conversely, let A be an IP-set containing the infinite IP-system $\Lambda = \mathrm{IP}\{t_i\}_{i \in I}$. For every finite subset $\alpha \subset I$ put $\Lambda_\alpha = \mathrm{IP}\{t_i\}_{i \in I \setminus \alpha}$ and $K_\alpha = \mathrm{cls}\ \Lambda_\alpha \subset \beta T$. Let $K = \bigcap_\alpha K_\alpha$. We claim that K is a subsemigroup of $\beta T \setminus T$. In fact let $p, q \in K$ and let $B \subset T$ be such that $pq \in \bar{B}$. Let α be an arbitrary finite subset of I. Since $p \in K$ and since right multiplication by q is continuous there exists $t \in \Lambda_\alpha$ with $tq \in \bar{B}$. Now $t = t_{i_1} t_{i_2} \ldots t_{i_k}$ for some finite subset $\beta = \{i_1, \ldots, i_k\} \subset I \setminus \alpha$. Since $q \in K_{\alpha \cup \beta}$ and since left multiplication by t is continuous there exists $s \in \Lambda_{\alpha \cup \beta}$ such that $ts \in \bar{B}$. But $ts \in \Lambda_\alpha$ and we conclude that $\bar{B} \cap K_\alpha \neq \emptyset$. Thus $pq \in K_\alpha$ and since α was arbitrary we have $pq \in K$. Clearly $K \subset \bar{A} \setminus A$ and by the following lemma our proof is completed.

2.1. Lemma. (Ellis) *Let K be a compact Hausdorff semigroup with continuous right multiplication. Then K contains an idempotent.*

Proof. Let $\mathscr{L} = \{L \subset K : L$ is closed and $L^2 \subset L\}$. By Zorn's lemma $\mathscr{L}$ contains a minimal element under inclusion, say L. Let $u \in L$ and notice that Lu is closed and satisfies $(Lu)^2 \subset Lu$. Since $Lu \subset L$ we have $Lu = L$. Thus the set $\{t \in L : tu = u\} = N$ is non-empty, closed and satisfies $N^2 \subset N$. Thus $N = L$ and we conclude that $u^2 = u$.

Of course the first statement of Theorem 2.1. is just Hindeman's theorem for any abelian group T. In fact all we have done is meaningful for an abelian semigroup T as well. For example taking $T = \mathbf{Z}^+$, the multiplicative semigroup of positive integers, we can prove the following:

2.3. Proposition. (B. Weiss) *Let $\mathbf{Z}^+ = \bigcup_{i=1}^N S_i$ be a partition of $\mathbf{Z}^+$. Then there exists an i $(1 \leqq i \leqq N)$ and an infinite set of positive integers $\{t_1, t_2, \ldots\}$ which are mutually disjoint and such that* $\mathrm{IP}\{t_1, t_2, \ldots\} \subset S_i$.

Proof. Let $\mathbf{Z}_{(n)}{}^+$ be the set of positive integers which are disjoint from $p_1, p_2, \ldots, p_n$, where p_j is the jth prime. Let K_n be the closure of $\mathbf{Z}_{(n)}{}^+$ in the semigroup $\beta\mathbf{Z}^+$ and put $K = \bigcap K_n$. Then K is a closed subsemigroup and by Lemma 2.2 there exists an idempotent $u \in K$. Since $u \in \overline{\mathbf{Z}^+} = \bigcup \bar{S}_i$ there exists an i such that $u \in \bar{S}_i$. We now repeat the process of constructing an IP-system, as in the first part of the proof of Theorem 2.1, with the additional requirement that t_n will be an element of $\mathbf{Z}_{(k_n)}{}^+$ where p_{k_n} is the largest among the prime factors of the numbers t_j, $1 \leqq j \leqq n - 1$.

3. Difference sets and Ramsey's theorem. We say that a subset $A \subset T$ is a *difference set* or a D-*set* if there exists an infinite sequence $\{t_k\}_{k=1}^\infty \subset T$ such that for every $i \neq j$ either $t_i^{-1}t_j$ or $t_it_j^{-1}$ is in A.

Clearly the requirements (i) and (ii) of a divisible property are satisfied by the property of being a D-set. Requirement (iii) follows from Ramsey's theorem. For let A be a D-set containing the differences of the infinite sequence $\{t_i\}_{i=1}^{\infty}$, and let $A = B_1 \cup B_2$; we can assume that B_1 and B_2 are disjoint. Define a coloring of the unordered pairs of elements of $\{1, 2, 3, \ldots\}$ as follows: The pair $\{i, j\}$ will have color 1 if either $t_i^{-1}t_j$ or $t_it_j^{-1}$ is in B_1. Otherwise it will have color 2. By Ramsey's theorem there is an infinite subset $\{n_1, n_2, \ldots\} \subset \{1, 2, \ldots\}$ such that all the pairs of the form $\{n_k, n_l\}$ have the same color. We deduce that either B_1 or B_2 is a difference set.

Our goal in the next theorem is to identify the kernel of the property of being a difference set. (This of course will provide an alternative proof to the fact that this property is divisible.)

