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LIFTING UNCONDITIONALLY CONVERGING SERIES AND SEMIGROUPS OF OPERATORS

MANUEL GONZÁLEZ AND ANTONIO MARTÍNEZ-ABEJÓN

We introduce and study two semigroups of operators $\mathcal{U}_+$ and $\mathcal{U}_-$, defined in terms of unconditionally converging series. We prove a lifting result for unconditionally converging series that allows us to show examples of operators in $\mathcal{U}_+$. We obtain perturbative characterisations for these semigroups and, as a consequence, we derive characterisations for some classes of Banach spaces in terms of the semigroups. If $\mathcal{U}_+(X,Y)$ is non-empty and every copy of c_0 in Y is complemented, then the same is true in X. We solve the perturbation class problem for the semigroup $\mathcal{U}_-$, and we show that a Banach space X contains no copies of ℓ_∞ if and only if for every equivalent norm $|\cdot|$ on X, the semiembeddings of $(X,|\cdot|)$ belong to $\mathcal{U}_+$.

1. INTRODUCTION

Tauberian operators, introduced by Kalton and Wilansky [**12**], are useful in Banach space theory because they preserve some isomorphic properties of sets in Banach spaces. For example, the second factor in the factorisation given in [**5**] is tauberian and, since tauberian operators preserve the relative weak compactness of bounded sets, it follows that weakly compact operators factorise through reflexive Banach spaces. We refer to the survey [**7**] for further information about tauberian operators.

The class of tauberian operators is a semigroup and has analogous properties to that of upper semi-Fredholm operators, replacing finite dimensional spaces by reflexive spaces. We refer to [**7, 10**] for details. Moreover, upper semi-Fredholm operators, tauberian operators and other semigroups of operators defined in terms of sequences admit perturbative characterisations [**10**].

Here we define two new semigroups of operators, denoted $\mathcal{U}_+$ and $\mathcal{U}_-$, in terms of the action of the operators over unconditionally converging series. We obtain a perturbative characterisation: An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{U}_+$ if and only if for every compact operator $K \in \mathcal{B}(X,Y)$ the kernel $N(T+K)$ contains no copies of c_0. As a consequence we derive an algebraic characterisation of operators in $\mathcal{U}_+$, we show that a Banach space X contains no copies of ℓ_∞ if and only if every semiembedding of X belongs to $\mathcal{U}_+$, and

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we characterise Banach spaces whose non-reflexive (respectively, infinite dimensional) subspaces contain c_0 in terms of $\mathcal{U}_+$.

We prove a lifting result for unconditionally converging series, analogous to Lohman's lifting for weakly Cauchy sequences [**15**]. As a consequence, we show that operators with closed range and kernel containing no copies of c_0 belong to $\mathcal{U}_+$, and operators with closed range and cokernel containing no complemented copies of ℓ_1 belong to $\mathcal{U}_-$. We refer to [**9**] for other lifting results for sequences.

We also prove that operators in $\mathcal{U}_+$ preserve an isomorphic property: if there exists an operator $T \in \mathcal{U}_+(X,Y)$ and every subspace of Y isomorphic to c_0 is complemented, then the same is true for X. Separable spaces, or more generally, weakly compactly generated spaces, satisfy this property.

For the dual semigroup $\mathcal{U}_-$ we also obtain a perturbative characterisation: $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{U}_-$ if and only if for every compact operator $K \in \mathcal{B}(X,Y)$ the cokernel $Y/\overline{R(T)}$ contains no complemented copies of ℓ_1. As a consequence, we derive characterisations of Banach spaces whose non-reflexive (respectively, infinite dimensional) quotients contain a complemented copy of ℓ_1 in terms of $\mathcal{U}_-$. We also solve the perturbation class problem for the semigroup $\mathcal{U}_-$: given an operator $K \in \mathcal{B}(X,Y)$, we have $T + K \in \mathcal{U}_-$ for every $T \in \mathcal{U}_-(X,Y)$ if and only if the conjugate K^* is unconditionally converging.

We use standard notations: X and Y are Banach spaces and B_X denotes the closed unit ball of X. The class of (bounded linear) operators from X to Y is $\mathcal{B}(X,Y)$, the dual of X is X^*, and given an operator $T \in \mathcal{B}(X,Y)$, we denote by $T^* : Y^* \longrightarrow X^*$ the conjugate operator of T, by $R(T)$ and $N(T)$ the range and kernel of T, and by $Y/\overline{R(T)}$ the cokernel of T. Moreover, $\mathbb{N}$ is the set of all positive integers. We identify X with a subspace of X^{**}.

