Can. J. Math., Vol. XXXIII, No. 1, 1981, pp. 59–67

ISOMETRIES OF $s_p(\alpha)$

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Introduction. Let $1 < p < \infty$, $p \neq 2$, and $\alpha > 0$. In what follows $s_p(\alpha)$ will denote the space of all real or complex sequences for which

(1.1) $\Sigma|x_k|^p + \alpha|x_{k+1} - x_k|^p$ is finite.

In this paper we show that the spaces $s_p(\alpha)$ are Banach spaces under the natural norm and in fact share many properties that the usual l_p spaces have. Our main results give characterizations of the surjective isometries of $s_p(\alpha)$. These turn out to be quite different than the results for l_p. For example, we show that for $\alpha \neq 1$, an operator T is a surjective isometry if and only if T is a modulus one multiple of the identity. The methods used are valid for both real and complex scalars. They involve the use of a disjoint support condition together with a property of semi inner products. In the complex case the information on isometries allows us to give complete descriptions of the Hermitian operators as well as the adjoint abelian operators. Surprisingly, we show that for $\alpha \neq 1$, these classes of operators coincide. This last result sheds some light on a question asked by the authors in a previous paper [**2**].

2. Basic properties of $s_p(\alpha)$. It is obvious from the definition that the space $s_p(\alpha)$ consists of exactly the same sequences as l_p. An application of Minkowski's inequality for sums shows that the function given by

$$(2.1) \qquad ||x|| = \left(\sum_k |x_k|^p + \alpha|x_{k+1} - x_k|^p\right)^{1/p}$$

satisfies the usual triangle inequality and hence is a norm. Furthermore, since

$$(2.2) \qquad ||x||_p \leqq ||x|| \leqq (1 + 2^p\alpha)^{1/p}||x||_p,$$

where $||x||_p$ denotes the usual l_p norm, we see that the spaces $s_p(\alpha)$ are isomorphic to l_p. We also note that every subspace M of $s_p(\alpha)$ with the property that $(x_k) \in M$ implies (x_k) has at least one zero between every pair of nonzero entries is isometric to l_p.

It is a straightforward computation to verify that Clarkson's inequalities [**3**] hold for the norm on $s_p(\alpha)$. Furthermore, using the fact that the usual norm on l_p is differentiable [**4**] we can show that the norm on $s_p(\alpha)$ is differentiable. Hence we record the following result without proof.

Received April 30, 1979 and in revised form September 17, 1979.

2.3. PROPOSITION. *The Banach spaces* $s_p(\alpha)$ *for* $1 < p < \infty$ *and* $\alpha > 0$ *are uniformly convex and smooth.*

Remark. The fact that the unit ball of $s_p(\alpha)$ is smooth is important for our methods in Section 3.

2.4. *Definition.* Let X be a normed linear space with real or complex scalars and let $\| \;\|$ denote the norm. A *semi inner product on* X *compatible with the norm* is a function $[\;,\;]$ on $X \times X$ to the field such that $[x, x] = \|x\|^2$, $[\;, y]$ is a linear functional, and $|[x, y]| \leqq \|x\| \, \|y\|$ for each $x, y \in X$.

Remark. This notion is due to G. Lumer [**7**] and is very useful in the study of isometries.

It is known that if X is a Banach space with a differentiable norm then there is exactly one semi inner product on X compatible with the norm [**7**].

2.5. PROPOSITION. *If* $\alpha > 0$, $1 < p < \infty$, *the unique semi inner product on* $s_p(\alpha)$ *compatible with the norm is given by*

$$(2.6) \qquad [x, y] = \sum_k x_k \bar{y}_k \left(\frac{|y_k|}{\|y\|}\right)^{p-2} + \alpha \sum_k (Dx)_k (\overline{Dy})_k \left(\frac{|(Dy)_k|}{\|y\|}\right)^{p-2}$$

where $(Dx)_k = x_{k+1} - x_k$.

