5

Conformal geometry

Conformal geometry is concerned with the properties of *angle-preserving* geometric transformations. Conformal geometry, as a branch of differential geometry, has a long story going back to the work of Cotton, Schouten and Weyl; see, for example, Cotton (1899), Schouten (1921) and Weyl (1918, 1968). It remains an active area of research; compare the monograph by Fefferman and Graham (2012).

The approach to the use of conformal methods in general relativity followed in this book goes back to the seminal work by R. Penrose in the 1960s; see Penrose (1963, 1964). Penrose's ideas allowed to reformulate, in a geometric manner, the study of the asymptotic behaviour of the gravitational field. Since then, conformal methods have provided a valuable tool for the analysis of global aspects of the Einstein field equations and their solutions. Conformal methods have also been useful in the construction of exact solutions to the Einstein field equations; see Stephani et al. (2003).

This chapter provides an introduction to the notions of conformal geometry to be used in the later parts of this book. The organisation of this chapter is geared towards applications.

5.1 Basic concepts of conformal geometry

This section discusses the basic notions of conformal geometry that will be used throughout this book.

5.1.1 Conformal rescalings and transformations

The key notion in conformal geometry is that of a ***conformal rescaling***. In what follows, let $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ denote two metrics on a manifold $\tilde{\mathcal{M}}$. The metrics

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$\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ are said to be ***conformally related*** (or simply ***conformal***) to each other if there exists a positive $\Xi \in \mathfrak{X}(\tilde{\mathcal{M}})$ such that

$$\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}. \tag{5.1}$$

The scalar Ξ is called the ***conformal factor***. Throughout this book, the symbol Ξ will be used to denote a generic conformal factor on a four-dimensional manifold.

The conformal rescaling in Equation (5.1) gives rise to an equivalence relation among the set of metrics over $\tilde{\mathcal{M}}$. The ***conformal class*** of a metric $\tilde{\boldsymbol{g}}$, to be denoted by $[\tilde{\boldsymbol{g}}]$, is the collection of metrics conformally related to $\tilde{\boldsymbol{g}}$. A conformal class is also called a ***conformal structure***. From Equation (5.1) it follows that the contravariant metrics $\tilde{\boldsymbol{g}}^\sharp$ and $\boldsymbol{g}^\sharp$ are related by

$$\boldsymbol{g}^\sharp = \Xi^{-2} \tilde{\boldsymbol{g}}^\sharp;$$

that is, $g^{ab} = \Xi^{-2} \tilde{g}^{ab}$, so as to ensure that $\tilde{g}_{ab}\tilde{g}^{bc} = \delta_a{}^c$ and $g_{ab}g^{bc} = \delta_a{}^c$.

Closely related to the notion of conformally related metrics is the concept of conformal transformations. To discuss this idea, let $\tilde{\mathcal{M}}$ and $\mathcal{M}$ denote two manifolds with metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$, respectively. A ***conformal transformation*** (also called ***conformorphism***) is a diffeomorphism $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ such that the pull-back of $\boldsymbol{g}$ is conformal to $\tilde{\boldsymbol{g}}$. That is, one has that

$$\varphi^* \boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}. \tag{5.2}$$

Notice that as φ is a diffeomorphism, then $(\varphi^*)^{-1}$ is well defined and the last expression could have been written, alternatively, as $\boldsymbol{g} = (\varphi^*)^{-1}\left(\Xi^2 \tilde{\boldsymbol{g}}\right)$.

A special case of the previous discussion occurs when $\tilde{\boldsymbol{g}}$ is a flat metric – in the Lorentzian four-dimensional case the Minkowski metric $\boldsymbol{\eta}$ and in the Riemannian three-dimensional case the Euclidean metric $\boldsymbol{\delta}$. In these cases one then says that $\boldsymbol{g}$ is ***conformally flat***. Determining whether a given conformal class contains the flat metric is a classical problem in conformal geometry; see Section 5.2.3.

The conformal group

As before, let $[\tilde{\boldsymbol{g}}]$ denote the conformal class of a Lorentzian metric $\tilde{\boldsymbol{g}}$ on a manifold $\tilde{\mathcal{M}}$. Consider a frame $\{\tilde{\boldsymbol{e}}_{\boldsymbol{a}}\}$ which is orthonormal with respect to $\tilde{\boldsymbol{g}}$. If $\{\tilde{\boldsymbol{\omega}}^{\boldsymbol{a}}\}$ denotes the associated coframe, one has that

$$\tilde{\boldsymbol{g}} = \eta_{\boldsymbol{ab}} \tilde{\boldsymbol{\omega}}^{\boldsymbol{a}} \otimes \tilde{\boldsymbol{\omega}}^{\boldsymbol{b}}, \qquad \text{that is,} \qquad \tilde{\boldsymbol{g}}(\tilde{\boldsymbol{e}}_{\boldsymbol{a}}, \tilde{\boldsymbol{e}}_{\boldsymbol{a}}) = \eta_{\boldsymbol{ab}}.$$

In order to investigate the type of transformations of $\{\tilde{\boldsymbol{\omega}}_{\boldsymbol{a}}\}$ which lead to another metric $\boldsymbol{g} \in [\tilde{\boldsymbol{g}}]$, write

$$\tilde{\boldsymbol{\omega}}^{\boldsymbol{a}} = K^{\boldsymbol{a}}{}_{\boldsymbol{c}} \boldsymbol{\omega}^{\boldsymbol{c}},$$

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with $(K^{\boldsymbol{a}}{}_{\boldsymbol{c}})$ denoting some transformation matrix and $\{\boldsymbol{\omega_a}\}$ another orthonormal frame. The condition on the matrix $(K^{\boldsymbol{a}}{}_{\boldsymbol{c}})$ so that it leads to another member of the conformal class, say, $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$, is then given by

$$\tilde{\boldsymbol{g}} = \eta_{\boldsymbol{ab}} K^{\boldsymbol{a}}{}_{\boldsymbol{c}} K^{\boldsymbol{b}}{}_{\boldsymbol{d}} \boldsymbol{\omega}^{\boldsymbol{c}} \otimes \boldsymbol{\omega}^{\boldsymbol{d}} = \Xi^{-2}\eta_{\boldsymbol{cd}} \boldsymbol{\omega}^{\boldsymbol{c}} \otimes \boldsymbol{\omega}^{\boldsymbol{d}}.$$