Let $V \subset \beta T$ be the set of all points $p \in \beta T$ such that there exist a net $\{t_\alpha\} \subset T$ and an element $q \in \beta T$ with $q = \lim t_\alpha$ and $p = \lim t_\alpha^{-1}q$. Let $Z = \bar{V}$.

3.1. THEOREM. *The set Z is the kernel of the divisible property of being a difference set.*

Proof. Let $A \subset T$ be such that $\bar{A} \cap Z \neq \emptyset$, then also $\bar{A} \cap V \neq \emptyset$ and there exists a point, say p, in this intersection. Since $p \in V$ there exist a net $\{t_\alpha\} \subset T$ and a point $q \in \beta T$ such that $\lim t_\alpha = q$ and $\lim t^{-1}q = p$. Choose an α_1 such that $\alpha \geqq \alpha_1$, implies $t_\alpha^{-1}q \in \bar{A}$, and $\alpha_2 > \alpha_1$, such that $\alpha \geqq \alpha_2$ implies $t_{\alpha_1}^{-1}t_\alpha \in A$. Now $\alpha_2 > \alpha_1$, hence $t_{\alpha_2}^{-1}q \in \bar{A}$ and therefore there exists $\alpha_3 > \alpha_2$ for which $\alpha \geqq \alpha_3$ implies $t_{\alpha_2}^{-1}t_\alpha \in A$.

Suppose we already have $\alpha_1 < \alpha_2 < \ldots < \alpha_k$ with

$$\alpha \geqq \alpha_j \text{ implies } t_{\alpha_{j-1}}^{-1} t_\alpha \in A \quad (1 < j \leqq k).$$

Now $\alpha_k > \alpha_1$ implies $t_{\alpha_k}^{-1}q \in \bar{A}$ and hence there exists $\alpha_{k+1} > \alpha_k$ for which $\alpha \geqq \alpha_{k+1}$ implies $t_{\alpha_k}^{-1}t_\alpha \in A$. By induction A is a difference set.

Conversely, suppose there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that for $i \neq j$ either $t_i^{-1}t_j$ or $t_j^{-1}t_i$ is in A. We choose a subnet $\{t_{i_\alpha}\}$ of $\{t_i\}$ for which the limits

$$\lim_\alpha t_{i_\alpha} = q, \quad \lim_\alpha t_{i_\alpha}^{-1}q = p, \quad \lim t_{i_\alpha}^{-1} = q', \quad \text{and } \lim t_{i_\alpha}q' = p'$$

exist. Let U and U' be neighborhoods of p and p' respectively. Then there exists α with $t_{i_\alpha}^{-1}q \in U$ and $t_{i_\alpha}q' \in U'$. Now choose β for which

$$t_{i_\alpha}^{-1}t_{i_\beta} \in U \quad \text{and} \quad t_{i_\alpha}t_{i_\beta}^{-1} \in U'.$$

One of these elements belongs to A and either $\bar{A} \cap U \neq \emptyset$ or $\bar{A} \cap U' \neq \emptyset$. Since p and p' are in V we have $\bar{A} \cap V \neq \emptyset$ and hence also $\bar{A} \cap Z \neq \emptyset$.

Let (X, T) be a minimal flow; a theorem due to W. A. Veech states that for every $x \in X$ $S_e(x) = Zx$ and if X is metric then $S_e(x) = Vx$ [**15**]. Here Z and V are as above and S_e is the equicontinuous structure relation on X; i.e., the smallest closed, invariant, equivalent relation R on X for which $(X/R, T)$ is equicontinuous. We shall use this fact later to gain some further information about difference sets.

3.2. PROPOSITION. *Every* IP-*set is a* D-*set*.

Proof. Let $\Lambda = IP\{t_i\}_{i\in I}$ be an infinite IP-system. Let $\{i_1, i_2, \ldots\}$ be a sequence in I and let $s_n = \prod_{j=1}^{n} t_{i_j}$. Now for $m > n$ we have

$$s_m s_n^{-1} = \prod_{j=n+1}^{m} t_{i_j} \in \Lambda.$$

Hence Λ contains the differences of the sequence $\{s_n\}_{n=1}^{\infty}$ and is therefore a D-set.

3.2. COROLLARY. *Let U be the set of idempotents in βT and let V be as above; then $U \subset \bar{V}$.*

4. Big sets, minimal sets, and v.d. Warden's theorem. We identify the set of all subsets of T with the sequence space $\Omega = \{0, 1\}^T$. A subset $A \subset T$ corresponds to the sequence $\xi \in \Omega$ where $\xi(t) = \chi_A(t)$. Let T act on Ω by translations, i.e., given $\xi \in \Omega$ and $t \in T$ we let $t\xi(s) = \xi(ts)$ $(s \in T)$. Notice, however, that $t\chi_A = \chi_{t^{-1}A}$ $(t \in T, A \subset T)$. Define an operation of the semigroup βT on Ω as follows: For $p \in \beta T$ and $A \subset T$ put $p*A = \{t \in T : tp \in \bar{A}\}$. Notice that for $t \in T$, $t*A = t^{-1}A$.