Proof. In what follows, $\|x\|_p$ denotes the usual l_p norm and $1/p + 1/q = 1$. From (2.6) it follows from Hölder's inequality for sums that

$$(2.7) \qquad \|y\|^{p-2}|[x, y]| \leqq \|x\|_p \|y\|_p^{p-1} + \|Dx\|_p \|Dy\|_p^{p-1}.$$

Apply Hölder's inequality again to the right side of (2.7) to get:

$$(2.8) \qquad \|y\|^{p-2}|[x, y]| \leqq (\|x\|_p^p + \|Dx\|_p^p)^{1/p}(\|y\|_p^{(p-1)q} + \|Dy\|_p^{(p-1)q})^{1/q}.$$

Hence,

$$(2.9) \qquad \|y\|^{p-2}|[x, y]| \leqq \|x\| \, \|y\|^{p/q},$$

and from this we see that

$$(2.10) \qquad |[x, y]| \leqq \|x\| \, \|y\|.$$

The other properties required of a semi inner product are obvious from the definition. The uniqueness follows from Proposition 2.3 and the remark prior to Proposition 2.5. With that the proof is finished.

The importance of the semi inner product in the study of isometries is due to the following theorem of D. Koehler and P. Rosenthal [**5**].

2.11. THEOREM. (Koehler–Rosenthal) *Let* X *be a normed linear space (real or complex) and let* U *be an operator mapping* X *into itself. Then* U

https://doi.org/10.4153/CJM-1981-007-4 Published online by Cambridge University Press

is an isometry if and only if there is a semi inner product compatible with the norm such that

(2.12) $[Ux, Uy] = [x, y]$ *for every* $x, y \in X$.

2.13. COROLLARY. *If T is an isometry of $s_p(\alpha)$ then $[Tx, Ty] = [x, y]$ for every x and y in $s_p(\alpha)$, where $[\ ,\ ]$ is the semi inner product given in* (2.6).

3. The structure of isometries on $s_p(\alpha)$**.** In this section we give a complete matricial description of the surjective isometries on $s_p(\alpha)$. The following lemma which is due to Lamperti [**6**] is needed in the sequel and we state it for completeness.

3.1. LEMMA (Lamperti). *Let $\Phi(t)$ be a continuous, strictly increasing function defined for $t \geqq 0$, with $\Phi(0) = 0$, and let z and w be complex numbers. If $\Phi(\sqrt{t})$ is a convex function of t, then*

$$\Phi(|z + w|) + \Phi(|z - w|) \geqq 2\Phi(|z|) + 2\Phi(|w|), \tag{3.2}$$

while if $\Phi(\sqrt{t})$ is concave the reverse inequality is true. Providing the convexity or concavity is strict, equality holds if and only if $zw = 0$.

3.3. LEMMA. *Let U be an isometry of $s_p(\alpha)$ and let (x_k) and (y_k) be sequences in $s_p(\alpha)$ such that*

$$\begin{cases} x_k y_k = 0 \\ x_k y_{k+1} = 0 \\ x_{k+1} y_k = 0. \end{cases} \tag{3.4}$$

If $x_k' = (Ux)_k$ and $y_k' = (Uy)_k$ then

$$\begin{aligned} x_k' y_k' &= 0 \\ x_k' y_{k+1}' &= 0 \\ x_{k+1}' y_k' &= 0. \end{aligned} \tag{3.5}$$

Proof. Let (x_k) and (y_k) be sequences in $s_p(\alpha)$ which satisfy (3.4). Then

$$\|x + y\|^p + \|x - y\|^p = 2\|x\|^p + 2\|y\|^p. \tag{3.6}$$

If U is the isometry then

$$\|U(x) + U(y)\|^p + \|U(x) - U(y)\|^p = 2\|U(x)\|^p + 2\|U(y)\|^p. \tag{3.7}$$

Let $x_k' = (Ux)_k$ and $y_k' = (Uy)_k$ and rewrite (3.7) in terms of the sums. This leads to

$$\begin{aligned} \sum_k (|x_k' + y_k'|^p + |x_k' - y_k'|^p - 2|x_k'|^p - 2|y_k'|^p) \\ + \alpha \sum_k{}' (|Dx')_k + (Dy')_k|^p + |(Dx')_k - (Dy')_k|^p \\ - 2|(Dx')_k|^p - 2|(Dy')_k|^p) = 0, \end{aligned} \tag{3.8}$$

where $(Dx')_k = x_{k+1}' - x_k'$.

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If we now apply Lamperti's Lemma to (3.8) we obtain:

$$\text{3.9)}\quad \begin{cases} x_k' y_k' = 0 \\ (Dx')_k (Dy')_k = 0, \end{cases}$$

and the desired result follows immediately.