2. The semigroups

Recall that a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X is *weakly unconditionally Cauchy* if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for all $x^* \in X^*$. A series is *unconditionally converging* if every subseries is convergent.

An operator $T \in \mathcal{B}(X,Y)$ is said to be *unconditionally converging*, denoted $T \in \mathcal{U}(X,Y)$, if it takes weakly unconditionally Cauchy series into unconditionally converging series. The following characterisation will be useful. We refer to [**18**, p.270] for a proof.

Proposition 2.1. *An operator $T \in \mathcal{B}(X,Y)$ is unconditionally converging if and only if given a subspace M of X, if the restriction $T|_M$ is an isomorphism then M contains no copies of c_0.*

The definition of the semigroup $\mathcal{U}_+$ is opposite in some sense to that of the uncon-

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ditionally converging operators.

DEFINITION 2.2: An operator $T \in \mathcal{B}(X,Y)$ belongs to the class $\mathcal{U}_+$ if for every weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$ in X, if $\sum_{n=1}^{\infty} Tx_n$ is unconditionally converging then $\sum_{n=1}^{\infty} x_n$ is unconditionally converging

REMARK 2.3. (a) It readily follows from the definition that the class $\mathcal{U}_+$ is stable under products and under unconditionally converging perturbations:

$$T \in \mathcal{U}_+(X,Y) \text{ and } S \in \mathcal{U}_+(Y,Z) \Rightarrow ST \in \mathcal{U}_+(X,Z);$$
$$T \in \mathcal{U}_+(X,Y) \text{ and } K \in \mathcal{U}(X,Y) \Rightarrow T + K \in \mathcal{U}_+(X,Y).$$

(b) Since a Banach space X contains no copies of c_0 if and only if every weakly unconditionally Cauchy series in X is unconditionally converging [**2**, Theorem 5], we have

$$T \in \mathcal{U}_+(X,Y) \Rightarrow N(T) \text{ contains no copies of } c_0.$$

Now we prove a lifting result for unconditionally converging series and derive some consequences. Given a subspace M of a Banach space X, we denote by $q_M : X \longrightarrow X/M$ the quotient map.

THEOREM 2.4. *Let M be a subspace of X containing no copies of c_0. If $\sum_{n=1}^{\infty} x_n$ is a weakly unconditionally Cauchy series in X, and $\sum_{n=1}^{\infty} q_M x_n$ is unconditionally converging, then $\sum_{n=1}^{\infty} x_n$ is unconditionally converging*

PROOF: Clearly, it is enough to show that $\sum_{n=1}^{\infty} x_n$ is convergent.

Suppose that $\sum_{n=1}^{\infty} x_n$ is non-convergent. Then there are a number $\delta > 0$ and integers $1 \leqslant m_1 \leqslant n_1 < m_2 \leqslant n_2 < \ldots$ so that, denoting $y_k := x_{m_k} + \ldots + x_{n_k}$, we have $\|y_k\| > \delta$.

The sequence (y_k) is weakly null and bounded away from 0. Therefore, using the Bessaga-Pelczynski selection principle [**2**, C.1], we can select a basic subsequence (z_n) of (y_k). This subsequence is equivalent to the unit vector basis of c_0 because $\sum_{n=1}^{\infty} z_n$ is weakly unconditionally Cauchy [**2**, Lemma 1].

Moreover, since $\sum_{n=1}^{\infty} q_M x_n$ is unconditionally converging, we have that $(q_M z_n)$ converges in norm to 0. Then we can find a sequence (w_n) in M such that $\lim_{n\to\infty} \|w_n - z_n\| = 0$, and a standard perturbation argument for basic sequences implies that a subsequence of (w_n) is equivalent to the basis of c_0, which gives a contradiction. ∎

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Corollary 2.5. *If $T \in \mathcal{B}(X,Y)$ has closed range and its kernel contains no copies of c_0, then $T \in \mathcal{U}_+$.*

Proof: It is enough to observe that T can be written as the composition of the quotient map $X \longrightarrow X/N(T)$ and an isomorphism (into). ∎

The following result is well-known. We prove it as an application of our lifting result.

Corollary 2.6. *The class of Banach spaces that contain no copies of c_0 has the three-space property.*

Proof: Let M be a subspace of X and assume M and X/M contain no copies of c_0. If we denote by $Q : X \longrightarrow X/M$ the quotient map, given a weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} x_n$, we have that $\sum_{n=1}^{\infty} Qx_n$ is weakly unconditionally Cauchy in X/M; hence unconditionally converging because X/M contains no copies of c_0, and applying the lifting result we conclude that $\sum_{n=1}^{\infty} x_n$ is unconditionally converging. ∎

The operators in $\mathcal{U}_+$ can be characterised by their action over sequences equivalent to the unit vector basis of c_0 and in terms of the kernels of the perturbations by operators in $\mathcal{U}$.

