MEASURABLE MAJORANTS IN L^1

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Given a probability space $(X, \mathcal{F}, \mu)$ and a σ-algebra $A \subset \mathcal{F}$, arguably the most powerful tool in gaining information about an $\mathcal{F}$-measurable function f from restricted knowledge of $\mathcal{A}$-measurability is that of the conditional expectation $E(f \mid \mathcal{A})$; written $E^{\mathcal{A}}f$ throughout the remainder of this note. Two properties of conditional expectation that may be exploited to gain information, but which also limit conditional expectation's use are the following.

(i) If v is a probability measure mutually absolutely continuous with respect to μ, then the conditional expectation described in terms of v will not in general be the same as the one developed in terms of μ.

(ii) $E^{\mathcal{A}}f$ derived its uses from the idea that it represents f *on the average* with respect to $\mathcal{A}$. Specifically, for each $A \in \mathcal{A}$, $\int_A f\,d\mu = \int_A E^{\mathcal{A}}f\,d\mu$. This means that except when f is $\mathcal{A}$-measurable, $E^{\mathcal{A}}f$ and f are never related by a pointwise inequality, and conditional expectation is of limited value in making pointwise estimates to the value of a function. In this note we shall examine the concept of measurable majorants of nonnegative L^1 functions. This concept has a source in the study of subinvariant functions for Markov operators. Also, recently, C. Akermann and N. Weaver have explored a similar behavior for nested von Neumann algebras. Before beginning our analysis of majorants we present a statement of the results from these diverse fields relevant to the present discussion. An excellent source for the study of Markov operators is [**4**]. The new development by Akermann and Weaver is found in [**1**].

PROPOSITION [**1**]. *Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras, let $\Phi : \mathcal{M} \to \mathcal{N}$ be a faithful conditional expectation, and let $x \in \mathcal{M}$ be positive. Then the sequence $\Phi(x^n)^{1/n}$ converges in the strong operator topology to x_+, a minimal majorant of x in $\mathcal{N}$.*

A result similar to the preceding one is presented below (Lemma 2.1). Our emphasis is in the L^1 setting, and the operator algebra point of view will not be stressed in this article. The measure theoretic setting of Markov operators is directly related to this investigation.

Let $(X, \mathcal{F}, \mu)$ be a complete probability space. Given a linear transformation Q on $L^1(X, \mathcal{F}, \mu)$ which is a Markov operator (i.e. *positive*: $f \geq 0$ a.e. $\Rightarrow Qf \geq 0$ a.e. and *contractive*: $\int_X |Qf|\,d\mu \leq \int_X |f|\,d\mu$ for all $f \in L^1$), and given a measurable function g for which $0 \leq g \leq 1$ a.e., there is a function g_0 which is minimal with respect to the properties

(i) $g \leq g_0 \leq 1$ a.e.,

(ii) $Q^* g_0 \leq g_0$ a.e. (*subinvariance* for Q^*)

(see [**4**, p. 19]). Now suppose that $\mathcal{A}$ is a sub-σ-algebra of $\mathcal{F}$ and let $E^{\mathcal{A}}$ denote the conditional expectation operator with respect to $\mathcal{A}$ on $L^1(X, \mathcal{F}, \mu)$. Then $E^{\mathcal{A}}$ is a rather basic Markov operator, and its adjoint is its restriction to L^∞, which we shall also denote by $E^{\mathcal{A}}$. But $E^{\mathcal{A}}$ has no subinvariant nonegative functions (in L^1 or L^∞) which are not invariant, i.e., if $f \geq 0$ and $E^{\mathcal{A}}f \leq f \in L^1$, then $E^{\mathcal{A}}f = f$. Also, f is $\mathcal{A}$ measurable if and only if $E^{\mathscr{A}}f = f$. Thus the subinvariance result mentioned above takes on the following form for conditional expectations.

PROPOSITION. *Let $g \in L^{\infty}_{+}(X, \mathscr{F})$. Then there is an $\mathscr{A}$ measurable function $g^{\mathscr{A}}$ such that $g^{\mathscr{A}}$ is minimal with respect to the property $g \leq g^{\mathscr{A}} \leq \|g\|_{\infty}$.*

In this note we shall examine the properties of this measurable majorant. We shall actually examine the existence and properties of this majorant for nonnegative L^1 functions. Unlike the L^{∞} case, the majorant need not be finite a.e. This analysis will then be applied to the question of existence of nonegative generators for the kernel of a conditional expectation, and to the classification of certain operator order ideals.

