POLYNOMIAL EXPANSION OF THE PLANETARY SECULAR TERMS: RELATIVISTIC AND LUNAR PERTURBATIONS. (*)

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ABSTRACT. The relativistic and lunar perturbations must be included in a realistic theory of the secular evolution of the planetary elements. In our general theory, we include the first order of these perturbations. Comparison with more elaborated studies shows that it is sufficient with respect to the accuracy of our theory.

INTRODUCTION

The long period variations of the orbital elements of the solar system are of increasing interest since the development of the Milankovitch theory of climate (Berger *et al.*, 1984). The early solution of Brouwer and Van Woerkom (1950) contains the first order linear terms and the second order terms in the Jupiter-Saturn couple computed by Hill (1897) up to degree 5 in the eccentricity. Bretagnon (1974) took into account all the second order terms of degree 3 in eccentricity and inclination for the 8 major planets.

We have undertaken a new general theory for the 8 planets, based on the works of Brumberg and Egorova (1971), Brumberg (1980), Chapront (1970), Abu El Ata and Chapront (1975), and on Duriez's theory of the 4 outer planets (1977, 1979). Special attention is given to estimating the accuracy of the solution, which includes all the terms of order 2 up to degree 5 in eccentricity and inclination. The relativistic and lunar perturbations must be included in a realistic solution (Bretagnon, 1984a).

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THE SECULAR EQUATIONS

The elliptic elements are denoted by the classical notations $a, e, i, \Omega, \varpi, \varepsilon$. We use also the variables $(p, q, z, \varsigma, \mathbf{k}, \mathbf{h}, \mathbf{q}, \mathbf{p})$ defined by:

$$\begin{aligned} a &= A(1+p)^{-2/3} \iff n = N(1+p) \\ \lambda &= Nt - \sqrt{-1}q = \int n\, dt + \varepsilon \\ z &= e \exp \sqrt{-1}\varpi = \mathbf{k} + \sqrt{-1}\mathbf{h} \\ \varsigma &= \sin \frac{i}{2} \exp \sqrt{-1}\Omega = \mathbf{q} + \sqrt{-1}\mathbf{p} \end{aligned} \tag{1}$$

where n is the mean motion,N the observed mean mean motion, and A the semi-major axis of reference related to N by the Kepler relation $N^2A^3 = n^2a = GM_\odot(1 + m/M_\odot)$ where G is the gravitation constant, $M_\odot$ the solar mass, and m the planetary mass.

The variations of the osculating elements (p, q, z, ς) are given by the classical Lagrange equations :

$$\begin{aligned} \frac{dp}{dt} &= \frac{-3\sqrt{-1}}{NA^2}(1+p)^{4/3}\frac{\partial R}{\partial p} \\ \frac{dq}{dt} &= \sqrt{-1}Np + \frac{\sqrt{-1}(1+p)^{1/3}}{NA^2}\left[3(1+p)\frac{\partial R}{\partial p} + \phi\psi z\left(\frac{\partial R}{\partial z} + \bar{z}\frac{\partial R}{\partial \bar{z}}\right) + \frac{1}{2\phi}\left(\varsigma\frac{\partial R}{\partial \varsigma} + \bar{\varsigma}\frac{\partial R}{\partial \bar{\varsigma}}\right)\right] \\ \frac{dz}{dt} &= \frac{\sqrt{-1}(1+p)^{1/3}}{NA^2}\left[2\psi\frac{\partial R}{\partial \bar{z}} - \phi\psi z\frac{\partial R}{\partial q} + \frac{z}{2\phi}\left(\varsigma\frac{\partial R}{\partial \varsigma} + \bar{\varsigma}\frac{\partial R}{\partial \bar{\varsigma}}\right)\right] \\ \frac{d\varsigma}{dt} &= \frac{\sqrt{-1}(1+p)^{1/3}}{2\phi NA^2}\left[\frac{\partial R}{\partial \bar{\varsigma}} - \varsigma\frac{\partial R}{\partial q} + \varsigma\left(-z\frac{\partial R}{\partial z} + \bar{z}\frac{\partial R}{\partial \bar{z}}\right)\right] \end{aligned} \tag{2}$$

where $\phi = \sqrt{1 - z\bar{z}}$ and $\psi = 1/(1+\phi)$. The disturbing function R is expanded in Poisson series depending on the variables $p_i, q_i, z_i, \bar{z}_i, \varsigma_i, \bar{\varsigma}_i$ and on the time t (Duriez, 1977; Laskar, 1985a).