Proposition 2.7. *For $T \in \mathcal{B}(X,Y)$ the following statements are equivalent:*

(a) $T \in \mathcal{U}_+(X,Y)$;

(b) *if $(x_n) \subset X$ is equivalent to the unit vector basis of c_0 then there exists $k \in \mathbb{N}$ such that $(Tx_n)_{n>k}$ is equivalent to the unit vector basis of c_0;*

(c) *there is no (normalised) sequence (x_n) in X equivalent to the unit vector basis of c_0 and such that $\lim_{k\to\infty} Tx_k = 0$.*

Proof: *(a)⇒(b)* Assume $(x_n)_n$ is a sequence in X equivalent to the unit vector basis of c_0, but $(Tx_n)_{n>k}$ is equivalent to this basis for no $k \in \mathbb{N}$. Then we can find a sequence of scalars (a_n) such that $|a_n| \leqslant 1$ for all n and a sequence of integers $1 \leqslant m_1 \leqslant n_1 < m_2 \leqslant n_2 < \cdots$ so that, denoting $y_k := a_{m_k}x_{m_k} + \ldots + a_{n_k}x_{n_k}$, we have that (y_k) is equivalent to the unit vector basis of c_0, but $\|Ty_k\| \to 0$. By passing to a subsequence we may assume $\|Ty_k\| < 2^{-k}$. Then $\sum_{k=1}^{\infty} Ty_k$ is unconditionally converging, and we conclude that $T \notin \mathcal{U}_+$.

(b)⇒(c) is trivial.

(c)⇒(a) Assume that $T \notin \mathcal{U}_+(X,Y)$. Then there is a weakly unconditionally Cauchy series $\sum_{k=1}^{\infty} z_k$ in X which is not unconditionally converging, such that $\sum_{k=1}^{\infty} Tz_n$ is unconditionally converging. Now, proceeding as in the proof of Theorem 2.4, we obtain a sequence (y_k) equivalent to the unit vector basis of c_0 and such that $\lim_{k\to\infty} Ty_k = 0$, and the normalised sequence given by $x_k := \|y_k\|^{-1} y_k$ shows that *(c)* fails. ∎

Theorem 2.8. *An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{U}_+$ if and only if for every compact operator $K \in \mathcal{B}(X,Y)$ the kernel $N(T+K)$ contains no copies of c_0.*

Proof: The direct implication was shown in Remark 2.3.

For the converse, assume T does not belong to $\mathcal{U}_+$. By Proposition 2.7, we can find (x_n) in X equivalent to the unit vector basis of c_0, and (f_n) bounded in X^* such that $f_i(x_j) = \delta_{ij}$ and $\|f_n\|\,\|Tx_n\| < 2^n$. Then

$$Kx := -\sum_{n=1}^{\infty} f_n(x)Tx_n$$

defines a compact operator $K \in \mathcal{B}(X,Y)$ such that $N(T+K)$ contains the subspace generated by (x_n). ∎

As a consequence of the perturbative characterisation, we derive "algebraic" characterisations.

Corollary 2.9. *For $T \in \mathcal{B}(X,Y)$, the following statements are equivalent:*

- *(a) $T \in \mathcal{U}_+(X,Y)$;*
- *(b) for every Banach space Z and every $L \in \mathcal{B}(Z,X)$, we have $TL \in \mathcal{U}(Z,Y)$ only if $L \in \mathcal{U}(Z,X)$;*
- *(c) for every subspace $M \subset X$, if the restriction $T|_M$ belongs to $\mathcal{U}(M,Y)$, then M contains no copies of c_0.*

Proof: *(a)⇒(b)* Assume $T \in \mathcal{U}_+(X,Y)$ and let $L \in \mathcal{B}(Z,X)$ be an operator such that $TL \in \mathcal{U}(Z,Y)$. Let $\sum z_k$ be a weakly unconditionally Cauchy series in Z. Thus $\sum TLz_k$ is unconditionally converging, and since $T \in \mathcal{U}_+(X,Y)$, the series $\sum Lz_k$ must be unconditionally converging.

(b)⇒(c) It is enough to observe that the inclusion operator $i_M : M \longrightarrow X$ belongs to $\mathcal{U}(M,X)$ if and only if M does not contain copies of c_0 [**2**, Theorem 5].