4.1. LEMMA. *Under this operation βT is the enveloping semigroup of (Ω, T).*

Proof. Let $A \subset T$ and $p \in \beta T$ be given. Let $\{t_i\}$ be a net in T which converges to p. We have for $s \in T$, $\chi_{p*A}(s) = 1 \Leftrightarrow s \in p*A \Leftrightarrow sp \in \bar{A} \Leftrightarrow$ eventually $st_i \in A \Leftrightarrow$ eventually $s \in t_i^{-1}A \Leftrightarrow$ eventually $t_i\chi_A(s) = 1 \Leftrightarrow$ $p\chi_A(s) = 1$. Thus $p\chi_A = \chi_{p*A}$ and this defines a homomorphism of βT onto the enveloping semigroup of (Ω, T). We show that this homomorphism is one-to-one. Suppose $p*A = q*A$ for every $A \subset T$. If $p \neq q$ then there exists an $A \subset T$ with $p \in \bar{A}$ and $q \notin \bar{A}$. This implies $e \in p*A$ and $e \notin q*A$, a contradiction; thus $p = q$ and the proof is completed.

We say that a subset A of T is *small* if the only minimal set in the orbit closure of χ_A in (Ω, T) is the singleton $\{\chi_\phi\}$. A is *big* if it is not small [**4**]. A subset $B \subset T$ is called *minimal* if χ_B is an almost periodic point of (Ω, T); i.e., χ_B belongs to a minimal subset of (Ω, T). Now A is big if and only if the orbit closure of χ_A in (Ω, T) contains a minimal subset which is not $\{\chi_\phi\}$. Since this orbit closure is equal to $(\beta T)\chi_A$, this is equivalent to the existence of a minimal idempotent $u \in \beta T$ such that $u*A = B$ is non-empty; i.e., A is big if and only if it is proximal to a minimal non-empty B.

Let $\{M_\alpha\}$ be the collection of minimal ideals in βT; let $Z = \text{cls}(\bigcup M_\alpha)$.

4.1. PROPOSITION. *A subset $A \subset T$ is big if and only if $\bar{A} \cap Z \neq \emptyset$. Thus being big is a divisible property with kernel Z.*

Proof. Suppose $\bar{A} \cap Z \neq \emptyset$; then there exists a minimal ideal M such that $\bar{A} \cap M \neq \emptyset$. Let $p \in \bar{A} \cap M$; then $e \in p*A = B$ and B is a minimal, non-empty set in the orbit closure of A in Ω. Conversely suppose A is big; then for some p in some minimal ideal M, $p*A = B$ is non-empty. Let $t \in B$; then $tp \in \bar{A} \cap M$ and $\bar{A} \cap Z \neq \phi$.

4.2. COROLLARY. *Let A be big then there exists $t \in T$ such that tA is* IP.

Proof. Let M be a minimal ideal such that $V = \bar{A} \cap M \neq \emptyset$. Then V is an open non-empty subset of the minimal flow (M, T). Hence there exist $t_1, \ldots, t_n \in T$ such that $M = \bigcup_{i=1}^n t_i V$. Let u be an idempotent in M and let $u \in t_j V$; then $u \in t_j A$ and $\overline{t_j A}$ is IP.

Let bT be the Bohr compactification of T. Then (bT, T) is the universal minimal equicontinuous T-flow. Let $\phi : \beta T \to bT$ denote the canonical homomorphism. We think of T as a subset of bT as well as of βT. One might suspect that a necessary and sufficient condition for a set $A \subset T$ to be a D-set is that $e \in \mathrm{cls}_{bT} A$. However there are examples of subsets $A \subset \mathbf{Z}$ which are dense in bT without being D-sets. (See for example [**10**].)

4.3. THEOREM. *Let $A \subset T$ be minimal; then A is a* D-*set if and only if $e \in \mathrm{cls}_{bT} A$.*

Proof. Let A be a D-set. There exists a sequence $\{t_i\}_{i=1}^\infty$ such that $t_i^{-1}t_j$ or $t_j^{-1}t_i$ is in A for every $i \neq j$. Let $\{t_{i\alpha}\}$ be a subnet of $\{t_i\}$ such that $\lim t_{i_\alpha} = x$ exists in bT. Since bT is a compact topological group $\lim t_{i_\alpha}^{-1} = x^{-1}$ and every symmetric neighborhood of $e = x \cdot x^{-1}$ in bT contains elements of the form $t_{i_\alpha}^{-1} t_{j\beta}$ and $t_{i_\alpha}^{-1} t_{i_\beta}$. Thus $e \in \mathrm{cls}_{bT} A$.

Conversely let $A \subset T$ be such that A is minimal and $e \in \mathrm{cls}_{bT} A$. There exists an idempotent u in some minimal ideal M in βT with $u*A = A$. Let $\{t_i\}$ be a net in A such that $\lim t_i = e$ in bT and one can assume that $\lim t_i = p \in \bar{A}$ exists in βT. Now $\phi(pu) = e$ and $\phi(u) = e$ implies $pu \in S_e(u)$ where S_e is the equicontinuous structure relation on the minimal flow (M, T). By [**15**], $S_e(u) = Zu$ where $Z \subset \beta T$ is the kernel of the divisible property of being a D-set. Hence there exists $q \in Z$ with $pu = qu$. Now $t_i \in A = u*A \Leftrightarrow t_i u \in \bar{A} \Leftrightarrow pu \in \bar{A} \Leftrightarrow qu \in \bar{A}$. Let $\{s_i\}$ be a net in T such that $\lim s_i = q$; then eventually $s_i u \in \bar{A} \Rightarrow s_i \in u*A = A \Rightarrow q \in \bar{A}$. Thus $\bar{A} \cap Z \neq \emptyset$ and A is a D-set.