1. Notation and terminology. All functions and sets encountered are by assumption or construction $\mathscr{F}$ measurable. When a function's existence is determined by an argument outside the usual measure-theoretic countable limit family of constructions, care will be taken to ascertain its measurability.

All function and set statements are to be interpreted as holding up to a μ-null set. In particular, statements such as "$S = T$" should be understood as "the symmetric difference of S and T has measure 0".

$\{f \geq \alpha\}$ is slang for $\{x \in X : f(x) \geq \alpha\}$, etc.

L^p_+ refers to those L^p functions which are nonnegative.

For a σ-algebra $\mathscr{S}$, $\mathscr{S}_+$ is the collection of all sets in $\mathscr{S}$ of positive measure.

$L^1(\mathscr{F}) = L^1(X, \mathscr{F}, \mu)$, etc.

For a given measurable function g we choose a measurable set suppt. g so that $g \neq 0$ a.e. on suppt. g and $g = 0$ a.e. off suppt. g. At no time in this article will supports be employed for more than a countable number of functions simultaneously.

2. Some results in this section are more or less well known results about conditional expectations. In some cases the proofs are included because the results are not routinely stated in many reference texts on conditional expectation. The statements of these results will be preceded by the symbol #.

2.1. LEMMA. *Let $f \in L^1_+(\mathscr{F})$. Then $(E^{\mathscr{A}}(f^n))^{1/n}$ is increasing* a.e.

Proof. Apply the conditional form of Hölder's inequality with $p = (n+1)/n$ and $q = n+1$:

$$E^{\mathscr{A}}f^n \leq (E^{\mathscr{A}}(f^n)^{(n+1)/n})^{n/(n+1)} \cdot (E^{\mathscr{A}}1)^{1/(n+1)} = (E^{\mathscr{A}}(f^{n+1}))^{n/(n+1)}. \quad \square$$

2.2. DEFINITION. Let $f \in L^1_+(\mathscr{F})$. Then

$$f^{\mathscr{A}} = \lim(E^{\mathscr{A}}(f^n))^{1/n}.$$

2.3. REMARKS. Note that $f^{\mathscr{A}}$ may be infinite on sets of positive measure. Indeed, any of the conditional moments $E^{\mathscr{A}}f^n$ for $n \geq 2$ may be infinite on sets of positive measure. In any case, $f^{\mathscr{A}}$ is a pointwise limit of $\mathscr{A}$-measurable functions and so it is $\mathscr{A}$ measurable as well.

In the special case that $\mathscr{A}$ is the trivial σ-algebra consisting of sets of measure 0 or 1 only $E^{\mathscr{A}}g = \int_X g\, d\mu$. In this case $\mathscr{A}$ measurable functions are constant, and Lemma 2.1 and Definition 2.2 combine to yield the classic statement.

For each measurable function f, $\|f\|_p$ is an increasing function of p, and $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$.

At this point it would be more accurate to use the notation $f^{\mathscr{A},\mu}$ since the construction depends on conditional expectation, which in turn depends not just on the σ-algebra, but on the measure as well. We shall see however, that the definition yields the same function for all equivalent measures.

Note that for a nonnegative function f, both f and $E^{\mathscr{A}}f$ have precisely the same integral over X. It follows that unless f is $\mathscr{A}$ measurable, both $\{f \geq E^{\mathscr{A}}f\}$ and $\{f \leq E^{\mathscr{A}}f\}$ have positive measure. The next result shows that $f^{\mathscr{A}} \geq f$. The proof is essentially a conditional form of the proof of Chebychev's inequality.

2.4. PROPOSITION. *Let $f \in L^1_+(\mathscr{F})$. Then $f \leq f^{\mathscr{A}}$, and if g is $\mathscr{A}$ measurable with $f \leq g$, then $f^{\mathscr{A}} \leq g$.*

Proof. Let $\epsilon > 0$ and let $A = \{f^{\mathscr{A}} < \infty\}$. Then $A \in \mathscr{A}$. Let $T = \{f > f^{\mathscr{A}} + \epsilon\}$. Then $T \subset A$ and

$$E^{\mathscr{A}}\chi_T \leq E^{\mathscr{A}}\left(\left(\frac{f}{f^{\mathscr{A}}+\epsilon}\right)^n\right)\cdot \chi_A = \frac{1}{(f^{\mathscr{A}}+\epsilon)^n}\cdot E^{\mathscr{A}}(f^n)\cdot \chi_A.$$

Now for any nonnegative function g, $g^{1/n} \to \chi_{\text{suppt.}\, g}$ a.e., so that

$$\chi_{\text{suppt.}\, E^{\mathscr{A}}\chi_T} \leq \frac{f^{\mathscr{A}}}{f^{\mathscr{A}}+\epsilon}\cdot \chi_A < 1.$$

Thus μ (suppt. $E^{\mathscr{A}}\chi_T) = 0$, and consequently $\chi_T = 0$; that is $\mu(T) = 0$. But ϵ was chosen arbitrarily, and so $f \leq f^{\mathscr{A}}$.