(c)⇒(a) Assume $T \notin \mathcal{U}_+(X,Y)$. By Theorem 2.8 there is a compact operator $K \in \mathcal{B}(X,Y)$ such that $M := N(T+K)$ contains a copy of c_0. As $Ti_M = -Ki_M$, we have that Ti_M is compact, hence $Ti_M \in \mathcal{U}$, and M contains a copy of c_0. ∎

Recall that $T \in \mathcal{B}(X,Y)$ is said to be upper semi-Fredholm if it has closed range and finite dimensional kernel, and T is said to be *tauberian* if its second conjugate $T^{**} \in \mathcal{B}(X^{**},Y^{**})$ satisfies $T^{**}(X^{**}\setminus X) \subset Y^{**}\setminus Y$. We denote by $\mathcal{F}_+(X,Y)$ and $\mathcal{T}_+(X,Y)$ the classes of upper semi-Fredhom operators and tauberian operators, respectively. Both classes admit a perturbative characterisation. The result for $\mathcal{F}_+$ is classic.

Proposition 2.10. [**10**, Theorem 1.a] *An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{F}_+$ (respectively, $\mathcal{T}_+$) if and only if for every compact operator $K \in \mathcal{B}(X,Y)$ the kernel $N(T+K)$ is finite dimensional (respectively, reflexive).*

Corollary 2.11. *For every pair of Banach spaces X, Y we have*

$$\mathcal{F}_+(X,Y) \subset \mathcal{T}_+(X,Y) \subset \mathcal{U}_+(X,Y).$$

REMARK 2.12. Although the components of the semigroup $\mathcal{F}_+$ are open subsets of $\mathcal{B}(X,Y)$, this is not always true for the components of $\mathcal{T}_+$ and $\mathcal{U}_+$. This fact can be seen using an example similar to one given in [**1**].

In the space $\ell_2(c_0) := \left\{(x_n) : x_n \in c_0 \textit{ and } \sum_{n=1}^{\infty} \|x_n\|^2 < \infty\right\}$, we consider the operator $T \in \mathcal{B}\big(\ell_2(c_0), \ell_2(c_0)\big)$ given by $T(x_n) := (x_n/n)$. It easily follows from the definition that T is tauberian, hence it belongs to $\mathcal{U}_+$. However, the operators $T_n \in \mathcal{B}\big(\ell_2(c_0), \ell_2(c_0)\big)$ given by

$$T_n(x_1, x_2, \ldots) := (x_1, \ldots, x_n, 0, 0, \ldots),$$

satisfy $\lim_n \|T - T_n\| = 0$ and $ker(T_n)$ contains a copy of c_0 for every n. Hence T belongs to the boundaries of $\mathcal{U}_+\big(\ell_2(c_0), \ell_2(c_0)\big)$ and $\mathcal{T}_+\big(\ell_2(c_0), \ell_2(c_0)\big)$.

As a consequence of the perturbative characterisations, we characterise some classes of Banach spaces. Recall that a Banach space X is said to be *hereditarily* c_0 if every infinite dimensional subspace of X contains copies of c_0.

PROPOSITION 2.13. *Let X be a Banach space.*

(a) *The space X is hereditarily c_0 if and only if for every Y we have $\mathcal{U}_+(X,Y) = \mathcal{F}_+(X,Y)$.*
(b) *Non-reflexive subspaces of X contain copies of c_0 if and only if for every Y we have $\mathcal{U}_+(X,Y) = \mathcal{T}_+(X,Y)$.*
(c) *Reflexive subspaces of X are finite dimensional if and only if for every Y we have $\mathcal{T}_+(X,Y) = \mathcal{F}_+(X,Y)$.*

PROOF: (a) If X is not hereditarily c_0, then it contains an infinite dimensional subspace M containing no copies of c_0. By Corollary 2.5, the quotient map q_M belongs to $U_+ \setminus \mathcal{F}_+$. The direct implication follows directly from Theorem 2.8.

The proof of the other parts is analogous. ∎

We consider now Banach spaces X such that all their subspaces isomorphic to c_0 are complemented. This is the case when X is separable, weakly compactly generated, or more generally, when X is weakly compactly determined [**6**, Lemma VI.2.4]. Next we show that operators in $\mathcal{U}_+$ preserve this class.