The latter expression suggests writing

$$K^{\boldsymbol{a}}{}_{\boldsymbol{b}} = \Xi^{-1}\Lambda^{\boldsymbol{a}}{}_{\boldsymbol{b}},$$

where $(\Lambda^{\boldsymbol{a}}{}_{\boldsymbol{b}})$ is a Lorentz transformation; that is, $\Lambda^{\boldsymbol{a}}{}_{\boldsymbol{c}}\Lambda^{\boldsymbol{b}}{}_{\boldsymbol{d}}\eta_{\boldsymbol{ab}} = \eta_{\boldsymbol{cd}}$. The ***group of (four-dimensional) Lorentz transformations*** will be denoted by $O(1,3)$. It follows that at a point $p \in \tilde{\mathcal{M}}$ the group of transformations taking a $\tilde{\boldsymbol{g}}$-orthonormal frame to a frame which is orthonormal with respect to another metric in the conformal class $[\boldsymbol{g}]$, the so-called ***conformal group*** $CO(1,3)$, is given by $CO(1,3) = \mathbb{R}^+ \times O(1,3)$. The previous discussion can be adapted to the case of three-dimensional Riemannian metrics. In that case, the conformal group, denoted by $CO(3)$, is given by $CO(3) = \mathbb{R}^+ \times O(3)$, where $O(3)$ denotes the ***group of three-dimensional orthogonal transformations (rotations)***.

5.1.2 Conformal extensions and conformal compactifications

If a smooth mapping $\varphi : \tilde{\mathcal{M}} \to \mathcal{M}$ satisfying condition (5.2) is injective but not surjective (i.e. $\varphi(\tilde{\mathcal{M}}) \subsetneq \mathcal{M}$), then one says that $\mathcal{M}$ is a ***conformal extension*** of $\tilde{\mathcal{M}}$. An important type of conformal extensions are the so-called conformal compactifications. A ***conformal compactification*** of a manifold $\tilde{\mathcal{M}}$ with metric $\tilde{\boldsymbol{g}}$ is a conformal transformation $\varphi : \tilde{\mathcal{M}} \to \mathcal{U}$ where $\mathcal{U}$ is a relatively compact (i.e. the closure of $\mathcal{U}$ is compact), connected, open set of a manifold $\mathcal{M}$ such that

$$\boldsymbol{g} = (\varphi^*)^{-1}(\Xi^2\tilde{\boldsymbol{g}}) \quad \text{in } \mathcal{U},$$

with a conformal factor Ξ such that:

(i) $\Xi > 0$ in $\mathcal{U}$.
(ii) $\Xi = 0$ on $\partial\mathcal{U}$, the boundary of the open set $\mathcal{U}$. The set $\partial\mathcal{U}$ is called the ***conformal boundary*** of $\tilde{\mathcal{M}}$.

Examples of conformal extensions will be discussed in Chapter 6.

5.2 Conformal transformation formulae

The discussion of Section 2.4.4 can be applied to obtain the transformation formulae relating the curvature tensors of the Levi-Civita connections $\tilde{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ of two metrics $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ related to each other by Equation (5.1).

5.2.1 Transformation formulae for the connection

As a first step, one needs to find the specific form of the transition tensor $Q_a{}^c{}_b$ – see Equation (2.13). The first observation is that as the connections $\tilde{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are torsion free, it follows from Equation (2.15) that the transition tensor is symmetric; that is, one has that

$$Q_a{}^c{}_b = Q_{(a}{}^c{}_{b)}.$$

Using formula (2.14) one has that

$$\nabla_a g_{bc} - \tilde{\nabla}_a g_{bc} = -Q_a{}^d{}_b g_{dc} - Q_a{}^d{}_c g_{bd}.$$

From $\nabla_a g_{bc} = 0$ and $\tilde{\nabla}_a g_{bc} = \tilde{\nabla}_a(\Xi^2 \tilde{g}_{bc}) = 2\Xi\tilde{\nabla}_a\Xi\tilde{g}_{bc}$ (as $\tilde{\nabla}_a\tilde{g}_{bc} = 0$) one finds that

$$2(\Xi^{-1}\nabla_a\Xi)g_{bc} = Q_a{}^d{}_b g_{dc} + Q_a{}^d{}_c g_{bd}.$$

Two further companion equations can be obtained from the latter by permuting cyclically the indices ${}_{abc}$. Adding two of them and subtracting the third one, one can solve for $Q_a{}^c{}_b$ to find

$$Q_a{}^c{}_b = \Xi^{-1}(\nabla_a\Xi\delta_b{}^c + \nabla_b\Xi\delta_a{}^c - \nabla_d\Xi g^{dc} g_{ab}).$$

This last expression can be rewritten in a more concise form as

$$Q_a{}^c{}_b = S_{ab}{}^{cd}(\Xi^{-1}\nabla_d\Xi), \tag{5.3}$$

where

$$S_{ab}{}^{cd} \equiv \delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - g_{ab}g^{cd}.$$

To simplify the presentation of the various transformation formulae, let

$$\Upsilon_a \equiv \Xi^{-1}\nabla_a\Xi, \qquad \Upsilon_a{}^c{}_b \equiv S_{ab}{}^{cd}\Upsilon_d.$$

Hence, one can write schematically that

$$\boldsymbol{\nabla} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\boldsymbol{\Upsilon}), \tag{5.4}$$

and Equation (5.3) yields $Q_a{}^c{}_b = \Upsilon_a{}^c{}_b$. The tensor $\boldsymbol{S}$ appeared in Section 2.5.2 in the decomposition of the Riemann tensor; see Equation (2.21b). Using Equation (5.1) one finds that

$$\delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - g_{ab}g^{cd} = \delta_a{}^c\delta_b{}^d + \delta_a{}^d\delta_b{}^c - \tilde{g}_{ab}\tilde{g}^{cd}.$$

Hence, the tensor $\boldsymbol{S}$ is independent of the representative of the conformal class; that is, it is an invariant of $[\tilde{\boldsymbol{g}}]$.

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5.2.2 Transformation formulae for the curvature

Combining the results of Section 2.4.4 with the expression for the transition tensor of Equation (5.3) one obtains a transformation rule for the Riemann tensor:

$$R^c{}_{dab} - \tilde{R}^c{}_{dab} = 2(\nabla_{[a}\Upsilon_{b]}{}^c{}_d + \Upsilon_{[a}{}^c{}_{|e|}\Upsilon_{b]}{}^e{}_d). \tag{5.5}$$

Some of the transformation formulae for the various concomitants of the Riemann tensor are dimension dependent; thus, they are analysed separately.

