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Trivial Units in Group Rings

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Abstract. Let G be an arbitrary group and let U be a subgroup of the normalized units in $\mathbb{Z}G$. We show that if U contains G as a subgroup of finite index, then $U = G$. This result can be used to give an alternative proof of a recent result of Marciniak and Sehgal on units in the integral group ring of a crystallographic group.

In the last section of their paper [1], Marciniak and Sehgal discuss units in the integral group ring of a crystallographic group. Their new insight is that the given action of such a group Γ as isometries of $\mathbb{R}^n$ extends to an action of the group of normalized units $\mathcal{U}_1(\mathbb{Z}\Gamma)$ as affine transformations of $\mathbb{R}^n$. (Normalized units are units whose value under the augmentation map is 1.) They prove that if Γ is torsion free and the group of normalized units acts freely on the ambient affine space, then all such units are trivial, *i.e.*, $\mathcal{U}_1(\mathbb{Z}\Gamma) = \Gamma$. The first step is to observe that if P is an integral point in a compact fundamental domain for Γ and $u \in \mathcal{U}_1(\mathbb{Z}\Gamma)$, then there is some $g \in \Gamma$ such that $gu(P)$ is one of the finitely many integral points in the domain. Since the action is fixed point free, the index $[\mathcal{U}_1(\mathbb{Z}\Gamma) : \Gamma]$ is finite. Then the authors use a nontrivial geometric argument to deduce that $\mathcal{U}_1(\mathbb{Z}\Gamma)$ coincides with Γ.

In this short note, we give an algebraic proof of a stronger general result about unit groups in integral group rings which have finite index over the base group.

Theorem *Let G be an arbitrary group and let U be a subgroup of $\mathcal{U}_1(\mathbb{Z}G)$. If U contains G and $[U : G]$ is finite, then $U = G$.*

Let $\Delta(G)$ denote the finite conjugate subgroup of the group G. We first reduce to the case that $G = \Delta(G)$.

Lemma *Let G be a group and let H be a subgroup of G with $[G : H]$ finite. If $\alpha \in \mathcal{U}_1(\mathbb{Z}G)$ has 1 in its support and normalizes H, then α lies in $\mathbb{Z}\Delta(G)$.*

Proof Let $h \in H$. Then $\alpha h \alpha^{-1} = k$ for some $k \in H$. Since 1 is in the support of $h^{-1}\alpha h$ and $h^{-1}\alpha h = h^{-1}k\alpha$ it is also in the support of $h^{-1}k\alpha$. Therefore there are only finitely many possibilities for $h^{-1}k$ and, hence, only finitely many possibilities for $h^{-1}\alpha h$. It follows that α is centralized by a subgroup of finite index in G. We conclude that each element of the support of α lies in $\Delta(G)$. ∎

Now suppose U and G are groups as described in the theorem. Certainly G contains a subgroup H which is normal and of finite index in U. According to the lemma,

$$U = G \cdot \big(U \cap \mathbb{Z}\Delta(G)\big).$$

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Notice that $U \cap \mathbb{Z}\Delta(G) = U \cap \mathcal{U}_1\big(\mathbb{Z}\Delta(G)\big)$ and

$$\big[U \cap \mathcal{U}_1\big(\mathbb{Z}\Delta(G)\big) : G \cap \mathcal{U}_1\big(\mathbb{Z}\Delta(G)\big)\big] \leq [U : G].$$

But $G \cap \mathcal{U}_1\big(\mathbb{Z}\Delta(G)\big) = \Delta(G)$. Hence

$$\big[U \cap \mathcal{U}_1\big(\mathbb{Z}\Delta(G)\big) : \Delta(G)\big] < \infty.$$

In this manner, we see that it suffices to prove the theorem with $\Delta(G)$ replacing G.

Our main argument will require a basic result about units, due to Higman and Berman, and a basic tool, the trace associated with Hattori, Stallings, and Bass.

Theorem **[2, Corollary II.1.2], [3, Proposition 1.4]** *If $\alpha \in \mathbb{Z}G$ has 1 in its support and $\alpha^n = 1$ for some positive integer n, then $\alpha = \pm 1$.*

We review some facts about the trace in a form suitable for our purposes. Let G be a group and let $\mathbb{F}$ be a field of characteristic $p > 0$. If $\mathcal{C}$ denotes a conjugacy class of G we define the trace $\operatorname{tr}_{\mathcal{C}} \colon \mathbb{F}G \to \mathbb{F}$ by

$$\operatorname{tr}_{\mathcal{C}}\Big(\sum_{g \in G} \alpha_g g\Big) = \sum_{g \in \mathcal{C}} \alpha_g.$$

If $\mathcal{D}$ is another conjugacy class and it contains an element whose pth power is in $\mathcal{C}$, we will write $\mathcal{D}^p = \mathcal{C}$. The simple but powerful equation which leads to so many group ring consequences is

$$\operatorname{tr}_{\mathcal{C}}(\alpha^p) = \sum_{\mathcal{D}^p = \mathcal{C}} \big(\operatorname{tr}_{\mathcal{D}}(\alpha)\big)^p$$

for all $\alpha \in \mathbb{F}G$. (A proof and applications can be found in [3, Section 1.7].)