PROPOSITION 2.14. *Assume that $\mathcal{U}_+(X,Y)$ is non-empty. If every subspace of Y isomorphic to c_0 is complemented, then the same is true for X.*

PROOF: Let M be a subspace of X isomorphic to c_0. Taking $T \in U_+(X,Y)$, by Proposition 2.7 there is a subspace N of M such that M/N is finite dimensional and the restriction $T|_N$ is an isomorphism. Since N is isomorphic to c_0, we have that $T(N)$ is complemented. Now, if L is a complement of $T(N)$, then $T^{-1}(L)$ is a complement of N; hence N and M are complemented subspaces of X. ∎

Given a semigroup $\mathcal{S}$ of operators, Lebow and Schechter [**13**] define the *perturbation class* $\mathcal{PS}$, for spaces X, Y such that $\mathcal{S}(X,Y)$ is not empty, as follows:

$$\mathcal{PS}(X,Y) := \Big\{A \in \mathcal{B}(X,Y) : T + A \in \mathcal{S} \text{ for every } T \in \mathcal{S}(X,Y)\Big\}.$$

Recall that an operator $T \in \mathcal{B}(X,Y)$ is said to be *strictly singular* if there is no infinite dimensional M of X such that the restriction $T|_M$ is an isomorphism. The perturbation class for the upper semi-Fredholm operators contains the strictly singular operators, and it is a well-known open problem whether these classes coincide, even in the case $X = Y$ [**17**].

We have seen in Remark 2.3 that the perturbation class for $\mathcal{U}_+(X,Y)$ contains $\mathcal{U}(X,Y)$. Next we show that they coincide whenever subspaces of X and Y isomorphic to c_0 are complemented.

PROPOSITION 2.15. *Assume that $\mathcal{U}_+(X,Y)$ is nonempty and that every subspace of Y isomorphic to c_0 is complemented. Then we have $\mathcal{PU}_+(X,Y) = \mathcal{U}(X,Y)$.*

PROOF: Take $A \in \mathcal{B}(X,Y)$ which is not in $\mathcal{U}$. It follows from Proposition 2.1 that there exists a subspace M of X isomorphic to c_0 such that the restriction $A|_M$ is an isomorphism. From the hypothesis and Proposition 2.14, by passing to a complemented subspace of M we may assume that

$$X = U \oplus M \text{ and } Y = V \oplus A(M),$$

with U and V isomorphic to X and Y, respectively.

Now we can define an operator $T \in \mathcal{B}(X,Y)$ such that $T|_M = A|_M$, $T(U) \subset V$ and $T|_U$ belongs to $\mathcal{U}_+$. Clearly $T \in \mathcal{U}_+$. However, $T + A \notin \mathcal{U}_+$. ∎

Now we introduce the dual semigroup.

DEFINITION 2.16: An operator $T \in \mathcal{B}(X,Y)$ belongs to the class $\mathcal{U}_-$ if its conjugate operator T^* belongs to $\mathcal{U}_+$.

We denote by $\mathcal{U}^d$ the dual operator ideal of $\mathcal{U}$; that is,

$$\mathcal{U}^d(X,Y) := \Big\{T \in \mathcal{B}(X,Y) : T^* \in \mathcal{U}\Big\}.$$

The following characterisation of the operators in $\mathcal{U}^d$ will be useful. We refer to [**18**, Lemma p.272] for a proof.

PROPOSITION 2.17. *An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{U}^d$ if and only if there is no subspace M of X isomorphic to ℓ_1 such that the restriction $T|_M$ is an isomorphism and $T(M)$ is complemented in Y.*

REMARK 2.18. (a) It is not difficult to derive from the previous result that the dual X^* of a Banach space X contains no copies of c_0 if and only if X contains no complemented copies of ℓ_1 [**2**, Theorem 4].

(b) Similarly as in the case of $\mathcal{U}_+$, the class $\mathcal{U}_-$ satisfies the following properties:

$$T \in \mathcal{U}_-(X,Y) \text{ and } S \in \mathcal{U}_-(Y,Z) \;\Rightarrow\; ST \in \mathcal{U}_-(X,Z).$$
$$T \in \mathcal{U}_-(X,Y) \text{ and } K \in \mathcal{U}^d(X,Y) \;\Rightarrow\; T+K \in \mathcal{U}_-(X,Y).$$
$$T \in \mathcal{U}_-(X,Y) \;\Rightarrow\; Y/\overline{R(T)} \text{ contains no complemented copies of } \ell_1.$$

(c) An operator $T \in \mathcal{B}(X,Y)$ with closed range and cokernel $Y/R(T)$ containing no complemented copies of ℓ_1 belongs to $\mathcal{U}_-$. This is consequence of Corollary 2.5 and duality.