The 4-dimensional case

In the *four-dimensional case* one has that the Ricci and Schouten tensors and Ricci scalar of the connections $\tilde{\boldsymbol{\nabla}}$ and $\boldsymbol{\nabla}$ are related to each other, respectively, by the expressions

$$R_{ab} - \tilde{R}_{ab} = -\frac{2}{\Xi}\nabla_a\nabla_b\Xi - g_{ab}g^{cd}\left(\frac{1}{\Xi}\nabla_c\nabla_d\Xi - \frac{3}{\Xi^2}\nabla_c\Xi\nabla_d\Xi\right), \tag{5.6a}$$

$$L_{ab} - \tilde{L}_{ab} = -\frac{1}{\Xi}\nabla_a\nabla_b\Xi + \frac{1}{2\Xi^2}\nabla_c\Xi\nabla^c\Xi\, g_{ab}, \tag{5.6b}$$

$$R - \frac{1}{\Xi^2}\tilde{R} = -\frac{6}{\Xi}\nabla_c\nabla^c\Xi + \frac{12}{\Xi^2}\nabla_c\Xi\nabla^c\Xi. \tag{5.6c}$$

Using the tensor $S_{ab}{}^{cd}$, one can rewrite the transformation formula for the Schouten tensor, Equation (5.6b), in the alternative form

$$L_{ab} - \tilde{L}_{ab} = \nabla_a\Upsilon_b + \frac{1}{2}S_{ab}{}^{cd}\Upsilon_c\Upsilon_d. \tag{5.7}$$

By letting $\vartheta \equiv \Xi^{-1}$, the transformation rule for the Ricci tensor can be rewritten as

$$6\nabla_a\nabla^a\vartheta - R\vartheta = -\tilde{R}\vartheta^3.$$

Using the irreducible decomposition of the Riemann tensor, Equation (2.21b), as a definition for the Weyl tensor, together with Equations (5.5) and (5.6b), one finds that

$$C^c{}_{dab} = \tilde{C}^c{}_{dab}.$$

In other words, *the Weyl tensor is an invariant of the conformal class* $[\tilde{\boldsymbol{g}}]$. Using this invariance and the transformation law for the connection, a calculation leads to the important identity

$$\nabla_a(\Xi^{-1}C^a{}_{bcd}) = \Xi^{-1}\tilde{\nabla}_a C^a{}_{bcd}. \tag{5.8}$$

A further tensor which will play a role in the present treatment of conformal geometry is the so-called ***Cotton tensor*** of $\tilde{\boldsymbol{\nabla}}$. This tensor is defined as

$$\tilde{Y}_{abc} \equiv \tilde{\nabla}_a\tilde{L}_{bc} - \tilde{\nabla}_b\tilde{L}_{ac}.$$

Notice that by construction $\tilde{Y}_{abc} = \tilde{Y}_{[ab]c}$. The Cotton tensor is closely related to the Weyl tensor. To see this, consider the second Bianchi identity

$$\tilde{\nabla}_{[e}\tilde{R}^{a}{}_{|b|cd]} = 0, \tag{5.9}$$

satisfied by the Riemann tensor of the metric $\tilde{\boldsymbol{g}}$; see Section 2.4.3. Now, as seen in Section 2.5.2, for a Levi-Civita connection, the Riemann tensor $\tilde{R}^{a}{}_{bcd}$ can be decomposed in terms of the Weyl tensor $C^{a}{}_{bcd}$ and the Schouten tensor $\tilde{L}_{ab}$ as

$$\tilde{R}^{a}{}_{bcd} = C^{a}{}_{bcd} + 2(\tilde{g}^{a}{}_{[c}\tilde{L}_{d]b} - \tilde{g}_{b[c}\tilde{L}_{d]}{}^{a}). \tag{5.10}$$

Substituting the latter into Equation (5.9) one obtains

$$2(\tilde{g}_{b[c}\tilde{\nabla}_{e}\tilde{L}_{d]}{}^{a} - \tilde{g}^{a}{}_{[c}\tilde{\nabla}_{e}\tilde{L}_{d]b}) = \tilde{\nabla}_{[e}C^{a}{}_{|b|cd]}.$$

Contracting the indices ^{a} and ${}_{e}$ one obtains

$$\tilde{\nabla}_{c}\tilde{L}_{db} - \tilde{\nabla}_{d}\tilde{L}_{cb} = \tilde{\nabla}_{a}C^{a}{}_{bcd}. \tag{5.11}$$

That is,

$$\tilde{Y}_{cdb} = \tilde{\nabla}_{a}C^{a}{}_{bcd}. \tag{5.12}$$

In particular, one sees that if $C^{a}{}_{bcd}=0$, then $\tilde{Y}_{cdb}=0$. Moreover, as a consequence of the first Bianchi identity for the Weyl tensor, $\tilde{Y}_{[abc]}=0$. The Riemann tensor $R^{a}{}_{bcd}$ of the connection $\boldsymbol{\nabla}$ satisfies equations analogous to (5.9) and (5.10). It follows by the same computation described above that

$$\nabla_{c}L_{db} - \nabla_{d}L_{cb} = \nabla_{a}C^{a}{}_{bcd}. \tag{5.13}$$

Alternatively, defining the Cotton tensor of $\boldsymbol{\nabla}$, $Y_{cdb} \equiv \nabla_{c}L_{db} - \nabla_{d}L_{cb}$, one can write

$$Y_{cdb} = \nabla_{a}C^{a}{}_{bcd}. \tag{5.14}$$

Combining Equations (5.8), (5.12) and (5.14) one finds that the transformation rule for the Cotton tensor is given by:

$$Y_{cdb} - \tilde{Y}_{cdb} = \Upsilon_{a}C^{a}{}_{bcd}. \tag{5.15}$$

The three-dimensional case

In the case of a three-dimensional manifold, let $\boldsymbol{h} = \Omega^{2}\tilde{\boldsymbol{h}}$ – throughout, the symbol Ω will be used to denote a generic conformal factor on a manifold of dimension three. One has that

$$r_{ij} - \tilde{r}_{ij} = -\frac{1}{\Omega}D_{i}D_{j}\Omega - h_{ij}h^{kl}\left(\frac{1}{\Omega}D_{k}D_{l}\Omega - \frac{2}{\Omega^{2}}D_{k}\Omega D_{l}\Omega\right), \tag{5.16a}$$

$$l_{ij} - \tilde{l}_{ij} = -\frac{1}{\Omega}D_{i}D_{j}\Omega + \frac{1}{2\Omega^{2}}D_{k}\Omega D^{k}\Omega\, h_{ij}, \tag{5.16b}$$

$$r - \frac{1}{\Omega^{2}}\tilde{r} = -\frac{4}{\Omega}D_{i}D^{i}\Omega + \frac{6}{\Omega^{2}}D_{i}\Omega D^{i}\Omega. \tag{5.16c}$$