We say that a subset $A \subset T$ is *weakly mixing* if the orbit closure of χ_A in (Ω, T) is weakly mixing.

4.4. COROLLARY. *Let $A \subset T$ be minimal and weakly mixing; then A is a* D-*set.*

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Proof. Let (X, T) be the orbit closure of χ_A in (Ω, T). Then (X, T) is a minimal weakly mixing flow and is therefore disjoint from (bT, T); i.e., $(X \times bT, T)$ is a minimal flow. Let $W = \{\xi \in X : \xi(e) = 1\}$ and let V be an arbitrary open set in bT. Then there exists $t \in T$ such that $t(\chi_A, e) \in W \times V$ i.e., $t \in V$ and $t\chi_A(e) = \chi_A(t) = 1$. Thus A is dense in bT and by Theorem 4.3, A is a D-set.

We say that a set $A \subset T$ has the *van der Warden property* if for every finite set $\{t_1, t_2, \ldots, t_k\} \subset T$ there exist a positive integer n and an element $t \in A$ such that $tt_i^n \in A$, $i = 1, \ldots, k$. Now it is a standard argument to show that requirement (iii) of a divisible property follows for the van der Warden property, from the fact that whenever $T = \bigcup_{i=1}^N S_i$ then at least one of the sets $S_i = (1 \leqq i \leqq n)$ has the van der Warden property. We prove this latter statement using the following theorem of topological dynamics which we cite without proof [**6**].

THEOREM. *Let (X, T) be a minimal flow, let $t_1, \ldots t_k \in T$ and let V* be a *non-empty open subset of X. Then there exists a positive integer n with*

$$V \cap t_1^n V \cap \ldots \cap t_k^n V \neq \emptyset.$$

4.5. THEOREM. *Let $A \subset T$ be big; then A has the van der Warden property.*

Proof. Since A is big there exists a minimal ideal $M \subset \beta T$ such that $V = \bar{A} \cap M \neq \emptyset$. Let $t_1, \ldots, t_k \in T$ be given; then applying the above theorem to the minimal flow (M, T) we find a positive integer n with

$$V \cap t_1^{-n} V \cap \ldots \cap t_k^{-n} V \neq \emptyset.$$

Let p be a point in this intersection; then for $1 \leqq i \leqq k$, $t_i^n p \in V$. Since $p \in V = \bar{A} \cap M$ there exists $t \in A$ such that $t_i^n t \in A$ for every i and A has the van der Warden property.

We can now deduce van der Warden's theorem which is the statement that the v.d. Warden property is divisible. For as was mentioned above it suffices to show that whenever $T = \bigcup_{i=1}^N S_i$ then at least one of the sets S_i has the van der Warden property. Let M be an arbitrary minimal ideal in βT; then since $M \subset \beta T = \bar{T} = \bigcup_{i=1}^N \bar{S}_i$, there exists an i with $\bar{S}_i \cap M \neq \emptyset$. Thus S_i is big and our statement follows from Theorem 4.5.

Let U_m be the set of all minimal idempotents in βT; i.e., the idempotents which belong to minimal ideals. Put $Z = \text{cls}\ U_m$; then Z is the kernel of some divisible property which we call MIP. If $A \subset T$ is MIP then it is both big and IP.

4.6. PROPOSITION. *$A \subset T$ is* MIP *if and only if χ_A is proximal in (Ω, T) to a point $\chi_B \in \Omega$ where $B \subset T$ is minimal and $e \in B$ (i.e., $\chi_B(e) = 1$).*

Proof. Let $A \subset T$ be MIP; then $\bar{A} \cap \text{cls } U_m \neq \emptyset$ and hence there exists an idempotent u in some minimal ideal M with $u \in \bar{A}$; i.e., $e \in u*A$. But $u*A = B$ is minimal with $\chi_B(e) = 1$ and χ_B is proximal to χ_A. Conversely, assume B is minimal, contains e and χ_B is proximal to χ_A. Then there exists a minimal ideal M such that $p*A = p*B$ for every $p \in M$. Since B is minimal, there exists an idempotent $u \in M$ with $u*B = B$. Thus $u*A = B$ and $e \in B$ implies $u \in \bar{A}$; the proof is completed.

