THE SUN'S GRAVITATIONAL QUADRUPOLE MOMENT INFERRED FROM THE FINE STRUCTURE OF THE ACOUSTIC AND GRAVITY NORMAL MODE SPECTRA OF THE SUN

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ABSTRACT. The fine structure of the acoustic and gravity mode multiplets of the Sun have been analyzed to infer the internal rotation of the Sun and upper limits of the internal magnetic field. Observed fine structure for 137 multiplets has been obtained (Hill 1984b, 1985a, 1985b) and the fine structure has been examined for dependence on the angular order, m, of the modes. The inferred angular velocity distribution, together with the estimated upper limits on the internal magnetic fields, yields a gravitational quadrupole moment, J_2, of $\approx 7.7 \times 10^{-6}$. This result is consistent with the result obtained by Hill, Bos and Goode (1982) and has important implications for planetary tests of theories of gravitation.

1. INTRODUCTION

One of the most important tests of theories of gravitation concerns the rates of perihelion precession of planetary orbits, particularly Mercury's orbit. In the parameterized post-Newtonian (PPN) representation of the metric, the predicted advance $\Delta\tilde{\omega}$ of the perihelion per orbital period, correcting for Newtonian perturbations from other planets, is $6\pi GM\lambda_p/[a(1-e^2)c^2]$, where λ_p is generally given as

$$\lambda_p = \frac{1}{3}(2 + 2\gamma - \beta) + \frac{R^2c^2}{2GMa(1 - e^2)} J_2 \quad , \tag{1}$$

where M and R are the mass and radius of the Sun, G is the gravitational constant, a is the semi-major axis, e is the eccentricity of the planetary orbit, c is the speed of light, β and γ are Eddington-Robertson parameters of the PPN formalism, and J_2 is the gravitational quadrupole moment of the Sun. The value of the term $(2 + 2\gamma - \beta)/3$ does vary from one general relativistic theory to another, and its determination is one of the central objectives in planetary studies. In the General Theory of Relativity, for example, $\frac{1}{3}(2 + 2\gamma - \beta) = 1$. A more

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general expression for λ_p includes additional PPN parameters. However, they have been omitted in Equation (1) because they are negligible at the current level of accuracy to which $\Delta\tilde{\omega}$ has been determined for Mercury, the primary observational result of interest here. For a discussion of this point and a review of the development of the PPN formalism, see Will (1980).

The parameter J_2 in Equation (1), normally presumed to arise from centrifugal distortion, is defined such that the solar component of the gravitational potential outside the Sun, written in spherical polar coordinates (r, θ, ϕ) with respect to the Sun's rotational axis, is

$$\phi = \frac{GM}{r} \left[1 - J_2 \left(\frac{R}{r}\right)^2 P_2(\cos\,\theta)\right] \quad . \qquad (2)$$

For Mercury, the predicted perihelion advance per unit time obtained from Equation (1), in seconds of arc per century, is:

$$\dot{\tilde{\omega}} = 42.95 \left[\frac{1}{3} (2 + 2\gamma - \beta) + 0.029\, J_2 \times 10^5\right] \quad . \qquad (3)$$

The measurement of J_2 independent of planetary observations is highly desirable in the determination of $(2 + 2\gamma - \beta)/3$ from $\dot{\tilde{\omega}}$. The discovery of global solar oscillations by Hill and his collaborators and Severny, Kotov and Tsap (1976) has opened an implicit new route for an independent determination of J_2. For a review see Hill (1978) and Severny, Kotov and Tsap (1976). The potential now exists for characterizing the deep interior of the Sun more directly than was previously possible. In particular, rotational splitting of otherwise degenerate nonaxisymmetrical modes provides averages of the interior angular velocity Ω. Different modes weight Ω differently. With a sufficient variety of weights the variation of Ω with position can be determined and an improved estimate of the dynamical contribution to J_2 can be made.

