Two Triplets of Circum-Hyperbolas.

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Part I.

1. Let one of the series of circles, which can be drawn so as to touch the sides AB, AC of a triangle, touch those sides in K, L; and let $AK = AL = \delta$.

Then the points K, L are

$$b(c-\delta),\ a\delta,\ 0\ ;\quad c(b-\delta),\ 0,\ a\delta\ ;$$

and the lines BL, CK are given by

$$a\alpha\delta = c(b-\delta)\gamma,\quad a\alpha\delta = b(c-\delta)\beta.$$

Hence eliminating δ we get for the locus of P the hyperbola

$$(b-c)\beta\gamma + a\gamma\alpha - a\alpha\beta = 0. \qquad (A_1) \qquad \text{(i.)}$$

The allied hyperbolas are

$$-b\beta\gamma + (c-a)\gamma\alpha + b\alpha\beta = 0, \qquad (B_1)$$
$$c\beta\gamma - c\gamma\alpha + (a-b)\alpha\beta = 0. \qquad (C_1).$$

2. We shall in the main confine our attention to the consideration of the conic A_1.

3. Its centre is evidently the mid point of BC and the four fixed points through which the triple system passes are A, B, C and the Gergonne-point of ABC (or $a\alpha(s-a) = b\beta(s-b) = c\gamma(s-c)$).

4. The polar of any point $(\alpha', \beta', \gamma')$ with respect to (i.) is

$$a\alpha(\gamma' - \beta') + \beta[(b-c)\gamma' - a\alpha'] + \gamma[(b-c)\beta' + a\alpha'] = 0. \quad \ldots \quad \text{(ii.)}$$

The tangent at A therefore is $\beta = \gamma$, *i.e.*, the bisector of the $\angle A$ touches the curve at A.

The tangents at B, C are

$$a\alpha - (b-c)\gamma = 0,\ a\alpha + (b-c)\beta = 0,$$

hence they intersect on the external bisector of $\angle A$.

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The tangent at the Gergonne-point is

$$a(b-c)(s-a)^2\alpha + b^2(s-b)^2\beta - c^2(s-c)^2\gamma = 0,$$

if this line, and the allied ones, cut the sides in p, q, r, then Ap, Bq, Cr cointersect in the point

$$a^2(s-a)^2\alpha = b^2(s-b)^2\beta = c^2(s-c)^2\gamma \text{ *} \qquad \ldots \quad \text{(iii.)}$$

The polar of the incentre is $\beta(s-b) = \gamma(s-c)$, hence Ap_1, Bq_1, Cr_1 cointersect in $\qquad \alpha(s-a) = \beta(s-b) = \gamma(s-c).\dagger \quad \ldots \qquad \ldots \quad$ (iv.)

The polar of the orthocentre is

$$a\alpha\cos A(\cos B - \cos C) - b\cos B\beta(1-\cos A) + c\cos C\gamma(1-\cos A) = 0,$$

and Ap_2, Bq_2, Cr_2 meet in $a\cos A\alpha = b\cos B\beta = c\cos C\gamma. \quad \ldots \quad$ (v.)

The polar of the circumcentre is

$$a\alpha\cos\frac{A}{2}\sin\frac{B-C}{2} - b\beta\sin\left(C-\frac{A}{2}\right)\sin\frac{A}{2} + c\gamma\sin\left(B-\frac{A}{2}\right)\sin\frac{A}{2} = 0,$$

hence Ap_3, Bq_3, Cr_3 intersect in

$$a\alpha\operatorname{cosec}\left(A-\frac{B}{2}\right) = b\beta\operatorname{cosec}\left(B-\frac{C}{2}\right) = c\gamma\operatorname{cosec}\left(C-\frac{A}{2}\right). \quad \ldots \text{(vi.)}$$

The polars of the mid points of CA, AB are

$$a\alpha + (b-2c)\beta + c\gamma = 0, \quad (\because \text{ parallel to CA}),$$

and $\qquad a\alpha + b\beta - (2b-c)\gamma = 0, \quad (\because$ parallel to AB),

and these meet in $\qquad \dfrac{\beta}{b} = \dfrac{\gamma}{c} = \dfrac{-a\alpha}{(b-c)^2},$

i.e., on the symmedian line through A.