Now suppose that g is $\mathscr{A}$ measurable and $f \leq g$. Then for each $n \geq 1$, $f^n \leq g^n$, and so, since $E^{\mathscr{A}}g^n = g^n$, we obtain

$$(E^{\mathscr{A}}f^n)^{1/n} \leq (E^{\mathscr{A}}g^n)^{1/n} = g.$$

It then follows from the definition of $f^{\mathscr{A}}$ that $f^{\mathscr{A}} \leq g$. ☐

2.5. COROLLARY. *Let ν be a finite measure equivalent to μ. Then for every nonnegative function f, $f^{\mathscr{A},\nu} = f^{\mathscr{A},\mu}$.*

Proof. By the equivalence of the measures, "a.e." is unambiguous. Since $f \leq f^{\mathscr{A},\mu}$, we have $f^{\mathscr{A},\nu} \leq f^{\mathscr{A},\mu}$ a.e., and conversely. ☐

In light of 2.4 and 2.5 we shall refer to $f^{\mathscr{A}}$ as the $\mathscr{A}$ *majorant* of f.

EXAMPLE 1. Consider $f = \chi_F$ for $F \in \mathscr{F}$. Then

$$(E^{\mathscr{A}}(\chi_F)^n)^{1/n} = (E^{\mathscr{A}}\chi_F)^{1/n} \to \chi_{\text{suppt.}\, E^{\mathscr{A}}\chi_F};$$

that is

$$(\chi_F)^{\mathscr{A}} = \chi_{\text{suppt.}\, E^{\mathscr{A}}\chi_F}.$$

Thus the support of the conditional probability, $E^{\mathscr{A}}\chi_F$, of F is the smallest $\mathscr{A}$ set containing F.

EXAMPLE 2. Suppose that $\mathcal{A}$ is generated by a finite or countable partition $\{A_i\}$ of X. Then

$$E^{\mathcal{A}}f = \sum_i \frac{1}{\mu(A_i)} \cdot \left(\int_{A_i} f\, d\mu\right) \cdot \chi_{A_i},$$

and so

$$(E^{\mathcal{A}}f^n)^{1/n} = \sum_i \left(\frac{1}{\mu(A_i)}\right)^{1/n} \cdot \left(\int_{A_i} f^n\, d\mu\right)^{1/n} \cdot \chi_{A_i}.$$

Consequently,

$$f^{\mathcal{A}} = \sum_i \|f\chi_{A_i}\|_\infty \cdot \chi_{A_i}.$$

EXAMPLE 3. Let $d\mu = \frac{1}{2}\, dx$ on $X = [-1, 1]$, $\mathcal{F}$ the Lebesgue sets in X, and let $\mathcal{A}$ be the σ-algebra generated by the intervals $(-a, a)$ for $a \in (0, 1)$. Then

$$E^{\mathcal{A}}f(x) = \frac{f(x) + f(-x)}{2}.$$

For any positive numbers s and t, $(s^n + t^n)^{1/n} \to \max\{s, t\}$. (Note that a von Neumann algebra form of this innocent fact plays a central role in [**1**].) It follows that

$$f^{\mathcal{A}}(x) = \max\{f(x), f(-x)\}.$$

2.6. DEFINITION. For $f \geq 0$, the *$\mathcal{A}$ minorant of f* is defined as

$$f_{\mathcal{A}} = \frac{1}{(1/f)^{\mathcal{A}}},$$

where the obvious interpretation is to be used when the denominator is 0 or ∞.

One may verify in a routine manner that $f_{\mathcal{A}} \leq f$ and, if g is a nonnegative $\mathcal{A}$ measurable function with $g \leq f$, then $g \leq f_{\mathcal{A}}$. Also, since $\{f^{\mathcal{A}} < \infty\} \in \mathcal{A}$, it is easily seen that

$$f_{\mathcal{A}} = (f^{\mathcal{A}} - (f^{\mathcal{A}} - f)^{\mathcal{A}}) \cdot \chi_{\{f^{\mathcal{A}} < \infty\}}.$$

The properties of $\mathcal{A}$ majorants listed in the following proposition may all be proved by using the minimality of $f^{\mathcal{A}}$ among all $\mathcal{A}$ measurable $g \geq f$ a.e.