We gather the variables $p_i, q_i, z_i, \bar{z}_i, \varsigma_i, \bar{\varsigma}_i$ in a single vector V. The differential system (2) is then written in the short form:

$$\frac{dV}{dt} = \Lambda(V, t) \tag{3}$$

To integrate this system, we split V in two parts:

$$V = V_0 + \Delta V(V_0, t) \tag{4}$$

where V_0 denotes the secular part of the variables and ΔV the short period part, depending on V_0 and t. Substitution in (3) and identification by order gives then the secular differential system of order 2:

$$\frac{dV_0}{dt} = \langle\Lambda(V_0, t)\rangle + \left\langle\frac{\partial\Lambda}{\partial V_0}(V_0, t)\Delta_1 V(V_0, t)\right\rangle \tag{5}$$

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where $\langle x \rangle$ is the average of x over the time and $\Delta_1 V$ is the short period solution of order 1 (Duriez, 1977, 1979; Laskar, 1985a). The secular system of order 1 is given by the first part of (5); it does not contain the contributions of the short period terms to the secular terms which arise from the products of the second part of (5).

We performed a extensive computation of the secular system (5) up to degree 5 in the eccentricity and inclination variables (Laskar, 1985a). In the equation (5), the time does not appear any more, nor the variables q_i which are related to the longitudes. Besides, the secular part of p_i are constant and they are replaced by their numerical values in (5). The secular system (5) depends then only on the variables $z_i, \bar{z}_i, \zeta_i, \bar{\zeta}_i$.

We denote by α the secular part of $(z_1, \ldots, z_8, \zeta_1, \ldots, \zeta_8)$ and isolate in (5) the differential system giving the variations of α :

$$\frac{d\alpha}{dt} = \sqrt{-1}(\Phi_1 \alpha + \Phi_3(\alpha, \bar{\alpha}) + \Phi_5(\alpha, \bar{\alpha})) \tag{6}$$

This system contains 153 824 monomial terms. Φ_1 is a real matrix with constant coefficients, Φ_3 gathers the terms of degree 3 and Φ_5 the terms of degree 5.

INTEGRATION OF THE EQUATIONS

In order to avoid the problems of small divisors in the analytical integration of (6) (Laskar, 1984), we integrate (6) numerically integration which can be performed with a large step size of about 500 years. Moreover, we have also included in our differential system (6) the equations of (Kinoshita, 1977) for the secular evolution of precession and obliquity of the Earth which are then integrated together with the orbital elements.

A first numerical integration was made over 10 000 years on each side of J2000. The solution is then expanded in Taylor series by numerical differentiation with respect to the time . It allow us to check the accuracy of the solution by direct comparisons with the analytical ephemeris VSOP82 (Bretagnon, 1982) , and with the JPL numerical ephemeris DE102 (Newhall *et al.*, 1983). Besides, the computation of the polynomial expansion of the secular terms up to high powers of the time has been used to extend VSOP82 over 6000 years for the construction of ephemerides aimed at historical computations (Laskar, 1985b; Bretagnon *et al.*, 1985).

Over a longer span of time, the polynomial expansion in powers of the time is not useful. A representation of the solution as a quasi-periodic function should be preferred to give the main frequences of the solution. We have performed a numerical integration of the whole system (6) over 30 millions years which should be enough to derive a solution in a quasi-periodic form. This work is not finished yet and we shall present the results in a forthcoming paper.

RELATIVISTIC AND LUNAR PERTURBATIONS

In both numerical integrations, we have included some perturbations due to relativity and to the effect of the Moon. These perturbations are simplified models which contains only the first order terms. They do not consider the second order terms which could arise from the products of short period terms. We can check their accuracy by comparison with some more elaborated theories.