Since the usual unit vector basis for l_p is a basis for $s_p(\alpha)$, every isometry of $s_p(\alpha)$ has a matrix representation. We now apply the preceding lemma to obtain the general form of a matrix which represents an isometry of $s_p(\alpha)$. The matrix description below is given relative to the basis $e_1 = (1, 0, 0 \ldots)$, and $e_n = (0, 0 \ldots 1, 0, 0 \ldots)$ for $n > 1$, which we will refer to as the "standard basis."

3.19. LEMMA. *Let U be a surjective isometry of $s_p(\alpha)$. Then U has the following matrix description:*

$$\text{(3.11)}\quad U = \left| \begin{matrix} u_{11} & u_{12} & 0 & 0 & 0 & \cdot \\ 0 & u_{22} & 0 & 0 & 0 & \cdot \\ 0 & 0 & u_{33} & 0 & 0 & \cdot \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \\ & & & & & u_{nn} \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \end{matrix} \right|$$

Proof. Let the entries in the matrix of U be denoted by u_{kj}. Since the basis vectors e_n and e_{n+j} satisfy the hypothesis of Lemma (3.3) when $j \geqq 2$, we conclude that

$$\text{(3.12)}\quad \begin{aligned} u_{kn} u_{k,n+j} &= 0 \\ u_{kn} u_{k+1,n+j} &= 0 \qquad \text{for } k \geqq 1, 2, 3 \ldots, n = 1, 2 \ldots, \\ u_{k+1,n} u_{k,n+j} &= 0 \qquad \text{and } j = 2, 3 \ldots \end{aligned}$$

We make some observations concerning (3.12). From the first of the relations it follows that there are at most two non-zero elements per row. Also, once there is a non-zero element in the kn position, in the $k + 1$ row all the elements are zero from the $n + 2$ column on.

The general form (3.11) is obtained as follows. Since U is injective every column has at least one non-zero entry. Hence we know there is a least positive integer n for which $u_{n1} \neq 0$. We claim $u_{j1} = 0$ for all $j \neq n$. For suppose that there exists a positive integer m such that $u_{m1} \neq 0$. Then $u_{kj} = 0$ for $j \geqq 3$ and $k = n, m$ and $m + 1$. Hence the n, m and $m + 1$ rows are linearly dependent and this is impossible since U is surjective.

We now argue that $n = 1$. Otherwise we can conclude from (3.12) that $u_{n-1,2}$ and $u_{n+1,2}$ are the only possible non-zero entries in the $n - 1$ and $n + 1$ rows respectively which again contradicts the fact that U is surjective.

We have established that the matrix for U has the following form in case $n = 1$:

$$(3.13)\qquad U = \begin{bmatrix} u_{11} & u_{12} & 0 & \cdot & 0 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & & \\ 0 & u_{22} & 0 & & 0 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot & & \\ 0 & 0 & u_{33} & & 0 & \cdot\ \cdot & 0 & 0 & \cdot\ \cdot\ \cdot & & \\ 0 & 0 & 0 & & 0 & & & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & & & \\ \cdot & \cdot & \cdot & & \cdot & & & & & & \\ 0 & 0 & 0 & & 0 & \cdot\ \cdot\ \cdot & u_{nn} & 0 & \cdot\ \cdot\ \cdot & & \\ \cdot & \cdot & \cdot & & \cdot & \cdot\ \cdot\ \cdot & & & & & \\ \cdot & \cdot & \cdot & & \cdot & & 0 & u_{n+1,n+1} & 0 & 0 & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & & \cdot & & 0 & u_{n+2,n+1} & u_{n+2,n+2} & & \cdot \\ & & & & & & \cdot & \cdot & \cdot & & \cdot \\ & & & & & & \cdot & \cdot & \cdot & & \cdot \end{bmatrix}$$

Assume the form is correct for a given $n = k$, and suppose there exists $m > k + 1$ with $u_{m,k+1} \neq 0$. Then the $k + 1$, m and $m + 1$ rows would be linearly dependent, since their only non zero elements occur necessarily in the $k + 1$, $k + 2$ columns. This contradiction shows that $u_{k+1,k+1}$ is the only non-zero entry in the $k + 1$ column and by an argument exactly as was given for column one, we can show that $u_{k+2,k+2}$ is the first non-zero entry in the $k + 2$ column. It follows from (3.12) that $u_{k+2,k+2}$ is also the only nonzero entry in the $k + 2$ row, hence the form (3.13) must hold for $n = k + 1$. The fact that U has the general form, (3.11) follows by induction.