(d) We have that $T^* \in \mathcal{U}_-$ implies $T \in \mathcal{U}_+$, because T is a restriction of T^{**}. However, the converse implication is not true. If we consider a Banach space Z containing no copies of c_0 such that Z^{**} contains a copy of c_0, then the zero operator 0_Z in Z belongs to $\mathcal{U}_+$, but its second conjugate 0_Z^{**} does not.

We can take as Z the hereditarily reflexive predual of ℓ_1, obtained by Bourgain and Delbaen [**3**].

Now we give a perturbative characterisation for $\mathcal{U}_-$.

Theorem 2.19. *An operator $T \in \mathcal{B}(X,Y)$ belongs to $\mathcal{U}_-$ if and only if for every compact operator $K \in \mathcal{B}(X,Y)$, the cokernel $Y/\overline{R(T+K)}$ contains no complemented copies of ℓ_1.*

Proof: The direct implication follows from Remark 2.18. For the converse, assume $T \notin \mathcal{U}_-$; equivalently, $T^* \notin \mathcal{U}_+$. By Proposition 2.7, we can select a sequence (f_n) in Y^* equivalent to the unit vector basis of c_0 and such that $\lim_{n\to\infty} \|T^* f_n\| = 0$.

Using a result of Johnson and Rosenthal [**11**, Remark 3.1] we can select a subsequence (g_n) of (f_n) and a bounded sequence (y_n) in Y such that $g_k(y_l) = \delta_{kl}$. Moreover, we can assume that $\|T^* g_n\|\,\|y_n\| < 2^{-n}$. Therefore, the expression

$$Kx := \sum_{n=1}^{\infty} (T^* g_n)(x) y_n$$

defines a compact operator $K \in \mathcal{B}(X,Y)$ whose conjugate is given by

$$K^* f = \sum_{n=1}^{\infty} f(y_n) T^* g_n.$$

Thus (g_n) is contained in $N(T^* + K^*) = \left[Y/\overline{R(T+K)}\right]^*$; hence $Y/\overline{R(T+K)}$ contains a complemented copy of ℓ_1. ∎

Now we can give algebraic characterisations of $\mathcal{U}_-$. Note that the identity I_X of a Banach space X belongs to $\mathcal{U}^d$ if and only if X contains no complemented copies of ℓ_1.

Corollary 2.20. *For $T \in \mathcal{B}(X,Y)$, the following assertions are equivalent:*

(a) $T \in \mathcal{U}_-$;

(b) *for every Z and every $A \in \mathcal{B}(Y,Z)$, if $AT \in \mathcal{U}^d$ then $A \in \mathcal{U}^d$;*

(c) *for every closed subspace M of Y, if $q_M T \in \mathcal{U}^d$, then Y/M contains no complemented copies of ℓ_1.*

PROOF: *(a)⇒(b)* If $T \in \mathcal{U}_-$ and $AT \in \mathcal{U}^d$ then $T^* \in \mathcal{U}_+$ and $T^*A^* \in \mathcal{U}$. Theorem 2.9 gives that $A^* \in \mathcal{U}$, hence $A \in \mathcal{A}^d$.

(b)⇒(c) Assume M is a subspace of Y such that $q_M T \in \mathcal{U}^d$. Hypothesis *(b)* leads to $q_M \in \mathcal{U}^d$; equivalently, the dual of Y/M contains no copies of c_0.

(c)⇒(a) Assume $T \notin \mathcal{U}_-$. By Theorem 2.19, there is a compact operator $K \in \mathcal{B}(X,Y)$ such that $\left[Y/\overline{R(T+K)}\right]^*$ contains a copy of c_0. Let $M := \overline{R(T+K)}$. Since K is compact, $-q_M K = q_M T$ is compact, so $q_M T \in \mathcal{U}^d$, but $q_M \notin \mathcal{U}^d$. ∎

An operator $T \in \mathcal{B}(X,Y)$ is said to be *lower semi-Fredholm*, denoted $T \in \mathcal{F}_-$ [**13**] or *cotauberian* [**19**], denoted $T \in \mathcal{T}_-$, if T^* belongs to $\mathcal{F}_+$ or $\mathcal{T}_+$, respectively. Both classes are semigroups and admit perturbative characterisations.