2. MULTIPLET FINE STRUCTURE AND THE INTERNAL ROTATION OF THE SUN

The fine structure of the eigenfrequency spectrum of solar oscillations has been used to infer properties of the internal rotation of the Sun (Gough 1982; Hill, Bos and Goode 1982; Campbell _et al_. 1983; Duvall _et al_. 1984; Hill _et al_. 1984). The fine structure has also been used to infer upper limits on the magnitude of the internal magnetic field (Dziembowski and Goode 1984). The θ dependent part of an Ω that is symmetric about θ = π/2 for an axisymmetric system contributes odd-order terms in m to the fine structure; the lowest order is cubic in m (Hansen, Cox and Van Horn 1977; Hill 1984a) where m is the angular order of the eigenfunction for a given mode. This contribution, as the Sun is currently understood, is the primary source of the cubic and fifth-order terms in m. The internal magnetic fields contribute terms to the fine structure that are even-powers of m for an axisymmetric system (Dziembowski and Goode 1984). Such a description of the Sun is supported by considerable observational evidence about the properties of normal modes (Hill, Bos and Goode 1982; Bos and Hill 1983; Hill 1984b; Hill and Caudell 1985; Hill 1985a, 1985b; Hill, Alexander and Caudell 1985). The combined effect of the Coriolis and centrifugal forces in second- and higher-order perturbation theory is to generate m^3 and m^5 terms, but such terms are negligible compared to the observed magnitudes of the cubic terms (Hill et al. 1985). Hence, study of the θ dependence of Ω is enabled because its manifestation is strongly decoupled from manifestations of other effects in the fine structure.

3. OBSERVATIONS OF MULTIPLET FINE STRUCTURE

Numerous highly discrepant results on rotational splitting in the eigenfrequency spectrum of the Sun have been presented (see Hill 1984b, 1985a for summary). These will not be discussed here. In this analysis, the multiplet fine structure found in the results obtained by Hill (1984b, 1985a, 1985b) and Hill et al. (1985) is examined for internal consistency and implications for J_2. These results include observed values for the fine structure linear and quadratic in m for the low-degree 5 min modes (Hill 1985a), the low-order, low-degree acoustic modes (Hill 1984b), and the low-degree gravity modes (Hill, Bos and Goode 1982, Hill 1985b). In addition, measurement of the fine structure terms cubic- and fifth-order in m have been obtained by Hill et al. (1985) for the low-order, low-degree acoustic modes. The observations for the 5 min modes are the differential radius observations obtained by Bos and Hill (1983), where the fine structure for 54 multiplets was obtained (Hill 1985a). The m = 0 eigenfrequencies obtained by Hill (1985a) are in excellent agreement with those obtained by other observational techniques. The observations for the low-order, low-degree modes are also the differential radius observations obtained by Bos and Hill (1983) where the fine structure for 30 multiplets was obtained (Hill 1984b). These findings have been confirmed by Hill and Caudell (1985) using the 1978 diameter observations. In 1982, the fine structure of two gravity mode multiplets was obtained by Hill, Bos and Goode (1982), again based on the observations of Bos and Hill (1983). By comparing the differential velocity observations of Kotov et al. (1983) with the differential radius observations of Bos and Hill (1983), the fine structure for 31 gravity mode multiplets has been obtained by Hill (1985b). The successful combinations of these two sets of observations is made possible because of the work of Hill, Tash and Padin (1985).

The results of Hill, Bos and Goode (1982), Hill (1984b, 1985a, 1985b) and Hill et al. (1985), described above, combine to form a rather extensive set of analyses. In this series of works, a large number of different modes are examined, and the hypothesis that resolved members of multiplets have been observed and properly classified is put to numerous tests. The results of these analyses are found in 6 works: Hill, Bos and Goode (1982), Hill (1984b, 1985a, 1985b), Hill and Caudell (1985) and Hill, Alexander and Caudell (1985). Evidence of 146 multiplets was obtained in these works, based on 644 resolved modes that were classified. The boundaries in n and ℓ of these 146 multiplets are shown in Figure 1 as the enclosed areas in the eigenfrequency diagram.

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This eigenfrequency diagram presents the m = 0 eigenfrequencies for the standard solar model of Saio (1982). These six works also contain a series of independent tests, performed to determine the probabilities that multiplets have been detected and correctly classified therein. These tests are based on the observed symmetry and horizontal spatial properties of the eigenfunctions. Also in this series of tests, observations from two different years are compared to determine the level of internal consistency of multiplet detection and classification.