It is shown in [**6**] that an MIP subset of $\mathbf{Z}$ contains for every (k, r) a (k, r) Deuber system; i.e., a set of integers $p_0, \ldots, p_k$ along with all their integral combinations of the form

$$\begin{array}{lll} p_0 & & \\ p_1 + i_0 p_0 & |i_0| \leqq r & \\ p_2 + i_1 p_1 + i_0 p_0 & |i_0| \leqq r, & |i_1| \leqq r \\ \quad \cdot & & \\ \quad \cdot & & \\ \quad \cdot & & \\ p_k + i_{k-1} p_{k-1} + \ldots + i_0 p_0 & |i_0| \leqq r, \ldots, |i_{k-1}| \leqq r. & \end{array}$$

Clearly in every partition $\mathbf{Z} = \bigcup_{i=1}^{N} S_i$, one of the sets S_i is MIP and thus contains (k, r) Deuber systems for every (k, r). It was shown by Deuber [**2**], that this implies Rado's theorem about regular systems of equations. (Deuber proved a stronger result; namely that the property of containing arbitrary (k, r) Deuber systems is divisible.)

Let $T = \mathbf{Z}$ and let $A \subset \mathbf{Z}$; we say that A has *positive upper density* if there exists a double sequence n_l, m_l of integers with $n_l - m_l \to \infty$ such that

$$\frac{1}{n_l - m_l} \sum_{j=m_l+1}^{n_l} \chi_A(j)$$

converges to a positive limit. Clearly having a positive upper density is a divisible property. Let $\mathscr{M}$ be the set of all $\mathbf{Z}$-invariant probability measures on $\beta\mathbf{Z}$ and put

$$Z = \text{cls} \bigcup \{\text{Supp}(\mu) : \mu \in \mathscr{M}\}.$$

We denote by τ the element of $\beta\mathbf{Z}$ which corresponds to $1 \in \mathbf{Z}$. Thus $n \to \tau^n$ is the canonical embedding of $\mathbf{Z}$ into $\beta\mathbf{Z}$.

4.7. THEOREM. *The kernel of the divisible property of having positive upper density is Z.*

Proof. Let $A \subset \mathbf{Z}$ have positive upper density. Let $\mathscr{A}$ be the algebra of real functions generated by χ_A and the constant function 1. For $f \in \mathscr{A}$ let

$$m(f) = \lim_{l \to \infty} \frac{1}{n_l - m_l} \sum_{j=m+1}^{n_l} f(j)$$

where $\{n_l, m_l\}$ is the double sequence for which $m(\chi_A) > 0$. Clearly

$$|m(f)| \leqq ||f|| = \mathrm{Sup}_n |f(n)| \quad \text{and} \quad m(f_k) = m(f)$$

for every $f \in \mathscr{A}$ and $k \in \mathbf{Z}$ (where $f_k(n) = f(n + k)$). Thus m can be extended to a bounded $\mathbf{Z}$-invariant functional $\tilde{m}$ on $l^\infty(\mathbf{Z})$. This in turn corresponds to a probability measure μ on $\beta\mathbf{Z}$. Now $\tilde{m}(\chi_A) = m(\chi_A) = \int \chi_{\bar{A}} d\mu > 0$ implies that $\bar{A} \cap \mathrm{Supp}(\mu) \neq \emptyset$, hence $\bar{A} \cap Z \neq \emptyset$.

Conversely, assume that $A \subset \mathbf{Z}$ is such that $\bar{A} \cap Z \neq \emptyset$. Since $\bar{A}$ is open there exists an invariant probability measure μ on $\beta\mathbf{Z}$ such that $\bar{A} \cap \mathrm{Supp}(\mu) \neq \emptyset$. This implies that for some ergodic invariant probability measure ν, $\bar{A} \cap \mathrm{Supp}(\nu) = V \neq \emptyset$. Let $p \in V$ be a generic point for ν; then

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} \chi_{\bar{A}}(\tau^j p) = \int \chi_{\bar{A}} d\nu = a > 0.$$

For every $l > 0$ choose $m_l \in \mathbf{Z}$ for which

$$\left| \frac{1}{l} \sum_{j=1}^{l} \chi_{\bar{A}}(\tau^j \tau^{m_l}) - a \right| < \frac{1}{l}$$

and let $n_l = m_l + l$; then

$$\lim \frac{1}{l} \sum_{j=m_l+1}^{n_l} \chi_A(j) = a > 0$$

and A has a positive upper density.

The following theorem is due to Furstenberg [**5**].

THEOREM. *Let $(X, \mathscr{B}, \mu, \tau)$ be a measure preserving system and $B \in \mathscr{B}$ with $\mu(B) > 0$. For any integer $k > 1$ there exists an $n \neq 0$ with*

$$\mu(B \cap \tau^n B \cap \tau^{2n} B \cap \ldots \cap \tau^{(k-1)n} B) > 0.$$

Let $A \subset \mathbf{Z}$ have positive upper density; then there exists an invariant probability measure μ on $\beta\mathbf{Z}$ with $\mu(\bar{A}) > 0$. By Furstenberg's theorem, for every $k > 1$ there exists an n with

$$\mu(B \cap \tau^n B \cap \ldots \cap \tau^{(k-1)n} B) > 0$$

where $B = \bar{A} \cap \mathrm{Supp}(\mu)$. Let

$$p \in B \cap \tau^n B \ldots \cap \tau^{(k-1)n} B;$$

then $\tau^{-nj} p \in B$ for every $j \in \{0, 1, \ldots, (k-1)\}$ and if $m \in \mathbf{Z}$ is such that $\tau^{-nj}\tau^m \in \bar{A}$ then $m - nj \in A$ for $0 \leqq j \leqq k - 1$. Thus from Furstenberg's theorem we deduced Szamerédi's theorem: Let $A \subset \mathbf{Z}$ have positive upper density; then for every positive integer k, A contains an arithmetic progression of length k.