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where D_i denotes the Levi-Civita covariant derivative of the metric $\boldsymbol{h}$, and r_{ij}, l_{ij}, r correspond to its Ricci and Schouten tensors and its Ricci scalar, respectively. The transformation law of the Schouten tensor is of particular interest. *Comparing Equations (5.6b) and* (5.16b) *one sees that although the definition of the Schouten tensor is dimension dependent, its transformation formula is not.*

Letting $\vartheta \equiv \Omega^{-1/2}$, the transformation law for the Ricci scalar can be recast as

$$8D_iD^i\vartheta - r\vartheta = -\tilde{r}\vartheta^5. \tag{5.17}$$

This expression plays an important role in the discussion of the Einstein constraint equations; see Chapter 11.

Given the three-dimensional Schouten tensor $\tilde{l}_{ij}$, its associated Cotton tensor $\tilde{y}_{ijk}$ is given by

$$\tilde{y}_{ijk} \equiv \tilde{D}_i\tilde{l}_{jk} - \tilde{D}_j\tilde{l}_{ik}. \tag{5.18}$$

Using the transformation rule (5.16b), a computation shows that

$$y_{ijk} = \tilde{y}_{ijk}.$$

That is, in three dimensions the Cotton tensor is conformally invariant. Sometimes it is more convenient to work with its Hodge dual, the so-called ***Cotton-York tensor***, given by

$$\tilde{y}_{ij} = -\frac{1}{2}\tilde{y}_{klj}\epsilon_i{}^{kl}.$$

It can be readily verified that

$$y_{ij} = y_{ji}, \qquad y_i{}^i = 0, \qquad D^i y_{ij} = 0.$$

Moreover, the Cotton-York tensor satisfies the transformation rule

$$y_{ij} = \Omega^{-1}\tilde{y}_{ij}. \tag{5.19}$$

5.2.3 Characterising conformal flatness

Given a conformal class $[\boldsymbol{g}]$ on a manifold $\mathcal{M}$, an important question is whether the flat metric belongs to it, so that $\boldsymbol{g}$ is conformally flat. Conformally flat metrics are a source of geometric intuition in general relativity as they have a simpler curvature tensor depending on the Schouten tensor only. Conformal flatness is characterised by the following classical result:

Theorem 5.1 (***Weyl-Schouten theorem***)

(i) Let $(\mathcal{M}, \boldsymbol{g})$ be a manifold with metric of dimension $n \geq 3$. The metric $\boldsymbol{g}$ is conformally flat if and only if the Cotton tensor of $\boldsymbol{g}$ vanishes.

(ii) Let $(\mathcal{M}, \boldsymbol{g})$ be a manifold with metric of dimension $n \geq 4$. The metric $\boldsymbol{g}$ is conformally flat if and only if the Weyl tensor of $\boldsymbol{g}$ vanishes.

Proof A direct computation shows that if a metric is conformally flat, then both its Cotton and Weyl tensors vanish; this proves the *if* part.

In order to prove the *only if* part, one uses the fact that if the Weyl tensor vanishes then, for dimensions $n \geq 4$, the Cotton tensor vanishes; compare Equation (5.14). In view of Equation (5.15) one concludes that the vanishing of the Cotton tensor holds for any metric in the conformal class. From this point onwards, the proofs for the various dimensions are similar. For simplicity, only the four-dimensional case is considered.

Given a metric $\boldsymbol{g}$ in the conformal class, one needs to find a conformal factor Ξ such that $\boldsymbol{g} = \Xi^2 \boldsymbol{\eta}$ where $\boldsymbol{\eta}$ is the flat Minkowski metric. Motivated by the transformation law for the Schouten tensor, Equation (5.6b), consider the equation

$$\nabla_a \alpha_b + \alpha_a \alpha_b - \frac{1}{2} \alpha_c \alpha^c g_{ab} = -L_{ab}. \tag{5.20}$$

The latter can be read as an *overdetermined* partial differential equation for the covector α_a. Given a solution to Equation (5.20), an antisymmetrisation yields that $\nabla_{[a} \alpha_{b]} = 0$ so that α_a is a *closed covector*. Thus, locally α_a is exact and can be written as $\alpha_a = \nabla_a (\ln \Xi) = \Xi^{-1} \nabla_a \Xi$ for some function Ξ. Comparing Equation (5.20) with (5.6b) one concludes that the Schouten tensor of the metric $\tilde{\boldsymbol{g}} = \Xi^{-2} \boldsymbol{g}$ must vanish. As $C^a{}_{bcd} = 0$, the whole Riemann tensor of $\tilde{\boldsymbol{g}}$ must vanish. Consequently, one concludes that $\tilde{\boldsymbol{g}} = \boldsymbol{\eta}$.

Hence, to conclude the proof one needs to show that Equation (5.20) admits a solution under the assumption that $C^a{}_{bcd} = 0$ and $Y_{abc} = 0$. Applying ∇_c to Equation (5.20), antisymmetrising on ${}_{ca}$ and finally using the commutator of covariant derivatives, one finds the *integrability condition*

$$R^d{}_{bca} \alpha_d + 2 \alpha_{[a} \nabla_{c]} \alpha_b + 2 \alpha_e \nabla_{[c} \alpha^e g_{a]b} = 0. \tag{5.21}$$

Now, as $C^a{}_{bcd} = 0$ one has that

$$R^a{}_{bcd} = 2(\delta^a{}_{[c} L_{d]b} - g_{b[c} L_{d]}{}^a).$$

Using the latter expression for the Riemann tensor together with Equation (5.20), one finds that the integrability condition (5.21) is automatically satisfied. A general version of the Frobenius theorem ensures the existence of a solution α_b to Equation (5.20); see, for example, Choquet-Bruhat et al. (1982) or Spivak (1970). □

5.3 Weyl connections

As in the previous sections, let $\tilde{\boldsymbol{\nabla}}$ denote the Levi-Civita connection of a metric $\tilde{\boldsymbol{g}}$ on $\tilde{\mathcal{M}}$. Some of the applications of conformal geometry to be considered in this book give rise to connections which are not necessarily the Levi-Civita connection

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of a metric but, nevertheless, respect the conformal class. A ***Weyl connection*** is a torsion-free connection $\hat{\boldsymbol{\nabla}}$ such that

$$\hat{\nabla}_a \tilde{g}_{bc} = -2\, \tilde{f}_a \tilde{g}_{bc}, \tag{5.22}$$

for some arbitrary covector $\tilde{f}_a$.