Now we let I denote the identity operator in $s_p(\alpha)$ and V be given by:

$$(V)_{ij} = \begin{cases} -1 & i = 1, j = 1 \\ 1 & i = 1, j = 2 \\ 1 & i = j = k \quad \text{for } k \geqq 2 \\ 0 & \text{otherwise.} \end{cases}$$

3.14. THEOREM. *Let U be an isometry from $s_p(\alpha)$ onto $s_p(\alpha)$. Then there exists a modulus one scalar λ such that*

$$(3.15)\qquad U = \lambda I \textit{ for } \alpha \neq 1,$$

and

$$(3.16)\qquad U = \lambda I \textit{ or } U = \lambda V \textit{ for } \alpha = 1.$$

The converse is also true.

Proof. From Lemma (3.10), the matrix representation relative to the standard basis is given by

$$(3.17) \quad U = \begin{bmatrix} u_{11} & u_{12} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & u_{22} & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & u_{33} & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & 0 & & & & & & \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & 0 & & & & & & u_{nn} \\ \cdot & \cdot & & & & & & & \end{bmatrix}$$

From (3.17) it follows that for $n \neq 2$, u_{nn} is an eigenvalue of U and so

$$(3.18) \quad |u_{nn}| = 1 \text{ for } n = 1, 3, 4, 5, \ldots.$$

If we let λ be an arbitrary scalar and set

$$(3.19) \quad x = (0, \ldots, 0, \lambda, \lambda, 0, \cdot) = \lambda e_n + \lambda e_{n+1}$$

then using that fact that $\|Ux\| = \|x\|$ we obtain

$$(3.20) \quad u_{n+1,n+1} = u_{nn} \quad \text{for } n \geqq 3.$$

Let λ, μ be scalars and set

$$(3.21) \quad x = (\lambda, 0, 0, 0, \cdot) \quad \text{and}$$

$$(3.22) \quad y = (0, \mu, 0, 0 \cdot \cdot).$$

By Corollary (2.13) U must preserve the semi inner product and so

$$(3.23) \quad [y, x] = [Uy, Ux].$$

After some computation, (3.23) yields

$$(3.24) \quad u_{12} = \frac{\alpha(u_{22} - u_{11})}{1 + \alpha}.$$

This leads immediately to the observation that the vector

$$(3.25) \quad z = \left(\frac{\alpha}{1 + \alpha}, 1, 0, 0 \ldots\right)$$

is an eigenvector of U with eigenvalue u_{22}. Hence

$$(3.26) \quad |u_{22}| = 1.$$

If we let

$$(3.27) \quad y = (0, 1, 1, 0, 0, \ldots),$$

$$(3.28) \quad x = (0, 1, 0, \ldots),$$

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and utilize the fact that U must preserve the norm of x and y we are led to the following two equations.

$$(3.29)\quad \alpha = \left(\frac{\alpha}{1+\alpha}\right)^p |u_{22} - u_{11}|^p + \alpha\left|\frac{\alpha u_{11} + u_{22}}{1+\alpha}\right|^p + \alpha|u_{33} - u_{22}|^p,$$

$$(3.30)\quad \alpha = \left(\frac{\alpha}{1+\alpha}\right)^p |u_{22} - u_{11}|^p + \alpha\left|\frac{u_{22} + \alpha u_{11}}{1+\alpha}\right|^p.$$

We obtain immediately from (3.29) and (3.30) that

$$(3.31)\quad u_{33} = u_{22}.$$

We can now show that for $\alpha \neq 1$, $u_{22} = u_{11}$ and if $\alpha = 1$, $u_{22} = \pm u_{11}$. To do this, let $\sigma = \alpha/(1 + \alpha)$ and

$$(3.32)\quad x = (0, 1, -1, 0, 0, \ldots),$$

$$(3.33)\quad y = (1, 1, 0, 0 \ldots).$$

Again using the fact that U preserves the norm of these vectors we obtain the following two equations:

$$(3.34)\quad \sigma^p|u_{22} - u_{11}|^p + \alpha|(1 - \sigma)u_{22} + \sigma u_{11}|^p = \alpha,$$

$$(3.35)\quad \alpha(1 - \sigma)^p|u_{22} - u_{11}|^p + |(1 - \sigma)u_{11} + \sigma u_{22}|^p = 1.$$