Assume that G has a normal subgroup N such that G/N is torsion free. Then any G-conjugacy class which meets N is entirely contained in N; moreover, $\mathcal{C} \subseteq N$ and $\mathcal{D}^p = \mathcal{C}$ implies $\mathcal{D} \subseteq N$. We conclude that

$$\sum_{\mathcal{C} \subseteq N} \operatorname{tr}_{\mathcal{C}}(\alpha^p) = \sum_{\mathcal{D} \subseteq N} \big(\operatorname{tr}_{\mathcal{D}}(\alpha)\big)^p.$$

The sum has a simpler interpretation of use to us. Let $\varepsilon_N \colon \mathbb{Z}G \to \mathbb{Z}$ denote the truncated augmentation map,

$$\varepsilon_N\Big(\sum_{g \in G} \alpha_g g\Big) = \sum_{g \in N} \alpha_g.$$

Then, under the hypotheses on N, we have the p-power formula

$$\varepsilon_N(\alpha^p) \equiv \big(\varepsilon_N(\alpha)\big)^p \mod p$$

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for all $\alpha \in \mathbb{Z}G$.

Now let us prove the main theorem. We are assuming that $G = \Delta(G)$. There is no loss of generality in taking G finitely generated. Then the elements of finite order in G constitute a finite characteristic subgroup $\Delta^+(G)$ and $G/\Delta^+(G)$ is torsion free abelian. Let T be a transversal for $\Delta^+(G)$ in G which contains 1.

Recall that $U \leqslant \mathcal{U}_1(\mathbb{Z}G)$, that U contains G, and that $[U : G] < \infty$. Since each member in the support of an element of $\mathbb{Z}G$ is centralized by a subgroup of finite index in G, we see that U is a finitely generated finite conjugate group, as well. Certainly $\Delta^+(U) \supseteq \Delta^+(G)$. There are two cases to consider.

First suppose that $\Delta^+(U) = \Delta^+(G)$. Since $\Delta^+(G)$ is normal in U and $U/\Delta^+(G)$ is abelian, G is normal in U. If $U = G$ we are done. Suppose, to the contrary, that there is a prime p and $\alpha \in U \setminus G$ such that $\alpha^p \in G$. By assumption, $\varepsilon_G(\alpha) = 1$. Thus we can find some $t \in T$ so that $\varepsilon_{\Delta^+(G)}(\alpha t^{-1}) \not\equiv 0 \mod p$. Notice that $(\alpha t^{-1})^p \in G$ by the normality of G in U. An application of the p-power formula yields $\varepsilon_{\Delta^+(G)}\big((\alpha t^{-1})^p\big) \not\equiv 0 \mod p$. We conclude that $(\alpha t^{-1})^p \in \Delta^+(G)$. But this means αt^{-1} has finite order, whence we reach the contradiction $\alpha \in \Delta^+(G)T \subseteq G$.

Finally, suppose that $\Delta^+(U) > \Delta^+(G)$. By choosing an element in $\Delta^+(U)$ which has a minimal power larger than 1 in $\Delta^+(G)$ we see that there exists a prime p and $\alpha \in \Delta^+(U) \setminus \Delta^+(G)$ such that $\alpha^p \in \Delta^+(G)$. This time we know that

$$\varepsilon_{\Delta^+(G)}(\alpha^p) = 1 \not\equiv 0 \mod p,$$

so a second application of the p-power formula tells us that $\varepsilon_{\Delta^+(G)}(\alpha)$ is nonzero. Thus there is some $h \in \Delta^+(G)$ which lies in the support of α; equivalently, 1 is in the support of $h^{-1}\alpha$. But $h^{-1}\alpha$ has finite order as it is a member of $\Delta^+(U)$. By the theorem on units of Higman and Berman, $h^{-1}\alpha = \varepsilon_G(h^{-1}\alpha) \cdot 1$. Since $h^{-1}\alpha$ is a normalized unit, we obtain the contradiction $\alpha = h \in \Delta^+(G)$.

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