2.7. PROPOSITION. *Let f and g be nonnegative $\mathcal{F}$ measurable functions and let a and b be nonnegative $\mathcal{A}$ measurable functions. Then the following results hold.*

(i) $(f + g)^{\mathcal{A}} \leq f^{\mathcal{A}} + g^{\mathcal{A}}$.
(ii) $(f \cdot g)^{\mathcal{A}} \leq f^{\mathcal{A}} \cdot g^{\mathcal{A}}$.
(iii) $(a \cdot f + b)^{\mathcal{A}} = a \cdot f^{\mathcal{A}} + b$.
(iv) *If $\mathcal{B}$ is a σ-algebra with $\mathcal{B} \subset \mathcal{A}$, then $f^{\mathcal{A}} \leq f^{\mathcal{B}}$.*

As noted earlier, μ may be replaced by any equivalent finite measure without affecting $f^{\mathcal{A}}$. We shall exploit this fact to study and apply the concept of $\mathcal{A}$-majorant. For convenience, define $\mathcal{E}(\mu)$ as the set of all probability measures mutually absolutely continuous with μ. For $\nu \in \mathcal{E}(\mu)$, $E^{\mathcal{A}}_\nu$ is defined to be the conditional expectation with respect to $\mathcal{A}$ and ν. The symbol $E^{\mathcal{A}}$ will be reserved for the case where the measure used is μ.

Suppose that ν is any finite measure with $\nu \ll \mu$, and let $f \geq 0$. Then since $f \leq f^{\mathcal{A}} \in \mathcal{A}$, the following is true.

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2.8. LEMMA. *For every finite $\nu \ll \mu$, $E_\nu^{\mathscr{A}} f \le f^{\mathscr{A}}$.*

We shall make repeated use of the following change of measure formula for conditional expectations.

2.9. PROPOSITION. *Let $\nu \ll \mu$. Then*

$$E_\nu^{\mathscr{A}} f = \frac{E^{\mathscr{A}}\left(\frac{d\nu}{d\mu}\cdot f\right)}{E^{\mathscr{A}}\left(\frac{d\nu}{d\mu}\right)} \cdot \chi_{\text{suppt.}(d\nu/d\mu)}.$$

Proof. We shall make use of the following facts.

(i) suppt. $u \subset$ suppt. $E^{\mathscr{A}} u$, for every nonnegative function u;

(ii) suppt. $E^{\mathscr{A}}(u\,.\,f) \subset$ suppt. $E^{\mathscr{A}} u$, for any nonnegative u and f.

Let $u = \dfrac{d\nu}{d\mu}$ and let $f \in L^1(\nu)$. Then for $A \in \mathscr{A}$, we have

$$\begin{aligned}\int_A E_\nu^{\mathscr{A}} f\,d\nu &= \int_A f\,d\nu \\ &= \int_A u\,.\,f\,d\mu = \int_A E^{\mathscr{A}}(u\,.\,f)\,d\mu \\ &= \int_A (E^{\mathscr{A}}(u\,.\,f))\,.\,\chi_{\text{suppt. } E^{\mathscr{A}} u}\,d\mu \\ &= \int_A (E^{\mathscr{A}}(u\,.\,f))\,.\,\chi_{\text{suppt. } E^{\mathscr{A}} u}\frac{1}{u}\,d\nu \\ &= \int_A (E^{\mathscr{A}}(u\,.\,f))\,.\,\chi_{\text{suppt. } E^{\mathscr{A}} u}\,.\,E_\nu^{\mathscr{A}}\left(\frac{1}{u}\right)d\nu.\end{aligned}$$

As the first and last integrands in this chain are $\mathscr{A}$-measurable,

$$E_\nu^{\mathscr{A}} f = (E^{\mathscr{A}}(u\,.\,f))\,.\,\chi_{\text{suppt. } E^{\mathscr{A}} u}\,.\,E_\nu^{\mathscr{A}}\left(\frac{1}{u}\right).$$

Temporarily letting $f = 1$, we see that

$$\chi_{\text{suppt. } E^{\mathscr{A}} u}\,.\,E^{\mathscr{A}}\left(\frac{1}{u}\right) = \frac{1}{E^{\mathscr{A}} u}\cdot\chi_{\text{suppt. } E^{\mathscr{A}} u}.$$