4. THEORY

Splitting of adiabatic nonradial modes of stellar oscillation owing to slow differential rotation, based on first-order perturbation theory, can be written as

$$\nu_{n\ell m} - \nu_{n\ell} = -m\{\int K\Omega_0 dr + I_2^{\ell,m} \int K\Omega_2 dr + I_4^{\ell,m} \int K\Omega_4 dr\}(2\pi)^{-1} \quad (4)$$

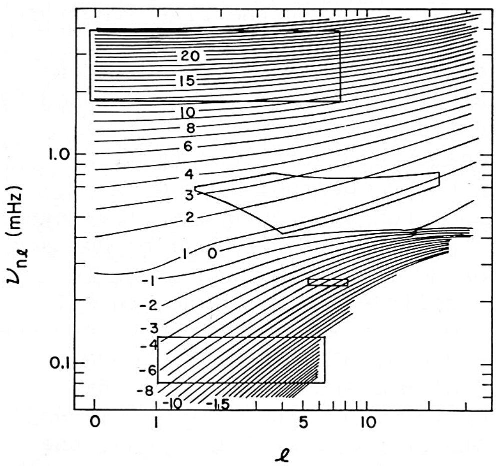

Fig. 1

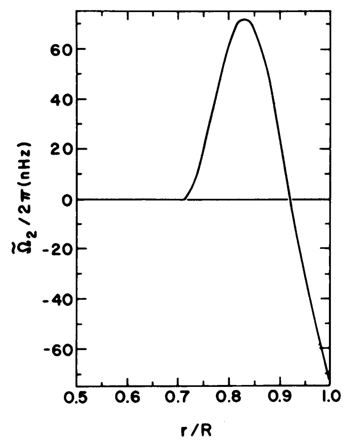

Fig. 2

Fig 1 Theoretical eigenfrequency spectrum for the standard solar model of Saio (1982). The acoustic and gravity modes, respectively, are denoted by positive and negative values, respectively, for the radial order. The enclosed areas contained the m = 0 modes of the 137 multiplets ($\ell \neq 0$) used in this analysis for $\tilde{J}_2$.

Fig 2 The inferred latitudinal-dependent, differential rotation as represented by $\tilde{\Omega}_2$ which is obtained by the inversion of the observed fine structure cubic in m. The analysis for the cubic term in m omitted the fifth order term in m.

The observed fine structure that is cubic in m has been obtained by Hill (1984b) and Hill et al. (1985) for the low-order, low-degree acoustic modes. In this analysis of the data, the fifth-order term was not included. Letting $\tilde{\Omega}_2$ be the inferred value based on the measured cubic term for the example of ℓ = 20, Hill et al. (1985) show that

$$\tilde{\Omega}_2 = \Omega_2 - 0.461\ \Omega_4 \quad . \tag{10}$$

The observed fine structure cubic in m where the fifth-order term has been omitted in the data analysis has been inverted by Hill et al. (1985). Their findings, shown in Figure 2, have the magnitude of $\tilde{\Omega}_2$ decreasing below the surface, passing through zero at ≈ .92 R and has an average value of ~ 10% of $\tilde{\Omega}_0$ for the lower two thirds of the convection zone. This implies that the lower two thirds of the convection zone is rotating faster than the corresponding equatorial region of the convection zone.

5.2. Inferred Radial Dependence of Ω_0

The observed fine structure that is first order in m has been obtained for the five min modes and gravity modes, making only quadratic fits to the observed multiplet eigenfrequency spectrum (Hill 1985a, 1985b). For the low-order, low-degree acoustic modes, a cubic fit to the observed fine structure was made by Hill (1984b) for the n= 1, 17 $\leq \ell \leq$ 22 multiplets and a quadratic fit to the remainder of the 30 multiplets classified. The extension of the cubic analysis for all 30 of the low-order, low-degree multiplets has been made by Hill et al. (1985). It has been observed by Hill et al. (1985) that the procedure used in the data analysis must be considered when interpreting the results of inverting the observed fine structure. Letting $\tilde{\Omega}_0$ be the inferred value based on the measured linear splitting where the cubic and fifth-order terms have been omitted in the data analysis, they find, for the example of ℓ = 20, that

$$\tilde{\Omega}_0 = \Omega_0 - 0.179\ \Omega_2 - 0.020\ \Omega_4 \quad . \tag{11}$$

When the linear and cubic terms have been included, but the fifth-order term omitted, they find, for the example of ℓ = 20, that

$$\tilde{\Omega}_0 = \Omega_0 + 0.250\ \Omega_2 - 0.218\ \Omega_4 \quad . \tag{12}$$