For relativity, we consider the first order terms in the motion of the perihelion (Brumberg, 1972; Lestrade and Bretagnon, 1982), limited to the post-newtonian approximation in $1/c^2$. In the isotropic coordinates, it adds to (6) the term:

$$\left.\frac{dz}{dt}\right|_R = \sqrt{-1}\frac{GM_\odot}{c^2} \times \frac{3n}{a} \times \frac{1}{1-e^2} \times z \tag{7}$$

The contribution of this term to the secular variations of the elliptical elements z_i, ζ_i are directly estimated by the comparison of the solution with and without the relativistic term (7). This contribution is given up to t^6 for the inner planets in Table 1. For comparison, we put in Table 2 the complete secular terms for the same variables, derived from our theory and VSOP82 (Laskar, 1985b). In Table 1, we also put the term in t and in t^2 computed by (Lestrade and Bretagnon, 1982). The constants they used were not exactly the same, and the coefficients of t^2 contains the second order terms coming from the products of newtonian and relativistic short period terms which do not exist in our solution. The discrepencies between the two solutions are very small and lead to differences smaller than 2×10^{-7} after 10 000 years in the eccentricities of Mercury and Mars, which is under our level of accuracy. Besides, the high powers of the relativistic contributions could be used to improve a newtonian solution.

The perturbation of the Moon is limited to a single constant term (Bretagon, 1984c):

$$\left.\frac{dz_3}{dt}\right|_L = \sqrt{-1}\delta_L\, z_3 \tag{8}$$

where $\delta = 3.192472 \times 10^{-7}$.

The same computations as with the relativity are made in this case and the results are presented in Table 3. The comparisons of the coefficients of t and t^2 are made with the results of (Bretagon, 1984b) which computed the same quantities using the complete theory of the Moon ELP-2000/82 (Chapront-Touzé, Chapront, 1983), and his planetary theory VSOP82. His values includes the contribution of the short period terms in the secular terms of the Moon (the comparison cannot be made with the terms in t^3 of (Bretagon, 1984b) which are not complete). In this case also, the agreement is good, despite the very simple form of (8).

CONCLUSION

An accurate secular theory of the 8 planets must include the relativistic and lunar perturbations. Their first order expression given in (7) and (8) are sufficient with respect to the actual level of precision of our theory. An improved theory should probably include the second order terms from the relativity, but principally a more elaborated representation of the lunar perturbations on the Earth-Moon center of mass.

	$k \times 10^{10}$	$h \times 10^{10}$	$q \times 10^{10}$	$p \times 10^{10}$
	MERCURY			
t	−41 825 896	9 306 191	0	0
	−41 826 163	9 306 388		
t^2	−2 522 666	−11 195 643	218 161	−362 819
	−2 520 879	−11 191 877	219 308	−362 065
t^3	1 497 089	−456 098	129 364	101 391
t^4	47 009	151 595	−38 383	23 255
t^5	−11 559	5 896	2 841	−9 469
t^6	−3 252	336	2 467	1 975
	VENUS			
t	−211 863	−187 861	0	0
	−211 875	−187 857		
t^2	139 204	146 492	−7 084	9 087
	138 041	146 104		
t^3	−83 173	32 988	−1 085	−2 811
t^4	−5 071	−22 829	1 625	769
t^5	5 865	335	−549	321
t^6	69	11	−195	−343
	EARTH			
t	−303 062	−69 618	0	0
	−303 066	−69 617		
t^2	66 348	−133 624	−947	2 750
	65 664	−132 712		
t^3	56 321	24 339	−1 485	−607
t^4	−6 409	17256	76	−639
t^5	−4 876	−1 706	110	150
t^6	446	−445	−10	124
	MARS			
t	248 226	559 105	0	0
	248 226	559 103		
t^2	−448 110	248 285	−516	2 048
	−446 324	247 322		
t^3	−108 734	−184 672	−2 558	−399
t^4	50 478	−31 382	−159	−2 204
t^5	6 041	9 807	1 474	−283
t^6	−1532	1 222	331	736

Table 1. Contribution of the relativity in the secular terms (Eq.7). The values in oldstyle are the corresponding values obtained by Lestrade and Bretagnon (1982), which include the second order terms due to the product of short period terms. The time t is measured in units of 10 000 julian years from J2000 (JD 2 451 545.0).