5. $\mathscr{A}$-interpolation sets. Let (X, T) be a flow with a dense orbit; say $x_0 \in X$ with $\overline{Tx_0} = X$. Let $\mathscr{C}(X)$ be the algebra of continuous complex valued functions on X. For each $F \in \mathscr{C}(X)$ there corresponds a function $f \in l^\infty(T)$ according to the formula $f(t) = F(tx_0)$ $(t \in T)$. In this way $\mathscr{C}(X)$ is mapped isometrically onto a norm closed selfadjoint T-invariant subalgebra $\mathscr{A}$ (shortly T-algebra), of $l^\infty(T)$. Conversely with every T-algebra $\mathscr{A} \subset l^\infty(T)$ we associate the space of maximal ideals $X = |\mathscr{A}|$. This can be given a structure of a flow, and the maximal ideal $\{f : f(e) = 0\} = x_0$ has a dense orbit in X. Let $\mathscr{A}$ be a given T-algebra. We say that a subset $A \subset T$ is an *$\mathscr{A}$-interpolation set* if for every function $\Phi \in l^\infty(A)$ there exists an $f \in \mathscr{A}$ with $f(t) = \Phi(t)\ \forall t \in A$. It is easy to check that A is an $\mathscr{A}$-interpolation set if and only if $\bar{A}x_0 \subset |\mathscr{A}|$ is homeomorphic to βA, the Stone–Čech compactification of A. Equivalently, if and only if the restriction to $\bar{A} \subset \beta T$ of the map $p \to px_0$ of βT onto $|\mathscr{A}|$ is a homeomorphism.

Let us consider two cases in which $\mathscr{A}$-interpolation sets were closely investigated. Let $G = \hat{T}$ be the compact dual group of T. Let $\mathscr{B}(T)$ be the algebra of all Fourier transforms of measures on G, and let $\mathscr{F}$ be the uniform closure of $\mathscr{B}(T)$ in $l^\infty(T)$. The $\mathscr{F}$-interpolation sets (which are also $\mathscr{B}(T)$-interpolation sets) are just the Sidon sets in T. By Drury's theorem the Sidon sets form an ideal. Thus "not being an $\mathscr{F}$-interpolation set" is a divisible property.

If we let $\mathscr{B}_d(T)$ be the algebra of Fourier transforms of discrete measures on G then $\mathscr{E} = \operatorname{cls} \mathscr{B}_d(T)$ is the T-algebra of almost periodic functions on T and our notion of an $\mathscr{E}$-interpolation set coincides with Ryll-Nardzewski's definition of an interpolation set (see for example [**8**]). (By a theorem of [**9**], this is also the same as $\mathscr{B}_d(T)$-interpolation set.) Now every lacunary subset of $\mathbf{Z}$ is an $\mathscr{E}$-interpolation set [**13**], while the set

$$\{2^j : j \in N\} \cup \{2^j + j : j \in N\}$$

which is the union of two lacunary subsets clearly fails to be an $\mathscr{E}$-interpolation set. Thus "not being an $\mathscr{E}$-interpolation set" is not a divisible property. We pose the following:

Problem. Give a characterization of those T-algebras $\mathscr{A}$ for which "not being an $\mathscr{A}$-interpolation set" is a divisible property.

In order to formulate our main result in this section we need two additional notions. Consider the space $\mathbf{C}^T$ of all complex valued functions on T, with the product topology. This is a Polish space and $l^\infty(T)$, considered as a subspace of $\mathbf{C}^T$, being σ-compact, is a Borel subset. We say that a T-algebra $\mathscr{A} \subset l^\infty(T)$ is a *Souslin algebra* [**14**] if $\mathscr{A}$ as a subset of $\mathbf{C}^T$ is Souslin.

Let (X, T) be a flow, $x_0 \in X$ with $\overline{Tx_0} = X$. Let $\mathscr{A}$ be the corresponding T-algebra. The map $t \to tx_0$ of T into X induces a topology on T which we denote by $J_{\mathscr{A}} = J_{(X,x_0)}$. Let $\tilde{\mathscr{A}}$ be the algebra of all bounded $J_{\mathscr{A}}$ continuous complex valued functions on T; then $\tilde{\mathscr{A}}$ is a T-algebra which clearly contains $\mathscr{A}$. Let $\tilde{X} = |\tilde{\mathscr{A}}|$ be the maximal ideal space of $\tilde{\mathscr{A}}$ and let $\tilde{x}_0$ be the evaluation at e. We have the following facts [**3**], [**11**].