The inversion presented here is based on the results where the cubic term has been included for the low-order, low-degree acoustic modes and the cubic term assumed negligible for the low-degree, five min modes and the gravity modes. The observations used are for the 30 low-order, low-degree acoustic modes (Hill 1984b), low-degree gravity modes (Hill, Bos, and Goode, 1982) and a rotational splitting for the n = 17, ℓ = 1 and 2 five min mode multiplets (Hill 1985a). The inversion technique is the same as that used by Hill et al. (1985), and the results for $\tilde{\Omega}_0 - 0.25\ \tilde{\Omega}_2$, which should be a good representation of with

$$K = \rho r^2[(\delta r)^2 - 2\delta r \delta r_h + \ell(\ell+1)(\delta r_h)^2]/\mathcal{I} \quad , \tag{5}$$

$$\mathcal{I} = \int \rho(r) r^2 [(\delta r)^2 + \ell(\ell+1)(\delta r_h)^2] dr, \tag{6}$$

where those terms that are important at the current level of observational accuracy have been retained, the displacement eigenfunction is $\vec{\xi} = \{\delta r, \delta r_h \frac{\partial}{h\partial\theta}, \frac{\delta r_h}{\sin\theta}\frac{\partial}{\partial\phi}\}Y_\ell^m$, ρ is the equilibrium value of the density at radius r, and Ω is written in a series of the orthogonal Legendre polynomials, $P_n(\cos\theta)$, as

$$\Omega(r,\theta) = \sum_{n=0}^{\infty} \Omega_n(r) P_n(\cos\theta) \quad . \tag{7}$$

Retaining the leading terms of the coefficients $I_n^{\ell,m}$ in Equation (4), these coefficients can be written as

$$I_2^{\ell,m} = -\frac{1}{4}[3m^2 - \ell(\ell+1)] \frac{(\ell+\frac{1}{2})}{(\ell-\frac{1}{2})_3} \quad , \tag{8}$$

$$I_4^{\ell,m} = \{\frac{7!!}{2^6}m^4 \frac{(\ell+\frac{1}{2})}{(\ell-\frac{3}{2})_5} + m^2[-\frac{45}{2^5}\frac{(\ell+\frac{1}{2})}{(\ell-\frac{1}{2})_3}] + \frac{9}{2^6}\} \quad , \tag{9}$$

where $(2n+1)!! = (2n+1)(2n-1)...3\cdot 1$ and $(a)_n = a(a+1)...(a+n-1)$. The splitting kernel, K, is the same as that used by Gough (1982), Hill, Bos, and Goode (1982), Campbell et al. (1983) and Hill et al. (1984) in the study of Ω. Examples of K can be found in Gough (1982) and Hill, Bos, and Goode (1982). With Equations (4)-(9) it is possible to show that, in first order perturbation theory, splitting terms due to the latitude-dependence of Ω are decoupled from the centrifugal and higher-order Coriolis terms. These latter effects give rise to splitting terms cubic and to higher, odd-order terms in m. Based on the magnitude of the observed first- and second-order terms in m, the magnitude of the third-order term, due to the combined Coriolis and centrifugal forces, is expected to be $\lesssim$ 1% of the observed cubic term reported by Hill (1984a, 1984b) for the n = 1, $\ell \approx 20$ acoustic modes. A similar result is expected for the fifth-order term. As noted in the Introduction, the effect of the internal magnetic fields is to contribute terms to the fine structure that are even-powers of m. Therefore, the cubic and fifth-order terms of m in Equation (4) are expected to be the primary contributors to such terms in the observed multiplet fine structure.

5. INVERSION OF FIRST-, THIRD-, AND FIFTH-ORDER TERMS IN m

5.1. Inferred Radial Dependence of $\tilde{\Omega}_2$

Ω_0, are shown in Figure 3. The basic features of this analysis are in agreement with those obtained by Hill et al. (1984) and with the fine structures of the 31 gravity mode multiplets classified by Hill (1985b).

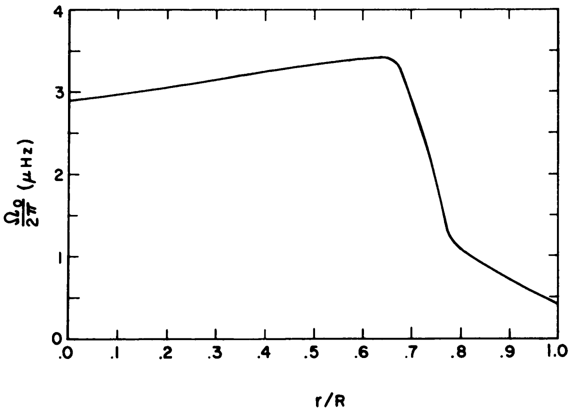

Fig 3 The inferred latitudinal-independent, differential rotation is represented by Ω_0 which is obtained by the inversion of the observed fine structure linear in m. The analysis for the linear term in m allowed for an Ω_2 and for a cubic term in splitting for the low-order, low-degree acoustic modes.