If $\alpha \neq 1$, multiply (3.35) by α and subtract the result from (3.34) to get

$$(3.36)\quad \left[\frac{\alpha^p - \alpha^2}{(1+\alpha)^p}\right]|u_{22} - u_{11}|^p = 0.$$

Since $p \neq 2$ and $\alpha \neq 1$ we get $u_{22} = u_{11}$.

If $\alpha = 1$, then (3.34) and (3.35) yield the same equation:

$$(3.37)\quad \left|\frac{u_{22} - u_{11}}{2}\right|^p + \left|\frac{u_{11} + u_{22}}{2}\right|^p = 1.$$

We apply Lamperti's Lemma (3.1) to (3.37) with $\Phi(t) = t^p$, $z = (u_{11} - u_{22})/2$ and $w = (u_{11} + u_{22})/2$. The result is,

$$(3.38)\quad \left|\frac{u_{11} - u_{22}}{2}\right| \left|\frac{u_{11} + u_{22}}{2}\right| = 0.$$

Hence $u_{11} = \pm u_{22}$ and the proof is finished.

4. Applications. In this section we restrict our attention to the complex $s_p(\alpha)$ spaces. Theorem (3.14) allows us to give a description of the one parameter groups of isometries on $s_p(\alpha)$.

4.1. PROPOSITION. *If $\{U_t\}$ is a strongly continuous group of surjective* *isometries acting on* $s_p(\alpha)$ *then there exists a real number* δ *such that*

(4.2) $\quad U_t = e^{i\delta t}I.$

(I is the identity operator.)

Proof. If $\{U_t\}$ is one parameter group of isometries then for each $t_0 \in R$,

(4.3) $\quad U_{t_0} = U_{(t_0/2)}U_{(t_0/2)} = U^2{}_{(t_0/2)}.$

Thus every element of the group is a square. From the description of the isometries given in Theorem 3.14 it is clear that

(4.4) $\quad U_t = \lambda_t I,$

where λ_t is a complex valued function of t. The strong continuity of the group implies that the map $t \to \lambda_t$ is continuous on R. Since $\lambda_{t+s} = \lambda_t\lambda_s$, it follows that

(4.5) $\quad \lambda_t = e^{i\delta t}$ for some real number δ.

The proof is complete.

Remark. An operator T acting on a complex Banach space is Hermitian (self conjugate) if T is the generator of a uniformly (strongly) continuous group of isometries ([**1**] and [**8**]).

4.6. COROLLARY. *An operator* T *on* $s_p(\alpha)$ *is Hermitian (self conjugate) if and only if there is a real number* δ *such that*

(4.7) $\quad T = \delta I.$

Remark. An operator T on a complex Banach space is said to be *adjoint abelian* [**9**], if there is a semi inner product [,] compatible with the norm for which $[Tx, y] = [x, Ty]$ for every x and y in the space.

In [**2**] the authors proved that if T is an adjoint abelian operator such that $T^2 = \mu I$ for some $\mu > 0$ then $T = \delta V$ for some real δ and some isometry V.

Since the square of an adjoint abelian operator is Hermitian, we can combine Corollary 4.6, Theorem 3.14, and the result in [**2**] to obtain the following.

4.8. COROLLARY. *If* $\alpha \neq 1$ *then* T *is adjoint abelian on* $s_p(\alpha)$ *if and only if there exists a real number* δ *such that*

(4.9) $\quad T = \delta I.$

Remark. Corollaries (4.6) and (4.8) imply that an operator T is adjoint abelian on $s_p(\alpha)$ $(\alpha \neq 1)$ if and only if T is Hermitian. This is the only case (except for Hilbert Space) known to the authors for which this phenomenon occurs.

Finally, we suspect that all isometries of $s_p(\alpha)$ must be surjective. However, this remains an open question. Stephen Cambell has recently answered this question in the negative.

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