The polar of the centroid is

$$a\alpha(b-c) + b\beta(b-2c) + c\gamma(2b-c) = 0,$$

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* The mode of procedure adopted in the following paragraphs is the same, viz., p, q, r points are on BC, CA, AB respectively.

† The polars of $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$ are respectively

$$\beta(s-c) = \gamma(s-b)$$

($\because$ this is the isogonal conjugate of the polar of the incentre),

$$a\alpha - (s-b)\beta + (s-b)\gamma = 0, \text{ and } a\alpha + (s-c)\beta - (s-c)\gamma = 0.$$

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hence Ap_4, Bq_4, Cr_4 cointersect in

$$a\alpha/(b+c-3a) = b\beta/(c+a-3b) = c\gamma/(a+b-3c). \quad \ldots \quad \text{(vii.)}$$

5. The circumcircle cuts (i.) in the additional point

$$-a\alpha\cos\frac{A}{2}\cos\frac{B-C}{2} = b\beta\sin\frac{A}{2}\cos\left(C-\frac{A}{2}\right) = c\gamma\sin\frac{A}{2}\cos\left(B-\frac{A}{2}\right),$$

hence Ap', Bq', Cr' meet in

$$a\alpha\sec\left(A-\frac{B}{2}\right) = b\beta\sec\left(B-\frac{C}{2}\right) = c\gamma\sec\left(C-\frac{A}{2}\right). \quad \ldots \quad \text{(viii.)}$$

6. If $$\sigma \equiv a\alpha + b\beta + c\gamma,$$

then the asymptotes are given by the equations

$$4abca(\beta+\gamma) = (b-c)(\sigma^2 - 4bc\beta\gamma). \quad \ldots \quad \text{(ix.)}$$

These cut CA in m_1, m_2 determined by

$$4abc\gamma\alpha = (b-c)(a\alpha + c\gamma)^2, \quad \ldots \quad \ldots \quad \text{(x.)}$$

whence $$\frac{a\alpha}{c\gamma} = \frac{\sqrt{b}-\sqrt{c}}{\sqrt{b}+\sqrt{c}}, \text{ or } = \frac{\sqrt{b}+\sqrt{c}}{\sqrt{b}-\sqrt{c}},$$

i.e., Bm_1, Bm_2 are isotomic conjugate lines with respect to CA.

7. The foci are determined from

$$\frac{a^2bc}{4\Delta^2}\alpha_1^2 + \frac{(b-c)^2}{4} = \frac{a^2bc}{4\Delta^2}\beta_1^2 - \frac{a^2c}{2\Delta}\beta_1 + \frac{a^2}{4}$$

$$= \frac{a^2bc}{4\Delta^2}\gamma_1^2 - \frac{a^2b}{2\Delta}\gamma_1 + \frac{a^2}{4}. \quad \ldots \quad \text{(xi.)}$$

8. Reverting to § 1, and calling Q, R, the points corresponding to P, we see that, for the same value δ, AP, BQ, CR meet in O, given by $$a\alpha/(a-\delta) = b\beta/(b-\delta) = c\gamma/(c-\delta);$$
hence the locus of O is the incentroidal axis

$$a\alpha(b-c) + b\beta(c-a) + c\gamma(a-b) = 0.$$

If $\delta = s$, *i.e.*, if the circles are the excircles, then O is the point

$$a\alpha/(s-a) = b\beta/(s-b) = c\gamma/(s-c).$$

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9. The isogonal transformation of (i.) is

$$(b-c)\alpha + a\beta - a\gamma = 0,$$

if this, and the allied lines for B_1, C_1, meet the sides in l, m, n, then Al, Bm, Cn meet in the incentre.