Using this in the penultimate displayed equation leads to the desired conclusion. □

As a special case, note that if ν is a finite measure equivalent to μ, then

$$E_\nu^{\mathscr{A}} f = \frac{E^{\mathscr{A}}\left(\frac{d\nu}{d\mu}\cdot f\right)}{E^{\mathscr{A}}\left(\frac{d\nu}{d\mu}\right)} = E^{\mathscr{A}}\left(\frac{d\nu/d\mu}{E^{\mathscr{A}}(d\nu/d\mu)}\cdot f\right),$$

and of course $E^{\mathscr{A}}\left(\dfrac{d\nu/d\mu}{E^{\mathscr{A}}(d\nu/d\mu)}\right)=1$ a.e. On the other hand, if $E^{\mathscr{A}}u=1$ a.e. for $u>0$, then for $d\nu=u\,d\mu$, $E^{\mathscr{A}}_{\nu}f=E^{\mathscr{A}}(u\,.\,f)$.

2.10. Lemma. *Let f be a strictly positive function in $L^1(X,\mathscr{F},\mu)$. Then $\left\{\dfrac{E^{\mathscr{A}}f^{n+1}}{E^{\mathscr{A}}f^n}\right\}$ is increasing a.e. and*

$$\lim_{n\to\infty}\frac{E^{\mathscr{A}}f^{n+1}}{E^{\mathscr{A}}f^n}=f^{\mathscr{A}}\text{ a.e.}$$

Proof. By the conditional expectation version of the Cauchy–Schwarz inequality, we have

$$\begin{aligned}(E^{\mathscr{A}}f^{n+1})^2&=(E^{\mathscr{A}}(\sqrt{f^{n+2}}\,.\,\sqrt{f^n}))^2\\&\le(E^{\mathscr{A}}f^{n+2})\,.\,(E^{\mathscr{A}}f^n);\end{aligned}$$

hence $\dfrac{E^{\mathscr{A}}f^{n+1}}{E^{\mathscr{A}}f^n}\le\dfrac{E^{\mathscr{A}}f^{n+2}}{E^{\mathscr{A}}f^{n+1}}$, which establishes the stated monotonicity. Let F be the pointwise limit of $\dfrac{E^{\mathscr{A}}f^{n+1}}{E^{\mathscr{A}}f^n}$. Then F is $\mathscr{A}$-measurable and (putting $n=0$) $F\ge E^{\mathscr{A}}f$. If, for some $n\ge0$ we have $F^n\ge E^{\mathscr{A}}f^n$, then

$$\begin{aligned}F^{n+1}\ge(E^{\mathscr{A}}f^n)\,.\,F&\ge(E^{\mathscr{A}}f^n)\,.\,\frac{(E^{\mathscr{A}}f^{n+1})}{(E^{\mathscr{A}}f^n)}\\&=E^{\mathscr{A}}f^{n+1}.\end{aligned}$$

This shows that $F\ge\lim_{n\to\infty}(E^{\mathscr{A}}f^n)^{1/n}=f^{\mathscr{A}}$. But for $d\nu_n=\dfrac{f^n}{E^{\mathscr{A}}f^n}\,d\mu$,

$$\begin{aligned}f^{\mathscr{A}}\ge E^{\mathscr{A}}_{\nu_n}f&=E^{\mathscr{A}}\left(\frac{f^n}{E^{\mathscr{A}}f^n}\cdot f\right)\\&=E^{\mathscr{A}}\left(\frac{f^{n+1}}{E^{\mathscr{A}}f^n}\right)\\&=\frac{E^{\mathscr{A}}f^{n+1}}{E^{\mathscr{A}}f^n}\to F,\end{aligned}$$

and consequently $f^{\mathscr{A}}=\lim\dfrac{E^{\mathscr{A}}f^{n+1}}{E^{\mathscr{A}}f^n}$. □

Given a set of $\mathscr{A}$-measurable functions $\mathscr{S}$ we will say that the $\mathscr{A}$-measurable function f is the *essential supremum of* $\mathscr{S}$; $f=\text{ess.}\sup\mathscr{S}$, if for every $g\in\mathscr{S}$, $f\ge g$ a.e. and if h is any $\mathscr{A}$-measurable function dominating all the members of $\mathscr{S}$, then $h\ge f$ a.e. We then may rephrase the equality established in Lemma 2.10 as follows.