1. $J_{(\tilde{X},\tilde{x}_0)} = J_{(X,x_0)}$ or equivalently $\tilde{\tilde{\mathscr{A}}} = \tilde{\mathscr{A}}$.
2. $|\tilde{\mathscr{A}}| = \tilde{X}$ is the Stone–Čech compactification of $(T, J_{\mathscr{A}})$.
3. For every $f \in \tilde{\mathscr{A}}$ let $\mathscr{A}_f$ be the T-algebra generated by $\mathscr{A}$ and f; then the homomorphism $|\mathscr{A}_f| \to |\mathscr{A}|$ is almost one to one.
4. If $|\mathscr{A}|$ is a minimal flow so is $|\tilde{\mathscr{A}}|$.
5. Let $\mathscr{E}$ be the T-algebra of almost periodic functions; then $\tilde{\mathscr{E}}$ is the T-algebra of almost automorphic functions.

5.1. THEOREM. *Let $\mathscr{A}$ be a T-algebra, (X, x_0) the corresponding pointed flow.*

(1) *Let $A \subset T$ be $J_{\mathscr{A}}$ closed and discrete; then A is an $\tilde{\mathscr{A}}$-interpolation set.*

(2) (Ryll-Nardzewski) *If $\mathscr{A}$ is Souslin then the union of an $\mathscr{A}$-interpolation set and a finite set is an $\mathscr{A}$-interpolation set.*

(3) *If $\mathscr{A}$ is Souslin then every $\mathscr{A}$-interpolation set is $J_{\mathscr{A}}$ closed and discrete.*

(4) *If $\mathscr{A}$ is Souslin so is $\tilde{\mathscr{A}}$.*

(5) *Let $\mathscr{A}$ be Souslin; then $A \subset T$ is an $\tilde{\mathscr{A}}$-interpolation set if and only if A is $J_{\mathscr{A}}$ closed and discrete. In particular the union of two $\tilde{\mathscr{A}}$-interpolation sets is an $\tilde{\mathscr{A}}$-interpolation set.*

(6) *Let $\mathscr{E}$ and $\tilde{\mathscr{E}}$ be the T-algebras of almost periodic and almost automorphic functions respectively. "Not being an $\tilde{\mathscr{E}}$-interpolation set" is a divisible property which coincides with the divisible property of "not being closed and discrete in the Bohr topology on T $(= J_{\mathscr{E}})$."*

(7) (Veech) *Let $\mathfrak{A}(u)$ be the universal minimal algebra corresponding to the minimal idempotent u; then there exists an $\mathfrak{A}(u)$-interpolation set A such that $A \cup \{e\}$ is not an $\mathfrak{A}(u)$-interpolation set. Thus $\mathfrak{A}(u)$ is not Souslin.*

Proof. (1) Let A be $J_{\mathscr{A}}$ closed and discrete. We construct inductively a sequence of $J_{\mathscr{A}}$ open and pairwise disjoint subsets L_n, $n = 1, 2, 3, \ldots$ such that $\cup L_n = T$ and such that for each n, $|L_n \cap A| \leqq 1$. Let $t_1 < t_2 < t_3 \ldots$ be an ordering of T. If $t_1 \in A$, then since A is $J_{\mathscr{A}}$ closed, $t_1x_0 \notin \overline{Ax_0}$ and we let U_1 be a closed subset of X such that $t_1x_0 \in \operatorname{int} U_1$, and $\overline{Ax_0} \cap U_1 = \emptyset$. If $t_1 \in A$ then by the discreteness of A there exists a closed set U_1 in X with

$$t_1x_0 \in \operatorname{int} U_1 \quad \text{and} \quad Ax_0 \cap U_1 = t_1x_0.$$

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Put $L_1 = \{t \in T : tx_0 \in \text{int } U_1\}$. Suppose $U_1, U_2, \ldots, U_n$ disjoint, closed subsets of X have been constructed with $\bigcup_{i=1}^n U_i \neq X$. We write

$$L_i = \{t \in T : tx_0 \in \text{int } U_i\}.$$

Let t_m be the first element of T which is not in $\bigcup_{i=1}^n L_i$. If $t_m \in A$ then there exists a closed subset U_{n+1} in X with

$$t_m x_0 \in \text{int } U_{n+1} \quad \text{and} \quad (\overline{Ax_0} \cup \bigcup_{i=1}^n U_i) \cap U_{n+1} = \emptyset.$$

(We note that once a countable subset of a completely regular space is zero-dimensional in the induced topology, the sets U_n can be chosen with the further property that $\partial U_n \cap T_{x_0} = \emptyset$ for all n.) If $t_m \in A$ we let U_{n+1} be a closed subset of X with $t_m x_0 \in \text{int } U_{n+1}$ and,

$$U_{n+1} \cap (\bigcup_{i=1}^n U_i) = \emptyset \quad \text{and} \quad U_{n+1} \cap Ax_0 = t_m x_0.$$

In either case let $L_{n+1} = \{t \in T : tx_0 \in \text{int } U_{n+1}\}$. We continue in this way to obtain the desired partition $T = \bigcup_{i=1}^\infty L_i$. All we have to observe now is that an arbitrary function on T which is constant on each L_i is $J_{\mathscr{A}}$ continuous. This completes the proof that A is an $\tilde{\mathscr{A}}$-interpolation set.

(2) This was proved in [**12**] for $\mathscr{A} = \mathscr{E}$, the T-algebra of almost periodic functions. However, the only fact about $\mathscr{E}$ used in the proof was that $\mathscr{E}$ is Souslin. Thus (2) holds for every Souslin T-algebra. (We remark that this is the only place in the proof of the theorem that we use the assumption that $\mathscr{A}$ is Souslin.)