	$k \times 10^{10}$	$h \times 10^{10}$	$q \times 10^{10}$	$p \times 10^{10}$
		MERCURY		
	2 007 233 137	406 156 338	456 355 046	
t	− 552 114 624	143 750 118	65 433 117	− 127 633 657
t^2	− 18 607 467	− 79 744 997	− 10 713 296	− 9 134 193
t^3	7 904 951	− 3 043 725	2 245 279	1 898 818
t^4	589 540	811 285	− 376 780	− 640 089
t^5	− 156 482	− 78 243	− 30 978	− 25 951
t^6	− 52 991	27 580	10 508	47 156
		VENUS		
	− 44 928 213	50 668 473	68 241 014	288 228 577
t	31 259 019	− 36 121 239	138 133 826	− 40 384 791
t^2	6 041 681	18 468 752	− 10 909 716	− 62 328 916
t^3	− 6 834 889	328 049	− 18 641 793	2 473 042
t^4	493 964	− 613 650	601 726	4 228 784
t^5	597 550	− 168 598	746 057	− 57 042
t^6	− 109 138	− 123 616	− 40 592	− 116 943
		EARTH		
	− 37 408 165	162 844 766	0	0
t	− 82 266 699	− 62 030 259	− 113 469 002	10 180 391
t^2	27 626 329	− 33 829 810	12 372 674	47 020 439
t^3	11 695 572	8 510 121	12 654 170	− 5 417 367
t^4	− 2 695 722	2 770 542	− 1 371 808	− 2 507 948
t^5	− 715 070	− 467 407	− 320 334	463 486
t^6	218 146	− 62 395	5 072	56 431
		MARS		
	853 656 025	− 378 997 324	104 704 257	122 844 931
t	376 330 152	624 657 465	17 138 526	− 108 020 083
t^2	− 246 579 527	155 295 878	− 40 776 201	− 19 221 776
t^3	− 36 760 524	− 63 487 894	− 13 883 445	8 718 504
t^4	11 112 422	− 6 592 895	916 176	3 090 121
t^5	259 071	729 862	1 759 071	37 687
t^6	7 855	113 707	112 984	8 722

Table 2. Secular terms for the inner planets. The time t is measured in units of 10 000 julian years from J2000 (JD 2 451 545.0).

	$k \times 10^{10}$	$h \times 10^{10}$	$q \times 10^{10}$	$p \times 10^{10}$
		MERCURY		
t	0	0	0	0
t^2	−1 710	5 690	6	58
	−1 712	5 695		
t^3	−114	−548	−173	−78
t^4	151	−568	−6	−105
t^5	−126	−130	1	8
t^6	−40	−62	15	23
		VENUS		
t	0	0	0	0
t^2	−18 608	72 095	−233	−174
	−18 480	71 366		
t^3	−4 4274	−13 952	636	274
t^4	8 931	−10 568	−149	217
t^5	4 594	109	−329	−203
t^6	−1 288	−276	99	−135
		EARTH		
t	−519 878	−119 425	0	0
	−518 985	−119 630		
t^2	136 257	−290 016	−105	236
	135 259	−287 427		
t^3	101 735	57 019	−402	−305
t^4	−17 564	25 773	336	−160
t^5	−6 293	−3 225	196	243
t^6	1 146	−739	−138	75
		MARS		
t	0	0	0	0
t^2	−3 584	22 600	982	−293
	−3 599	22 505		
t^3	−14 815	−3 170	−697	171
t^4	1 616	−5 460	−904	−78
t^5	1 346	89	243	−299
t^6	−108	226	267	85

Table 3. Contribution of the Moon in the secular terms (Eq.8). The values in oldstyle are the corresponding values obtained by Bretagnon (1984), which include the second order terms due to the product of short period terms. The time t is measured in units of 10 000 julian years from J2000 (JD 2 451 545.0).

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