2.11. Lemma. *Let $f\in L^1_+(X,\mathscr{F},\mu)$. Then*

$$f^{\mathscr{A}}=\text{ess.}\sup\{E^{\mathscr{A}}_{\nu}f:\nu\in\mathscr{E}(\mu)\}.$$

Proof. Using the ν_n's from Lemma 2.10, we see that the essential supremum is greater than or equal to $f^{\mathscr{A}}$ a.e. But we have also seen that $f^{\mathscr{A}} \geq E_\nu^{\mathscr{A}} f$, for every ν in $\mathscr{E}(\mu)$. □

3. Two applications. Conditional expectations are particularly pleasant projections on an L^1 space in that they are positive, contractive, and their ranges are L^1 subspaces of the original space. This point of view was used and developed extensively, notably in [**2**], [**6**] and [**3**]. However, the complementary projection $I - E^{\mathscr{A}}$ is only positive under rather restrictive conditions, and its range is far from being an L^1 space. Indeed the kernel of $E^{\mathscr{A}}$ generally possesses few of the algebraic amenities of an L^1 space. If $f \in \ker E^{\mathscr{A}}$ and $f \geq 0$ then $f = 0$; products of members of the kernel bear no special relationship to the kernel; etc. We will show now that the behavior of majorants can give at least one useful bit of information about these kernels. First note that $\ker E^{\mathscr{A}} = \{f - E^{\mathscr{A}} f : f \in L^1(\mathscr{F})\}$. Of course, there is nothing unique about representing a member of the kernel of $E^{\mathscr{A}}$ as $f - E^{\mathscr{A}} f$. Indeed, if g is an $\mathscr{A}$-measurable function for which $E^{\mathscr{A}} g$ exists and is finite a.e., then $f - E^{\mathscr{A}} f = (f + g) - E^{\mathscr{A}}(f + g)$. But this is as bad as it gets. If $f - E^{\mathscr{A}} f = h - E^{\mathscr{A}} h$, then $f - h = E^{\mathscr{A}}(f - h)$, which is $\mathscr{A}$-measurable. We ask whether, for a given real valued function $k \in \ker E^{\mathscr{A}}$, there is a nonnegative function p such that $k = p - E^{\mathscr{A}} p$.

3.1. PROPOSITION. *Let h be a real-valued member of* $\ker E^{\mathscr{A}}$. *Then there exists a nonnegative function p for which $h = p - E^{\mathscr{A}} p$ if and only if $(h^-)^{\mathscr{A}} < \infty$ a.e. In this case the function $p_0 = h + (h^-)^{\mathscr{A}}$ is the minimal nonnegative function for which $h = p_0 - E^{\mathscr{A}} p_0$.*

Proof. Suppose that $h \in \ker E^{\mathscr{A}}$, $f \geq 0$, $E^{\mathscr{A}} f < \infty$, and $h = f - E^{\mathscr{A}} f$. We may then write $f = h + a$, for some $\mathscr{A}$-measurable function a. Since $E^{\mathscr{A}} h = 0$ a.e., $a = E^{\mathscr{A}} f$; and since $f \geq 0$, it follows that $E^{\mathscr{A}} f \geq 0$, hence $a \geq 0$. But $h + a = f \geq 0$ so that $a \geq -h$ and $a \geq 0$; i.e., $a \geq h^-$. Thus $(h^-)^{\mathscr{A}} \leq a < \infty$. This also establishes the minimality of $h + (h^-)^{\mathscr{A}}$.

Conversely, suppose that $h \in \ker E^{\mathscr{A}}$ and $(h^-)^{\mathscr{A}} < \infty$ a.e. Then

$$h + (h^-)^{\mathscr{A}} \geq h + h^- = h^+ \geq 0$$

and

$$(h + (h^-)^{\mathscr{A}}) - E^{\mathscr{A}}(h + (h^-)^{\mathscr{A}}) = h. \quad \square$$

The preceding discussion was concerned with finiteness (a.e.) of majorants. The following material is concerned with having majorants in L^1.

For this illustration of the role of majorants, consider

$$\mathscr{K} = \{f : f \,.\, L^1(\mathscr{A}) \subset L^1(\mathscr{F})\}.$$

This space was studied in [**5**]. Its properties relevant to the present discussion are listed below.

3.2. PROPOSITION. ([**5**]).

(i) $L^\infty(\mathscr{F}) \subset \mathscr{K} \subset L^1(\mathscr{F})$.