(3) Let A be an $\mathscr{A}$-interpolation set. Since $\overline{Ax_0} \subset |\mathscr{A}|$ is homeomorphic to βA it is clear that A is $J_{\mathscr{A}}$ discrete. Now suppose $t \in T \backslash A$ and t is in the $J_{\mathscr{A}}$ closure of A. This means that $tx_0 \in \overline{Ax_0}$. But by (2), $B = A \cup \{t\}$ is an $\mathscr{A}$-interpolation set, a contradiction. We conclude that A is also $J_{\mathscr{A}}$ closed.

(4) By Theorem 4.5 of [**14**], the set of $J_{\mathscr{A}}$ continuous functions of T into $\mathbf{C}_n = \{z \in \mathbf{C} : |z| \leqq n\}$ is Souslin. Since $\mathbf{C} = \bigcup_n \mathbf{C}_n$ and since the countable union of Souslin sets is Souslin we conclude that $\tilde{\mathscr{A}}$ is Souslin.

(5) We have $J_{\tilde{\mathscr{A}}} = J_{\mathscr{A}}$ and $\tilde{\tilde{\mathscr{A}}} = \tilde{\mathscr{A}}$; hence by (1) every $J_{\mathscr{A}}$ discrete and closed subset is an $\tilde{\mathscr{A}}$-interpolation set. Conversely by (3) and (4) every $\tilde{\mathscr{A}}$-interpolation set is $J_{\mathscr{A}}$ closed and discrete.

(6) This follows since $\mathscr{E}$ is Souslin.

(7) This is Veech's argument to show that $\mathfrak{A}(u)$ is not Souslin [**14**].

Let $\mathscr{P}$ be a collection of subsets of T which has properties (i) and (ii) of a divisible property and also

(iii)′ $T = \bigcup_{i=1}^N A_i$ implies that at least one of the sets A_i is in $\mathscr{P}$.

We then say that $\mathscr{P}$ is *quasi-divisible*. We don't know whether every $\mathscr{A}$-interpolation property is quasidivisible.

5.2. THEOREM. *Let $\mathscr{A}$ be a proper T-algebra and let $\mathscr{P}$ be the property of* *"not being an $\mathscr{A}$-interpolation set." If $\mathscr{P}$ is not quasi-divisible then the natural homomorphism $\beta T \xrightarrow{\pi} |\mathscr{A}|$ is finite to one.*

Proof. Clearly $\mathscr{P}$ satisfies (i) and (ii). Suppose (iii)′ is not satisfied and let $T = \bigcup_{i=1}^{N} A_i$ where the A_i are $\mathscr{A}$-interpolation sets. Then $\pi|\bar{A}_i$ is one to one and our theorem is proved.

5.3. COROLLARY. (1) *Let $\mathscr{A}$ be a minimal T-algebra; then "not being an $\mathscr{A}$-interpolation set" is a quasi-divisible property.*

(2) *Let $\mathscr{W}$ be the T-algebra of weakly almost periodic functions; then "not being a $\mathscr{W}$-interpolation set" is a quasi-divisible property.*

(3) *T is not the union of finitely many Sidon sets.*

Proof. (1) There are 2^c minimal ideals in βT; each of them is mapped onto $|\mathscr{A}|$ by the natural homomorphism $\beta T \xrightarrow{\pi} |\mathscr{A}|$. Thus π is not finite to one and (1) follows.

(2) It follows, from [**16**], that $|\mathscr{W}|$ contains a unique minimal set. As above we conclude that $\beta T \xrightarrow{\pi} |\mathscr{W}|$ is not finite to one.

(3) Since $\mathscr{F} \subset \mathscr{W}$, every Sidon set is also a $\mathscr{W}$-interpolation set.

Problem. Is the union of two $\mathscr{W}$-interpolation sets a $\mathscr{W}$-interpolation set?

5.4. THEOREM. *Let $\tilde{\mathscr{E}}$ and $\mathscr{L}$ be the T-algebras of almost automorphic and point distal functions respectively.* (1) *An $\mathscr{L}$ interpolation set cannot contain an infinite* IP-*system.* (2) *An $\tilde{\mathscr{E}}$-interpolation set cannot contain an infinite* D-*system.*

Proof. Let $U \subset \beta T$ be the set of idempotents different from e. Then under the natural map $\beta T \xrightarrow{\pi} |\mathscr{L}|$, U is mapped onto a single point. Suppose A contains an infinite IP-system: it is then easy to see that A contains two disjoint IP-sets say $A_1, A_2 \subset A$. Let $u_1 \in \bar{A}_1 \cap U$ and $u_2 \in \bar{A}_2 \cap U$, then $\pi(u_1) = \pi(u_2)$ and $\pi|\bar{A}$ is not one to one, and A is not an $\mathscr{L}$-interpolation set. The proof of (2) is similar.

Problem. Is the union of two $\mathscr{L}$-interpolation sets an $\mathscr{L}$-interpolation set? Since $\mathscr{L} = \mathscr{L}$ the answer would be yes if $\mathscr{L}$ is Souslin [**14**].

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