10. If we take the polars of any point $(\alpha', \beta', \gamma')$ with respect to the triple system, we find that they meet in a point (π), viz.,

$$[-a\alpha'(s-a) + b\beta'(s-b) + c\gamma'(s-c)],$$
$$[\ \ a\alpha'(s-a) - b\beta'(s-b) + c\gamma'(s-c)],$$
$$[\ \ a\alpha'(s-a) + b\beta'(s-b) - c\gamma'(s-c)].$$

If the given point be the centroid, then π is the point (vii.): if it be the in-centre, then π is the point $\tan\frac{A}{2}$, $\tan\frac{B}{2}$, $\tan\frac{C}{2}$.

11. If $(\alpha', \beta', \gamma')$ be taken on the line

$$p\alpha + q\beta + r\gamma = 0,$$

then the polar of any point on it with reference to A_1 passes through the point $a\alpha/(b-c) = \beta = \gamma$, and therefore for the triple system the polars pass through the in-centre.

12. If $(\alpha', \beta', \gamma')$ be situated on the circumcircle then the envelope of its polar with regard to A_1 is the conic

$$a^2\alpha^2(b+c)^2 + (a^2 - bc + c^2)^2\beta^2 + (a^2 - b^2 + bc)^2\gamma^2$$
$$-2a\alpha\beta(b-c)(a^2 + bc + c^2) + 2a\alpha\gamma(b-c)(a^2 - b^2 - bc)$$
$$+2\beta\gamma[bc(b-c)^2 + a^2(b^2 - c^2 - a^2)] = 0.$$

13. We may note that the equation to A_1, referred to the mid-point of BC as centre and axes parallel to AB, AC, is

$$bx^2 - cy^2 = bc(c-b)/4.$$

Part II.

14. In the second system KL is an antiparallel to angle A, such that $AK = \lambda$, $AL = \mu$, hence

$$c\lambda = b\mu.$$

The equations to BL, CK are

$$(b-\mu)c\gamma = a\alpha\mu, \quad a\alpha\lambda = (c-\lambda)b\beta$$

whence P is $$a\alpha\lambda\mu = (c-\lambda)b\mu\beta = (b-\mu)c\lambda\gamma.$$

The locus of P is the rectangular hyperbola

$$(b^2-c^2)\beta\gamma + ab\gamma\alpha - ca\alpha\beta = 0. \qquad (A_2) \quad \ldots \quad \ldots \quad \text{(i.)}$$

15. The centre is at the mid point of BC.

16. The polar of any point $(\alpha', \beta', \gamma')$ with respect to A_2 is

$$a\alpha(b\gamma' - c\beta') + \beta[(b^2-c^2)\gamma' - ca\alpha'] + \gamma[(b^2-c^2)\beta' + ab\alpha'] = 0. \qquad \text{(ii.)}$$

The tangent at A is the symmedian line through A; at B and C the tangents are

$$ca\alpha = (b^2-c^2)\gamma, \quad -ab\alpha = (b^2-c^2)\beta,$$

these intersect in $$\frac{a\alpha}{b^2-c^2} = \frac{\beta}{-b} = \frac{\gamma}{c},$$

i.e., they are parallel.

The polar of the symmedian point is $\dfrac{\beta\cos B}{b^2} = \dfrac{\gamma\cos C}{c^2}$,

hence Ap_5, Bq_5, Cr_5 meet in $$\frac{\alpha\cos A}{a^2} = \frac{\beta\cos B}{b^2} = \frac{\gamma\cos C}{c^2}. \quad \ldots \quad \text{(iii.)}$$

The polar of the centroid is

$$a\alpha(b^2-c^2) + b\beta(b^2-2c^2) + c\gamma(2b^2-c^2) = 0. \quad \ldots \quad \text{(iv.)}$$

17. Let A′, B′, C′ be the extremities of the diameters through A, B, C, then the tangent at A′ is

$$a\alpha(b^2-c^2) + b^3\beta - c^3\gamma = 0, \quad \ldots \quad \ldots \quad \text{(v.)}$$

and Ap_6, Bq_6, Cr_6 meet in $a^3\alpha = b^3\beta = c^3\gamma$. $\quad \ldots \quad \ldots \quad$ (vi.)