(ii) *$f \in \mathscr{K}$ if and only if $E^{\mathscr{A}}|f| \in L^\infty$. Also $\|E^{\mathscr{A}}|f|\|_\infty$ defines a Banach space norm on $\mathscr{K}$.*

(iii) *$\mathscr{K}$ is an order ideal: $f \in \mathscr{K}$ and $|g| \leq |f| \Rightarrow g \in \mathscr{K}$.*

(iv) *(Extreme case.) $\mathscr{K} = L^1(\mathscr{F})$ if and only $\mathscr{A}$ is generated by a finite partition of X.*

(v) (*Extreme case.*) $\mathcal{K} = L^\infty(\mathcal{F})$ *if and only if there is a constant* C *such that* $|f| \le C \,.\, E^{\mathcal{A}}|f|$, *for every* $f \in L^1(\mathcal{F})$.

We will now examine this last condition with respect to $\mathcal{A}$ majorants. First note that for each order ideal such as $\mathcal{K}$, there is a *dual order ideal* $\mathcal{K}'$ defined as $\mathcal{K}' = \{f : f\,.\,\mathcal{K} \subset L^1(\mathcal{F})\}$. It follows from the general theory of order ideals that $\mathcal{K} = L^1(\mathcal{F})$ if and only if $\mathcal{K}' = L^\infty(\mathcal{F})$.

3.3. PROPOSITION. $\mathcal{K} = \{f : |f|^{\mathcal{A}} \in L^1\}$. *In particular* $\mathcal{K}' = L^1(\mathcal{F})$ *if and only if* $|f|^{\mathcal{A}} \in L^1(\mathcal{A})$, *for every* $f \in L^1(\mathcal{F})$.

Proof. For each $f \in \mathcal{K}'$, let $K_f' : \mathcal{K} \to L^1$ be the linear transformation of multiplication by f. Then a routine use of the closed graph theorem shows that each K_f' is continuous (with respect to the norm on $\mathcal{K}$ mentioned above). Suppose that $|f|^{\mathcal{A}} \in L^1$. Then, for each $g \in \mathcal{K}$,

$$\begin{aligned}\int_X |f\cdot g|\, d\mu &\le \int_X |f|^{\mathcal{A}}\,.\,|g|\, d\mu \\ &= \int_X |f|^{\mathcal{A}}\,.\,E^{\mathcal{A}}|g|\, d\mu \\ &\le \|\,|f|^{\mathcal{A}}\|_1\,.\,\|E^{\mathcal{A}}|g|\,\|_\infty,\end{aligned}$$

so that $f \in \mathcal{K}'$.

Conversely, suppose that $f \in \mathcal{K}'$, let $c = \|K_f'\|$, and let $A = \text{suppt.}\, E^{\mathcal{A}}|f|$. Then, via Lemma 2.10 and the monotone convergence theorem,

$$\int_X \frac{|f|^{n+1}}{E^{\mathcal{A}}|f|^n}\cdot \chi_A\, d\mu = \int_X \frac{E^{\mathcal{A}}|f|^{n+1}}{E^{\mathcal{A}}|f|^n}\chi_A\, d\mu \uparrow \int_X |f|^{\mathcal{A}}\, d\mu.$$

But

$$\begin{aligned}\int_X \frac{|f|^{n+1}}{E^{\mathcal{A}}|f|^n}\cdot \chi_A\, d\mu &= \int_X f\cdot \frac{|f|^n}{E^{\mathcal{A}}|f|^n}\cdot \chi_A\, d\mu, \\ &\le c,\end{aligned}$$

so that $|f|^{\mathcal{A}} \in L^1$. □

As a specific example of $\mathcal{A} \subset \mathcal{F}$ where $\mathcal{K}' = L^1$, note that if $E^{\mathcal{A}}f(x) = \dfrac{f(x)+f(-x)}{2}$ for $f \in L^1([-1,1], dx/2)$, then $|f| \le 2\,.\,E^{\mathcal{A}}|f|$. It is worth noting that whereas the extreme condition "$\mathcal{K} = L^1(\mathcal{F})$" allows an essentially measure independent characterization "$\mathcal{A}$ is generated by a finite partition of X", the same cannot be said for the opposite extreme. Indeed, using the example immediately above, let $dv = (1-x)(dx/2)$. Then $v \in \mathcal{E}(dx/2)$ and

$$\mathcal{E}_v^{\mathcal{A}}f(x) = \frac{(1-x)f(x) + (1+x)f(-x)}{2}.$$

In particular, for any continuous f, $E_v^{\mathcal{A}}f(1) = f(-1)$, and so we cannot bound nonnegative functions by their conditional expectations.