PROPOSITION 2.21. [**10**, Theorem 1.b] *Given $T \in \mathcal{B}(X,Y)$, we have that $T \in \mathcal{F}_-$ (respectively, $\mathcal{T}_-$) if and only if for every compact operator $K \in \mathcal{B}(X,Y)$ the cokernel $Y/\overline{R(T+K)}$ is finite dimensional (respectively, reflexive).*

COROLLARY 2.22. *For every pair of Banach spaces X, Y we have*

$$\mathcal{F}_-(X,Y) \subset \mathcal{T}_-(X,Y) \subset \mathcal{U}_-(X,Y).$$

Using the perturbative characterisations for the semigroups $\mathcal{F}_-$, $\mathcal{T}_-$ and $\mathcal{U}_-$ we can derive characterisations for some classes of Banach spaces. The proof is analogous to that of Proposition 2.13.

PROPOSITION 2.23. *Let X be a Banach space.*

(a) *Quotients of X containing no complemented copies of ℓ_1 are finite dimensional if and only if for every Y we have $\mathcal{U}_-(Y,X) = \mathcal{F}_-(Y,X)$.*

(b) *Quotients of X containing no complemented copies of ℓ_1 are reflexive if and only if for every Y we have $\mathcal{U}_-(Y,X) = \mathcal{T}_-(Y,X)$.*

(c) *Reflexive quotients of X are finite dimensional if and only if for every Y we have $\mathcal{T}_-(Y,X) = \mathcal{F}_-(Y,X)$.*

Now we study the perturbation class of $\mathcal{U}_-$. Recall that an operator $T \in \mathcal{B}(X,Y)$ is said to be *strictly cosingular* if a closed subspace N of Y is finite codimensional whenever $R(T) + N = Y$. The perturbation class for the lower semi-Fredholm operators contains the strictly cosingular operators, and it is an open problem whether they coincide [**17**].

We have observed in Remark 2.18 that the perturbation class for $\mathcal{U}_-$ contains $\mathcal{U}^d$. Next we show that these classes coincide for operators acting in the same space.

PROPOSITION 2.24. *The perturbation class of $\mathcal{U}_-$ coincides with $\mathcal{U}^d$.*

PROOF: Assume that $\mathcal{U}_-(X,Y)$ is not empty, and that $A \in \mathcal{B}(X,Y)$ does not belong to $\mathcal{U}^d$. It follows from Proposition 2.17 that there exists a subspace M of X isomorphic to ℓ_1 such that the restriction $A|_M$ is an isomorphism and $A(M)$ is complemented. As in the proof of Proposition 2.14, we get that M is also complemented in X, and using the argument in the proof of Proposition 2.15, we may assume that

$$X = U \oplus M \text{ and } Y = V \oplus A(M),$$

with U and V isomorphic to X and Y, respectively.

Now we can define an operator $T \in \mathcal{B}(X,Y)$ such that $T|_M = A|_M$, $T(U) \subset V$ and $T|_U$ belongs to $\mathcal{U}_-$. Clearly $T \in \mathcal{U}_-$. However, $T + A \notin \mathcal{U}_-$. ∎

3. Semiembeddings and semigroups

Here we show the relation between operators in $\mathcal{U}_+$ and semiembeddings. Recall that $T \in \mathcal{B}(X,Y)$ is said to be a *semiembedding* if T is injective and TB_X is closed. This concept, introduced in [**16**], has found applications in the study of the Radon-Nikodym property [**4**].

Semiembeddings are not stable under isomorphic renorming of the initial space [**16**], but it has been proved by Saint-Raymond (see [**4**, Proposition 1.6]) that an operator $T \in \mathcal{B}(X,Y)$ is a semi-embedding under some equivalent norm for X if and only if it is injective and its range $T(X)$ is an F_σ-set; that is, a countable union of closed sets. These operators are called *F_σ-embeddings.*

We shall present two examples of semiembeddings not belonging to $\mathcal{U}_+$, but we also show that F_σ-embeddings of X belong to $\mathcal{U}_+(X,Y)$ if (and only if) X contains no copies of ℓ_∞.

EXAMPLES. (a) The operator $S \in \mathcal{B}(\ell_\infty, \ell_2)$, given by $S(x_n) := (x_n/n)$, is an injective conjugate operator; hence it is a semiembedding. However, $S \notin \mathcal{U}_+$ because it carries the unit vector basis of $c_0 \subset \ell_\infty$ into a norm null sequence.

(b) For $1 \leqslant p < \infty$, the natural inclusion $i \in \mathcal{B}\big(L_\infty[0,1], L_p[0,1]\big)$ is a semiembedding; in fact, a sequence in the unit ball of L_∞ converging in the L_p-norm has a subsequence converging almost everywhere to a measurable function which belongs to the unit ball of L_∞ too.