5.3. Inferred Radial Dependence of $\tilde{\Omega}_4$

The fine structure fifth-order in m has been obtained and inverted by Hill et al. (1985). Although the observational results are not as extensive as those for the cubic terms in m, they find that $\tilde{\Omega}_4$ appears to have a radial dependence quite similar to that found for $\tilde{\Omega}_2$.

6. INFERRED DYNAMICAL CONTRIBUTION TO J_2

The inferred $\tilde{\Omega}_0$, $\tilde{\Omega}_2$ and $\tilde{\Omega}_4$ have been combined to obtain Ω_0 and Ω_2. The contributions of these two terms to J_2 have been calculated, and it is found that the contribution is

$$J_2 \approx 7.7 \times 10^{-6} \quad . \tag{13}$$

This result is in good agreement with the value of J_2 obtained by Hill, Bos, and Goode (1982). However, the observation base for the two results is much different. In 1982, the fine structure for 7 multiplets was available, whereas, the results given in Equation (12) are based on the observed fine structure of 34 multiplets and are consistent with the observed fine structure of 137 multiplets.

It is important to note that the observed fine structure for the 137 multiplets is internally consistent. This is measured by the ability

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to account for the observed fine structure by the $\tilde{\Omega}_2$ and $\tilde{\Omega}_0 - 0.25\,\tilde{\Omega}_2$ given in Figures 2 and 3. The results for recent inferred values of J_2 are given in Table 1.

7. CONCLUSIONS

An estimate of the dynamical contribution to the gravitational quadrupole moment J_2 of the Sun has been obtained based on the observed multiplet fine structure of 34 multiplets which is consistent with the rotational splitting observed for 137 multiplets. This set of multiplets includes 5 min modes, low-order, low-degree acoustic modes and gravity modes. Their observed fine structure are found to be internally consistent. Dziembowski and Goode (1984) have concluded that the effects of the magnetic fields are not important at the current level of observation. It is therefore concluded that there remains a discrepancy similar to that noted by Hill, Bos, and Goode (1982) between the work on J_2, the planetary observation on $\dot{\omega}$ and the General Theory of Relativity.

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TABLE 1
Gravitational Quadrupole Moment of the Sun

Rotational Splitting Fine Structure[a]	$J_2 \times 10^6$
Duvall *et al.* (1984)	0.17 ± 0.04
Hill, Bos and Goode (1982)	5.5 ± 1.3
Hill *et al.* (1984)[b]	4.5
current work	7.7 ± 1.8
Visual Solar Oblateness	
Hill and Stebbins, (1975)	1.0 ± 4.3
Dicke, Kuhn, and Libbrecht, (1985)[c]	4.5 ± 2.7
	7.5 ± 0.9

[a] The value obtained by Gough (1982) is not included because it was based on a preliminary set of multiplet classifications which was in error (Hill 1984b).

[b] Based on rotational curve of Hill et al. (1984).

[c] Two values are given based on whether or not a certain type of systematic error is taken into account.

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DISCUSSION

Grishchuk : what is the contribution of the Sun's quadrupole to the mo-motion of the perihelion.

Hill : about 1% of the relativistic value.

Grishchuk : then how does this match with the estimates of $2/3(2+2\gamma-\beta)=$ $=1\pm 0.002$?

Will : I will answer this question in my report.

Eichhorn : in 1977 and 1982, Dicke published estimates for J_2 that are now considered as wrong. How sure are your own estimates ?

Hill : the results of 1977 and 1982 are based on simplified models, while our estimates are model independent and are based on direct numerical integration.

Reasenberg : however, do your error estimates include the errors due to possible model dependence ?

Hill : yes.

Chechelnitsky : your graphs show smooth dependence of the angular velocity on coordinates. But the observations of Kotov in Crimea observatory have shown that this dependence is small.

Hill : these observations concern the solar surface. The contribution of the surface of the Sun to J_2 is smaller than 10^{-7} and is therefore negligible. Our analysis shows that there is no correlation between the surface and the internal structure.

Chechelnitsky : how does your model explain the 28, 31, etc... periods in Sun radiation.

Hill : we did not analyse them. Our objective was only to estimate $\bar{\omega}$.

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