The transition tensor $Q_a{}^c{}_b$ relating the connections $\tilde{\boldsymbol{\nabla}}$ and $\hat{\boldsymbol{\nabla}}$ can be obtained using an argument similar to the one employed in Section 5.2.1 to compute the transition tensor of a conformal rescaling. One finds that

$$Q_a{}^c{}_b = S_{ab}{}^{cd} \tilde{f}_d.$$

Schematically one writes

$$\hat{\boldsymbol{\nabla}} - \tilde{\boldsymbol{\nabla}} = \boldsymbol{S}(\tilde{\boldsymbol{f}}).$$

If the covector $\tilde{\boldsymbol{f}}$ is exact, so that on suitable open sets it can be written in the form $\boldsymbol{f} = -\Xi^{-1}\mathbf{d}\Xi$ with some smooth function $\Xi > 0$, then the Weyl connection $\hat{\boldsymbol{\nabla}}$ is, in fact, the Levi-Civita connection of the metric $\boldsymbol{g} = \Xi^2 \tilde{\boldsymbol{g}}$.

The condition $\hat{\nabla}_a \delta_b{}^c = 0$ satisfied by a generic connection together with the relation $\delta_b{}^c = \tilde{g}_{bd}\tilde{g}^{dc}$ and the defining property of a Weyl connection, Equation (5.22), show that

$$\hat{\nabla}_a \tilde{g}^{bc} = 2\, \tilde{f}_a \tilde{g}^{bc}.$$

Using the above expressions one readily obtains that

$$\begin{aligned}\hat{\nabla}_e S_{ab}{}^{cd} &= \hat{\nabla}_e(\delta_a{}^c \delta_b{}^d + \delta_b{}^c \delta_a{}^d - \tilde{g}_{ab}\tilde{g}^{cd}) \\ &= -\hat{\nabla}_e(\tilde{g}_{ab}\tilde{g}^{cd}) = -\hat{\nabla}_e \tilde{g}_{ab}\tilde{g}^{cd} - \tilde{g}_{ab}\hat{\nabla}_e \tilde{g}^{cd} = 0.\end{aligned}$$

In what follows, let $\hat{R}^a{}_{bcd}$ denote the Riemann tensor of the Weyl connection $\hat{\boldsymbol{\nabla}}$. This tensor possesses the basic symmetry $\hat{R}^a{}_{bcd} = -\hat{R}^a{}_{bdc}$. As the connection $\hat{\boldsymbol{\nabla}}$ has vanishing torsion, it follows that $\hat{R}^a{}_{bcd}$ satisfies the *first* and *second Bianchi identities* in the form:

$$\hat{R}^a{}_{[bcd]} = 0, \tag{5.23a}$$

$$\hat{\nabla}_{[e} \hat{R}^a{}_{|b|cd]} = 0. \tag{5.23b}$$

5.3.1 Weyl propagation

To investigate the relation between Weyl connections and the conformal class $[\tilde{\boldsymbol{g}}]$, consider a curve γ with parameter s $\in I \subseteq \mathbb{R}$ on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ with tangent $\dot{\boldsymbol{x}} \in T(\tilde{\mathcal{M}})$. A vector $\boldsymbol{u} \in T(\tilde{\mathcal{M}})$ is said to be ***Weyl propagated*** along γ if it is parallely propagated along γ with respect to a Weyl connection $\hat{\boldsymbol{\nabla}}$; that is, $\boldsymbol{u}$ satisfies the equation

$$\hat{\boldsymbol{\nabla}}_{\dot{\boldsymbol{x}}} \boldsymbol{u} = 0.$$

Writing the latter in terms of the Levi-Civita connection $\tilde{\boldsymbol{\nabla}}$ one has that

$$\begin{aligned}\dot{x}^a\tilde{\nabla}_a u^b &= -\dot{x}^a S_{ac}{}^{be}u^c\tilde{f}_e,\\ &= \tilde{g}_{cd}u^c\dot{x}^d\tilde{f}^b - u^c\tilde{f}_c\dot{x}^b - \dot{x}^c\tilde{f}_c u^b.\end{aligned}$$

In index-free notation one has that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{u} = \tilde{\boldsymbol{g}}(\boldsymbol{u},\dot{\boldsymbol{x}})\tilde{\boldsymbol{f}}^\sharp - \langle\tilde{\boldsymbol{f}},\boldsymbol{u}\rangle\dot{\boldsymbol{x}} - \langle\tilde{\boldsymbol{f}},\dot{\boldsymbol{x}}\rangle\boldsymbol{u}.$$

Let $\{\boldsymbol{e_a}\}$ denote an arbitrary frame which is Weyl propagated along γ so that $\hat{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{e_a} = 0$. Letting $\tilde{g}_{\boldsymbol{ab}} \equiv \tilde{\boldsymbol{g}}(\boldsymbol{e_a},\boldsymbol{e_b})$, a computation then shows that

$$\hat{\nabla}_{\dot{\boldsymbol{x}}}\tilde{g}_{\boldsymbol{ab}} = \hat{\nabla}_{\dot{\boldsymbol{x}}}(\tilde{\boldsymbol{g}}(\boldsymbol{e_a},\boldsymbol{e_b})) = -2\langle\tilde{\boldsymbol{f}},\dot{\boldsymbol{x}}\rangle\tilde{g}_{\boldsymbol{ab}}. \tag{5.24}$$

Consequently, one obtains $\hat{\nabla}_{\dot{\boldsymbol{x}}}(\ln\tilde{g}_{\boldsymbol{ab}}) = -2\langle\tilde{\boldsymbol{f}},\dot{\boldsymbol{x}}\rangle$. The latter equation can be solved to give

$$\tilde{g}_{\boldsymbol{ab}}(\eta) = \tilde{g}_{\boldsymbol{ab}}(\eta_\star)\,\exp\left(-2\int_{\mathrm{s}_\star}^{\mathrm{s}}\langle\tilde{\boldsymbol{f}},\dot{\boldsymbol{x}}\rangle\mathrm{ds}'\right)$$

along the curve $x(\mathrm{s})$. *Thus, one finds that Weyl connections respect the conformal class in the sense that parallel propagation of a metric using a Weyl connection leads to a metric in the same conformal class.* Notice also that Equation (5.24) allows one to conclude that if the frame is *orthogonal* at some point along the curve, then it is orthogonal elsewhere on γ – the normalisation, however, is lost.

