SUMS OF COMPLEXES IN TORSION-FREE ABELIAN GROUPS

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The number of elements in the sum A + B of two complexes A and B of a group G which have multiple representations a + b = a' + b' has been investigated by Scherk and Kemperman [1]. Kemperman [2] appealed to transfinite techniques (to order G) to prove:

> If G is a torsion-free abelian group with finite subsets A and B with $|B| \geq 2$, then at least two elements c of A + B admit exactly one representation c = a + b.

Entringer [3] gave a proof of this result using only finite induction. It will be shown below that if d(A) is the dimension or rank of A in the torsion-free abelian group G, then there are at least d(A) elements of G which have a unique representation as a sum a + b where a is in A and b is in B.

A finite set $A = \{a_1, \ldots, a_t\}$ of non-zero elements of an abelian group is linearly independent provided $\sum_{i=1}^{t} m_i a_i = 0$, with the m_i integers, implies $m_i a_i = 0$ for all i. The maximal number of linearly independent elements of A will be denoted by d(A). Then d(A) is the rank of the subgroup generated by A. The element a of A will be called the "A-component" of the sum a + b in A + B, and $U_B(A)$ will denote the set of all elements of A which are the A-component of an element in A + B which admits but one representation in A + B.

THEOREM. *If* G *is a torsion-free abelian group with finite non-empty subsets* A *and* B, *then* $U_B(A)$ *contains a maximal linearly independent subset of* A.

Proof. The result of Kemperman [2] and Entringer [3] mentioned above implies that $U_B(A)$ is non-empty. It is sufficient to show that the set A depends linearly on $U_B(A)$. Then recourse to the Steinitz

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Exchange Theorem for torsion-free abelian groups, Fuchs [4, Lemma 8.3], will establish that $d(A) \leq d(U_B(A))$ and hence that $U_B(A)$ contains a maximal linearly independent subset of A.

To complete the proof then, it must be shown that A depends on $U_B(A)$. Toward this end, order $A = \{a_1, \ldots, a_t\}$ such that $a_1, a_2, \ldots, a_n$ are not in $U_B(A)$, i.e., $U_B(A) = \{a_{n+1}, \ldots, a_t\}$. For each a_i, $i = 1, \ldots, n$, choose any $b_{i,1}$ in B and write $a_i + b_{i,1} = a_{i,2} + b_{i,2}$ for some $a_{i,2}$ in A and $b_{i,2}$ in B, with $b_{i,1} \neq b_{i,2}$. Similarly, write $a_i + b_{i,2} = a_{i,3} + b_{i,3}$, and so on. Since B is finite, some $b_{i,r} = b_{i,s}$ where $r < s$. Add the equations

$$a_i + b_{i,r} = a_{i,r+1} + b_{i,r+1}$$

$$a_i + b_{i,r+1} = a_{i,r+2} + b_{i,r+2}$$

$$\vdots$$

$$a_i + b_{i,s-1} = a_{i,s} + b_{i,r}$$

to obtain the equation $(s - r)a_i = \sum_{j=1}^{s-r} a_{i,r+j}$. The set of equations so obtained form a system:

$$(*) \qquad \sum_{j=1}^{t} c_{ij} a_j = 0, \qquad i = 1, \ldots, n$$

where

$$\text{(i)} \qquad \sum_{j=1}^{t} c_{ij} = 0, \qquad i = 1, \ldots, n;$$

$$\text{(ii)} \qquad c_{ij} \geq 0, \qquad i \neq j;$$

$$\text{(iii)} \qquad c_{ii} < 0, \qquad i = 1, \ldots, n.$$

It will now be shown by induction on n that the existence of such a system of equations implies that each of the elements a_i, $i = 1, \ldots, n$, depends linearly on $\{a_{n+1}, \ldots, a_t\}$. If $n = 1$, then $-c_{11}a_1 = \sum_{j=2}^{t} c_{ij}a_j$ and the assertion holds. If $n \geq 2$, obtain a new system which is equivalent to (*) by a pivot operation:

$$\sum_{j=1}^{t} c_{1j}a_j = 0; \quad \sum_{j=2}^{t} d_{ij}a_j = 0, \quad i = 2, \ldots, n,$$

where $d_{ij} = c_{i1}c_{1j} - c_{11}c_{ij}$. Then again

$$\text{(i')} \quad \sum_{j=2}^{t} d_{ij} = 0, \quad i = 2, \ldots, n;$$

$$\text{(ii')} \quad d_{ij} \geq 0, \quad i \neq j;$$

$$\text{(iii')} \quad d_{ii} < 0, \quad i = 2, \ldots, n.$$

It is routine to verify that (i') and (ii') hold. If $d_{kk} \geq 0$, then for all $j = 2, \ldots, t$ it must be that $d_{kj} = 0$ and hence that $c_{k1}c_{1j} = c_{11}c_{kj}$. Now $c_{k1} = 0$ implies $d_{kk} < 0$. So $c_{1j} = 0$ for $j \neq k$, $j = 2, \ldots, t$. But then $c_{11} = -c_{1k} \neq 0$ and $c_{11}(a_1 - a_k) = 0$. Since this is an impossibility, (iii') is established.

By the induction assumption, each a_j, $j = 2, \ldots, n$ depends on $\{a_{n+1}, \ldots, a_t\}$. Since $\sum_{j=1}^{t} c_{ij}a_j = 0$, some non-zero multiple of a_1 is a linear combination of $\{a_2, \ldots, a_t\}$ and so a_1 also depends on $\{a_{n+1}, \ldots, a_t\}$. The theorem is proved.

COROLLARY. _If_ $A_1, A_2, \ldots, A_n$ _are finite non-empty complexes of a torsion-free abelian group_ G _such that_ $d(A_1) \geq d(A_2) \geq \ldots \geq d(A_n)$, _then there are at least_ $d(A_1)$ _elements of_ G _having a unique_ representation in $A_1 + A_2 + \ldots + A_n$.

That there may be exactly $d(A_1)$ elements of G having unique expressions in $A_1 + \ldots + A_n$ is easily seen by letting $A_1 = A_2 = \ldots = A_n = \{a_1, \ldots, a_t\}$ be an independent set in G. Then $a_i + a_i + \ldots + a_i$, $i = 1, \ldots, t$ are the only such elements in G.

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