We conclude this note with a brief examination of some of the curious implications of the condition $|f| \leq C \,.\, E^{\mathscr{A}}|f|$, for all $f \in L^1$. Let us assume that this condition holds. Then for each nonnegative f in L^1, $f^{\mathscr{A}} \leq C \,.\, E^{\mathscr{A}}f$. But then for any $\nu \in \mathscr{E}(\mu)$,

$$E_\nu^{\mathscr{A}}f \leq C \,.\, E^{\mathscr{A}}f. \quad (\text{Recall that } E^{\mathscr{A}} = E_\mu^{\mathscr{A}}.)$$

Equivalently, for every strictly positive L^1 function g,

$$\frac{E^{\mathscr{A}}(f \,.\, g)}{E^{\mathscr{A}}g} \leq C \,.\, E^{\mathscr{A}}f.$$

But this shows that

$$E^{\mathscr{A}}(f \,.\, g) \leq C \,.\, (E^{\mathscr{A}}f) \,.\, (E^{\mathscr{A}}g),$$

which quickly leads to

$$E^{\mathscr{A}}(|f \,.\, g|) \leq C \,.\, (E^{\mathscr{A}}|f|) \,.\, (E^{\mathscr{A}}|g|),$$

for all f and g in L^1. σ-algebras $\mathscr{A}$ with this "Hölder's-like inequality" property seem worthy of study and classification. One potentially interesting property of such σ-algebras is seen by first noting that when every f in L_1^+ yields $f^{\mathscr{A}}$ in L_+^1, then of course all such $f^{\mathscr{A}}$ are finite a.e. This shows (via Proposition 3.1) that for every real-valued $h \in \ker E^{\mathscr{A}}$, $h = p - E^{\mathscr{A}}p$ for the nonnegative function $p = h + (h^-)^{\mathscr{A}}$.

3.4. Proposition. *Suppose that* $|f|^{\mathscr{A}} \in L^1$, *for every* f *in* L^1. *Then there is a constant* $d > 0$ *such that, for every real-valued* $h \in \ker E^{\mathscr{A}}$, *we have* $-d \,.\, (h^+)^{\mathscr{A}} \leq h \leq d \,.\, (h^-)^{\mathscr{A}}$.

Proof. Let h be a real valued function in $\ker E^{\mathscr{A}}$, and let $p = h + (h^-)^{\mathscr{A}}$. Further, there is a constant c such that $f \leq c \,.\, E^{\mathscr{A}}f$ a.e., for every nonnegative function f. It follows that

$$h + (h^-)^{\mathscr{A}} = p \leq c \,.\, E^{\mathscr{A}}p = c \,.\, (h^-)^{\mathscr{A}},$$

and so $h \leq d \,.\, (h^-)^{\mathscr{A}}$, where $d = c - 1$. Since the same analysis is applicable to $-h$, we see that

$$-h \leq d \,.\, ((-h)^-)^{\mathscr{A}} = d \,.\, (h^+)^{\mathscr{A}}$$

and so $-d \,.\, (h^+)^{\mathscr{A}} \leq h \leq d \,.\, (h^-)^{\mathscr{A}}$. □

3.5. Proposition. *Suppose that* $|f|^{\mathscr{A}} \in L^1$, *for every* f *in* L^1. *Then there is an integer* N *such that no collection of more than* N *mutually disjoint sets is independent of* $\mathscr{A}$. *Moreover, any function independent from* $\mathscr{A}$ *is a simple function.*

Proof. Let c be a constant such that $|f| \leq c \,.\, E^{\mathscr{A}}|f|$ a.e., for all f. If $F \in \mathscr{F}_+$ is independent of $\mathscr{A}$, then $E^{\mathscr{A}}\chi_F = \mu(F)$ a.e., so that $1 \leq c \,.\, \mu(F)$. If $\{F_i : i \in I\}$ is a collection of mutually disjoint sets in $\mathscr{F}_+$ independent of $\mathscr{A}$, then the cardinality of $I \leq c \,.\, \sum_{i \in I} \mu(F_i) \leq c$.

This establishes the first assertion. For the second, suppose that f is real-valued and independent of $\mathscr{A}$. Then

$$|f| \leq c \,.\, E^{\mathscr{A}}|f| = c \,.\, \int_X |f| \, d\mu,$$

so that $f \in L^\infty$. But the finiteness result just established shows that for any countable partition $\{S_i\}$ of $[-\|f\|_\infty, \|f\|_\infty]$, all but finitely many of the sets $f^{-1}(S_i)$ have measure 0.

This ensures that f is a simple function. This observation extends to complex-valued functions immediately. □

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