5.3.2 Transformation formulae for the curvature

The transformation formulae between the curvature tensors of the Levi-Civita connection $\tilde{\boldsymbol{\nabla}}$ and the Weyl connection $\hat{\boldsymbol{\nabla}}$ follow directly from the general discussion of Section 2.4.4.

In what follows let $\tilde{f}_a{}^c{}_b \equiv S_{ab}{}^{cd}\tilde{f}_d$. If $\hat{R}^a{}_{bcd}$ denotes the Riemann tensor of $\hat{\boldsymbol{\nabla}}$, then one has that

$$\hat{R}^a{}_{bcd} - R^a{}_{bcd} = 2(\tilde{\nabla}_{[c}\tilde{f}_{d]}{}^a{}_b + \tilde{f}_e{}^a{}_{[c}\tilde{f}_{d]}{}^e{}_b), \tag{5.25a}$$

$$\begin{aligned} &= 2(\delta^a{}_{[c}\tilde{\nabla}_{d]}\tilde{f}_b + \tilde{\nabla}_{[c}\tilde{f}^a\tilde{g}_{d]b} - \delta^a{}_b\tilde{\nabla}_{[c}\tilde{f}_{d]}\\ &\quad -\delta^a{}_{[c}\tilde{f}_{d]}\tilde{f}_b + \tilde{g}_{b[c}\tilde{f}_{d]}\tilde{f}^a + \delta^a{}_{[c}\tilde{g}_{d]b}\tilde{f}_e\tilde{f}^e).\end{aligned} \tag{5.25b}$$

Note that the above transformation law involves both the symmetric and antisymmetric parts of the covariant derivative $\tilde{\nabla}_a\tilde{f}_b$.

A transformation formula for the Ricci tensor $\hat{R}_{bd} \equiv \hat{R}^a{}_{bad}$ can be obtained directly from Equation (5.25b):

$$\hat{R}_{cd} - \tilde{R}_{cd} = -3\tilde{\nabla}_d\tilde{f}_c + \tilde{\nabla}_c\tilde{f}_d + 2\,\tilde{f}_c\tilde{f}_d - \tilde{g}_{cd}\left(\tilde{\nabla}_e\tilde{f}^e + 2\,\tilde{f}_e\tilde{f}^e\right). \tag{5.26}$$

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Now, as there is no canonical metric to lower or raise indices in expressions involving a Weyl connection, it is conventional to choose a representative of the conformal class, say, $\tilde{\boldsymbol{g}}$, and use it to compute traces. In this spirit one defines the Ricci scalar of the Weyl connection via $\hat{R} \equiv \tilde{g}^{ab}\hat{R}_{ab}$. It can then be directly computed that

$$\hat{R} - \tilde{R} = -6\tilde{\nabla}_a \tilde{f}^a - 6\,\tilde{f}_a \tilde{f}^a. \tag{5.27}$$

Combining the transformation formula for the Riemann tensor, Equation (5.25a), with the irreducible decomposition of the Riemann tensor $\tilde{R}^a{}_{bcd}$ given by Equation (2.21b), one can find an analogous decomposition for the Riemann tensor $\hat{R}^a{}_{bcd}$ of $\hat{\boldsymbol{\nabla}}$:

$$\begin{aligned}\hat{R}^c{}_{dab} &= C^c{}_{dab} + 2S_{d[a}{}^{ce}\hat{L}_{b]e}, \\ &= C^c{}_{dab} + 2(\tilde{g}^c{}_{[a}\hat{L}_{b]d} - \delta^c{}_d \hat{L}_{[ab]} - \tilde{g}_{d[a}\hat{L}_{b]}{}^c),\end{aligned} \tag{5.28a}$$

where

$$\hat{L}_{ab} = \frac{1}{2}\left(\hat{R}_{(ab)} - \frac{1}{2}\hat{R}_{[ab]} - \frac{1}{6}\tilde{g}_{ab}\hat{R}\right)$$

is the ***Schouten tensor of the Weyl connection*** $\hat{\boldsymbol{\nabla}}$. *This definition is independent of the choice of the representative of the conformal class.* Making use of the transformation laws for the Ricci tensor and scalar, Equations (5.26) and (5.27), one finds that

$$\tilde{L}_{ab} - \hat{L}_{ab} = \tilde{\nabla}_a \tilde{f}_b - \tilde{f}_a \tilde{f}_b + \frac{1}{2}\tilde{g}_{ab}\tilde{f}^c \tilde{f}_c, \tag{5.29a}$$

$$= \tilde{\nabla}_a \tilde{f}_b - \frac{1}{2}S_{ab}{}^{cd}\tilde{f}_c \tilde{f}_d, \tag{5.29b}$$

$$= \hat{\nabla}_a \tilde{f}_b + \frac{1}{2}S_{ab}{}^{cd}\tilde{f}_c \tilde{f}_d. \tag{5.29c}$$

Finally, it is observed that letting $\hat{R}_{abcd} \equiv \tilde{g}_{ae}\hat{R}^e{}_{bcd}$, it follows from the discussion in the previous paragraphs that

$$\hat{R}_{abcd} = \hat{R}_{[ab]cd} + 2\tilde{g}_{ab}\hat{\nabla}_{[c}f_{d]}, \tag{5.30a}$$

$$= \hat{R}_{[ab]cd} - 2\tilde{g}_{ab}\hat{L}_{[cd]}. \tag{5.30b}$$

These formulae show in an explicit way how the usual symmetries of the curvature tensor are obstructed by the covector defining a Weyl connection.