However, i is not $\mathcal{U}_+$ since given a sequence of pairwise disjoint, measurable subsets $C_n \subset [0,1]$ with $\mu(C_n) > 0$, the corresponding sequence of characteristic functions χ_{C_n} is equivalent in L_∞ to the unit vector basis of c_0, but $\lim\limits_{n\to\infty} \|\chi_{C_n}\|_p = \lim\limits_{n\to\infty} \mu(C_n)^{1/p} = 0$.

THEOREM 3.1. *A Banach space X contains no copies of ℓ_∞ if and only if for every equivalent norm $|\cdot|$ in X, the semiembeddings of $(X, |\cdot|)$ into any Banach space belong to $\mathcal{U}_+$.*

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PROOF: Suppose $T \in \mathcal{B}(X,Y)$ is a semiembedding and $T \notin \mathcal{U}_+(X,Y)$. By Proposition 2.7 there exists a sequence $(x_n) \subset X$ equivalent to the unit vector basis of c_0 and such that $\sum_{n=1}^{\infty} \|Tx_n\| < \infty$.

We select a constant M such that for every finite sequence of scalars $\{t_1, \dots, t_n\}$ with $\max_{1\leqslant i\leqslant n} |t_i| \leqslant M$ we have $t_1x_1 + \cdots + t_nx_n \in B_X$. Since $T(B_X)$ is closed, for every $(t_i) \in \ell_\infty$ with $\|(t_i)\|_\infty \leqslant M$ we have that $\sum_{i=1}^{\infty} t_iTx_i$ is absolutely convergent to some vector $y \in TB_X$. Then

$$(t_n) \in \ell_\infty \longrightarrow T^{-1}\left(\sum_{n=1}^{\infty} t_nTx_n\right) \in X$$

defines an operator $R \in \mathcal{B}(\ell_\infty, X)$ such that $\|R\| \leqslant M^{-1}$ and $R\,|_{c_0}$ is an isomorphism. By a result of Rosenthal (see [**14**, Proposition 2.f.4]), there exists an infinite subset $A \subset \mathbb{N}$ so that $R\,|_{\ell_\infty(A)}$ is an isomorphism. Hence X contains a copy of ℓ_∞.

Conversely, if X contains a copy of ℓ_∞, then X is isomorphic to $\ell_\infty \times Y$ for some Y. Now, if we endow the products $\ell_\infty \times Y$ and $\ell_2 \times Y$ with the supremum norms, we have that $T\big((x_n), y\big) := \big((x_n/n), y\big)$ defines a semiembedding of $\ell_\infty \times Y$ into $\ell_2 \times Y$ which is not in $\mathcal{U}_+$. ∎

COROLLARY 3.2. *Assume that X contains no copies of ℓ_∞.*

- *(a) Every semiembedding $T : X \to Y$ belongs to $\mathcal{U}_+$.*
- *(b) If there exists a semiembedding $T : X \to Y$ and every subspace of Y isomorphic to c_0 is complemented, then the same is true for X.*

Next we show that operators of $\mathcal{U}_+$ defined on $C[0,1]$ or $L_\infty[0,1]$ preserve a copy of the whole space.

PROPOSITION 3.3. *Suppose X is $C[0,1]$ or $L_\infty[0,1]$. Then for every $T \in \mathcal{U}_+(X,Y)$ there exists a subspace M of X isomorphic to X, such that the restriction $T\,|_M$ is an isomorphism.*

PROOF: Assume I_n is a disjoint sequence of closed, non-empty subintervals of $[0,1]$. We denote by C_n the subspace of functions with (essential) support contained in I_n.

If $T\,|_{C_n}$ is an isomorphism for some n, then have finished. Otherwise we can select a sequence of normalised functions $f_n \in C_n$, with $\lim_n \|Tf_n\| = 0$. Since (f_n) is equivalent to the unit vector basis of c_0, we obtain $T \notin \mathcal{U}_+$, a contradiction. ∎

Finally we state some open questions.

PROBLEMS. Suppose X is $C[0,1]$ or $L_\infty[0,1]$, and Y is any Banach space.

- (a) Is it true that $T \in \mathcal{U}_+(X,Y)$ if there is no normalised disjoint sequence (f_n) in X such that $\lim_{n\to\infty} \|Tf_n\| = 0$?
- (b) Is it true that $\mathcal{U}_+(X,Y)$ is open in $\mathcal{B}(X,Y)$?

We refer to [8] for a positive answer to similar questions for tauberian operators on $L_1[0,1]$.

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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Cantabria
39071 Santander, Spain

Departamento de Matemáticas
Facultead de Ciencias
Universidad de Oviedo
33007 Oviedo, Spain