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The line (v.) has the point $-(b^2+c^2)/a,\ b,\ c$ on it.

The tangents themselves meet in

$$a^2\alpha/\cos A = b^2\beta/\cos B = c^2\gamma/\cos C, \quad \ldots \quad \ldots \quad \text{(vii.)}$$

which is the inverse of point (iii.) above.

The tangent at the orthocentre is given by

$$2R\cos^2 A\sin(B-C)\alpha + b\beta\cos^2 B - c\gamma\cos^2 C = 0, \quad \ldots \quad \text{(viii.)}$$

hence Ap_7, Bq_7, Cr_7 meet in

$$a\alpha\cos^2 A = b\beta\cos^2 B = c\gamma\cos^2 C. \quad \ldots \quad \ldots \quad \text{(ix.)}$$

18. The curve A_2 obviously cuts the circumcircle in a fourth point determined by producing the join of the orthocentre and the mid point of BC to meet the circle.

The polar of this point is

$$\alpha(b^2-c^2) + ab\beta\cos^2 C - ca\gamma\cos^2 B = 0, \quad \ldots \quad \text{(x.)}$$

hence Ap_8, Bq_8, Cr_8 meet in

$$a\alpha\sec^2 A = b\beta\sec^2 B = c\gamma\sec^2 C. \quad \ldots \quad \ldots \quad \text{(xi.)}$$

19. The asymptotes are

$$a\alpha(b-c) + b\beta(b+c) - c\gamma(b+c) = 0, \quad (a)$$

$$a\alpha(b+c) + b\beta(b-c) - c\gamma(b-c) = 0. \quad (b)$$

The (a) set pass through the point

$$a\alpha/(b+c) = b\beta/(c+a) = c\gamma/(a+b);$$

hence they are readily constructed.

20. The polar of $(-a,\ b,\ c)$ is $\beta\cos C = \gamma\cos B$, *i.e.*, the diameter of the circumcircle through A produced.

The polars of $(a,\ -b,\ c)$, $(a,\ b,\ -c)$ are

$$b\alpha - c\cos B\beta + b\cos B\gamma = 0,$$

$$c\alpha + c\cos C\beta - b\cos C\gamma = 0,$$

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hence they meet BC where the symmedian line through A meets it.

21. The condition that the polars of a point $(\alpha', \beta', \gamma')$ with respect to A_1 and A_2 should be parallel is that the point should lie on the median parallel to BC.

22. The co-ordinates of the point of intersection of the polars of $(\alpha', \beta', \gamma')$ with regard to A_1, A_2 are

$$-\alpha'(a\alpha' + b\beta' + c\gamma'),\ \beta'(a\alpha' + b\beta' - c\gamma'),\ \gamma'(a\alpha' - b\beta' + c\gamma').$$

Hence for the centroid, the point is $(-3bc, ca, ab)$.

23. The polar of $(\alpha', \beta', \gamma')$ with regard to A_1 meets the polars with regard to B_2, C_2, in

$$a\alpha(-a^2 + bc + c^2) = b\beta(a^2 - bc + c^2) = c\gamma(a^2 + bc - c^2)\ ;$$
$$a\alpha(-a^2 + b^2 + bc) = b\beta(a^2 - b^2 + bc) = c\gamma(a^2 + b^2 - bc).$$

24. The equation to A_2 with reference to axes parallel to AB, AC through the mid point of BC is

$$x^2 - y^2 = cx - by.$$