5.4 Spinorial expressions

This section discusses the spinorial counterparts of the tensorial expressions obtained in the previous sections of this chapter.

5.4.1 Conformal rescalings

As in previous sections, let $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ denote two metrics on $\tilde{\mathcal{M}}$ related to each other by the conformal rescaling (5.1). Following the discussion of Chapter 3, the spinorial counterparts of $\tilde{\boldsymbol{g}}$ and $\boldsymbol{g}$ are given by

$$\tilde{g}_{AA'BB'} = \tilde{\epsilon}_{AB}\tilde{\epsilon}_{A'B'}, \qquad g_{AA'BB'} = \epsilon_{AB}\epsilon_{A'B'};$$

compare Equation (3.15). Hence, it is natural to consider the transformation laws

$$\epsilon_{AB} = \Xi\tilde{\epsilon}_{AB}, \qquad \epsilon^{AB} = \Xi^{-1}\tilde{\epsilon}^{AB},$$
$$\epsilon_{A'B'} = \Xi\tilde{\epsilon}_{A'B'}, \qquad \epsilon^{A'B'} = \Xi^{-1}\tilde{\epsilon}^{A'B'}.$$

Let $\{\tilde{o}_A, \tilde{\iota}_A\}$ and $\{o_A, \iota_A\}$ denote two spin bases satisfying, respectively, the conditions

$$\tilde{\epsilon}_{AB} = \tilde{o}_A\tilde{\iota}_B - \tilde{\iota}_A\tilde{o}_B, \qquad \epsilon_{AB} = o_A\iota_B - \iota_A o_B.$$

There are several possible transformation rules between the two spin bases which are consistent with the above equations and with the rescaling (5.1). Namely, one has:

$$o_A = \tilde{o}_A, \quad \iota_A = \Xi\tilde{\iota}_A, \quad o^A = \Xi^{-1}\tilde{o}^A, \quad \iota^A = \tilde{\iota}^A, \tag{5.31a}$$

$$o_A = \Xi\tilde{o}_A, \quad \iota_A = \tilde{\iota}_A, \quad o^A = \tilde{o}^A, \quad \iota^A = \Xi^{-1}\tilde{\iota}^A, \tag{5.31b}$$

$$o_A = \Xi^{1/2}\tilde{o}_A, \quad \iota_A = \Xi^{1/2}\tilde{\iota}_A, \quad o^A = \Xi^{-1/2}\tilde{o}^A, \quad \iota^A = \Xi^{-1/2}\tilde{\iota}^A. \tag{5.31c}$$

The choice of the most convenient transformation rule depends on the nature of the application at hand; see, for example, Chapter 10.

Transformation rules for the connection and curvature

In what follows let $\Upsilon_{AA'} \equiv \Xi^{-1}\nabla_{AA'}\Xi$ denote the spinorial counterpart of the covector Υ_a. Let also $\Upsilon_a{}^c{}_b \equiv S_{ab}{}^{cd}\Upsilon_d$. Its spinorial counterpart is given by

$$\Upsilon_{AA'}{}^{CC'}{}_{BB'} = \delta_A{}^C\delta_{A'}{}^{C'}\Upsilon_{BB'} + \delta_B{}^C\delta_{B'}{}^{C'}\Upsilon_{AA'} - \epsilon_{AB}\epsilon_{A'B'}\Upsilon^{CC'}.$$

By rewriting

$$\delta_A{}^C\delta_{A'}{}^{C'}\Upsilon_{BB'} + \delta_B{}^C\delta_{B'}{}^{C'}\Upsilon_{AA'} = \delta_A{}^C\epsilon_{A'}{}^{C'}\epsilon_{B'}{}^{D'}\Upsilon_{BD'} + \delta_B{}^C\epsilon_{B'}{}^{C'}\epsilon_{A'}{}^{D'}\Upsilon_{AD'},$$

and using the Jacobi identity (3.5), one finds that

$$\Upsilon_{AA'}{}^{CC'}{}_{BB'} = \Upsilon_{AA'}{}^C{}_B\delta_{B'}{}^{C'} + \bar{\Upsilon}_{A'A}{}^{C'}{}_{B'}\delta_B{}^C,$$

where

$$\Upsilon_{AA'}{}^C{}_B \equiv \delta_A{}^C\Upsilon_{BA'}.$$

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The reduced coefficient $\Upsilon_{AA'}{}^{C}{}_{B}$ can be used to obtain the transformation laws relating the covariant derivatives of spinors. In particular, one has for arbitrary spinors κ_A, $\mu_{A'}$, ξ^A and $\eta^{A'}$ that

$$\begin{aligned}\tilde{\nabla}_{AA'}\kappa_B &= \nabla_{AA'}\kappa_B + \Upsilon_{BA'}\kappa_A,\\ \tilde{\nabla}_{AA'}\mu_{B'} &= \nabla_{AA'}\mu_{B'} + \Upsilon_{AB'}\mu_{A'},\\ \tilde{\nabla}_{AA'}\xi^B &= \nabla_{AA'}\xi^B - \delta_A{}^B\Upsilon_{CA'}\xi^C,\\ \tilde{\nabla}_{AA'}\eta^{B'} &= \nabla_{AA'}\eta^{B'} - \delta_{A'}{}^{B'}\Upsilon_{AC'}\eta^{C'}.\end{aligned}$$

These expressions can be extended, in a direct way, to higher valence spinors. For the curvature spinors, it can be verified that

$$\begin{aligned}\tilde{\Psi}_{ABCD} &= \Psi_{ABCD},\\ \tilde{\Phi}_{AA'BB'} &= \Phi_{AA'BB'} + \Xi^{-1}\nabla_{A(A'}\nabla_{B')B}\Xi.\end{aligned}$$

5.4.2 Weyl connections

In what follows, let $\hat{\nabla}_{AA'}$ denote the spinorial counterpart of the Weyl connection $\hat{\boldsymbol{\nabla}}$ defined by Equation (5.22). To determine expressions for $\hat{\nabla}_{AA'}\tilde{\epsilon}_{BC}$ and $\hat{\nabla}_{AA'}\tilde{\epsilon}^{BC}$ one notices that the spinorial version of Equation (5.22) is

$$\hat{\nabla}_{AA'}(\tilde{\epsilon}_{BC}\tilde{\epsilon}_{B'C'}) = -2\,\tilde{f}_{AA'}\tilde{\epsilon}_{BC}\tilde{\epsilon}_{B'C'},$$

so that

$$\tilde{\epsilon}_{B'C'}\hat{\nabla}_{AA'}\tilde{\epsilon}_{BC} + \tilde{\epsilon}_{BC}\hat{\nabla}_{AA'}\tilde{\epsilon}_{B'C'} = -2\,\tilde{f}_{AA'}\tilde{\epsilon}_{BC}\tilde{\epsilon}_{B'C'}.$$

The latter is satisfied if one sets

$$\hat{\nabla}_{AA'}\tilde{\epsilon}_{BC} = -\tilde{f}_{AA'}\tilde{\epsilon}_{BC}.$$

From this expression and using that $\hat{\nabla}_{AA'}\delta_B{}^C = 0$, one can readily compute $\hat{\nabla}_{AA'}\tilde{\epsilon}^{BC}$. One finds that

$$\hat{\nabla}_{AA'}\tilde{\epsilon}^{BC} = \tilde{f}_{AA'}\tilde{\epsilon}^{BC}.$$

Decomposition of the spin connection coefficients of a Weyl connection

Let $\{\tilde{\epsilon}_{\boldsymbol{A}}{}^A\}$ denote a spin basis with respect to $\tilde{\epsilon}_{AB}$. Following the general discussion on spin connection coefficients of Section 3.2.2 – compare Equation (3.33) – the spinorial counterparts of the connection coefficients $\tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ and $\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}}$ admit the decompositions

$$\begin{aligned}\tilde{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} &= \tilde{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} + \bar{\tilde{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}}\delta_{\boldsymbol{C}}{}^{\boldsymbol{B}},\\ \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} &= \hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} + \bar{\hat{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}}\delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}.\end{aligned}$$

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The spinorial counterpart of the equation

$$\hat{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} = \tilde{\Gamma}_{\boldsymbol{a}}{}^{\boldsymbol{b}}{}_{\boldsymbol{c}} + \delta_{\boldsymbol{a}}{}^{\boldsymbol{b}}\tilde{f}_{\boldsymbol{c}} + \delta_{\boldsymbol{c}}{}^{\boldsymbol{b}}\tilde{f}_{\boldsymbol{a}} - \eta_{\boldsymbol{ac}}\tilde{f}^{\boldsymbol{b}}$$

is given by

$$\begin{aligned}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} &= \Gamma_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} + \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{CC'}} \\ &\quad + \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{AA'}} - \epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{A'C'}}\tilde{f}^{\boldsymbol{BB'}}.\end{aligned}$$

Now, by rewriting

$$\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{CC'}} = \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\epsilon_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\epsilon_{\boldsymbol{C'}}{}^{\boldsymbol{D'}}\tilde{f}_{\boldsymbol{CD'}}, \quad \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{AA'}} = \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}\epsilon_{\boldsymbol{C'}}{}^{\boldsymbol{B'}}\epsilon_{\boldsymbol{A'}}{}^{\boldsymbol{D'}}\tilde{f}_{\boldsymbol{AD'}}$$

and using the Jacobi identity (3.5), one finds that

$$\begin{aligned}&\delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{CC'}} + \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{AA'}} - \epsilon_{\boldsymbol{AC}}\epsilon_{\boldsymbol{A'C'}}\tilde{f}^{\boldsymbol{BB'}} \\ &\qquad = \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{CA'}} + \delta_{\boldsymbol{C}}{}^{\boldsymbol{B}}\delta_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{AC'}}.\end{aligned}$$

Hence,

$$\begin{aligned}\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{BB'}}{}_{\boldsymbol{CC'}} &= (\tilde{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} + \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\tilde{f}_{\boldsymbol{CA'}})\delta_{\boldsymbol{C'}}{}^{\boldsymbol{B'}} \\ &\quad + (\bar{\tilde{\Gamma}}_{\boldsymbol{AA'}}{}^{\boldsymbol{B'}}{}_{\boldsymbol{C'}} + \delta_{\boldsymbol{A'}}{}^{\boldsymbol{B'}}\tilde{f}_{\boldsymbol{AC'}})\delta_{\boldsymbol{C}}{}^{\boldsymbol{B}},\end{aligned}$$

so that

$$\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} = \tilde{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{B}}{}_{\boldsymbol{C}} + \delta_{\boldsymbol{A}}{}^{\boldsymbol{B}}\tilde{f}_{\boldsymbol{CA'}}. \tag{5.32}$$

In particular, as $\tilde{\Gamma}_{\boldsymbol{AA'BC}} = \tilde{\Gamma}_{\boldsymbol{AA'(BC)}}$, it follows that

$$\hat{\Gamma}_{\boldsymbol{AA'}}{}^{\boldsymbol{Q}}{}_{\boldsymbol{Q}} = \tilde{f}_{\boldsymbol{AA'}}.$$

Decomposition of the curvature tensors

The discussion of the decomposition of the spinorial counterpart of a general Riemann tensor given in Section 3.2.3 can be applied to the case of a Weyl connection. In particular, if $\hat{R}^{AA'}{}_{BB'CC'DD'}$ denotes the spinorial counterpart of the Riemann tensor of a Weyl connection $\hat{\boldsymbol{\nabla}}$, one has that Equation (3.35) gives, in the present context, the decomposition

$$\hat{R}_{AA'BB'CC'DD'} = \epsilon_{A'B'}\hat{R}_{ABCC'DD'} + \epsilon_{AB}\bar{\hat{R}}_{A'B'CC'DD'},$$

where

$$\begin{aligned}\hat{R}_{ABCC'DD'} &\equiv \hat{R}_{(AB)CC'DD'} + \frac{1}{2}\epsilon_{AB}(\hat{\nabla}_{CC'}\tilde{f}_{DD'} - \hat{\nabla}_{DD'}\tilde{f}_{CC'}), \\ &= \hat{R}_{(AB)CC'DD'} - \frac{1}{2}\epsilon_{AB}(\hat{L}_{CC'DD'} - \hat{L}_{DD'CC'}),\end{aligned}$$

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and $\hat{L}_{AA'BB'}$ denotes the spinorial counterpart of the Schouten tensor of $\hat{\boldsymbol{\nabla}}$. A more detailed expression is given by

$$\hat{R}_{ABCC'DD'} = -\Psi_{ABCD}\epsilon_{C'D'} + \hat{L}_{BC'DD'}\epsilon_{AC} - \hat{L}_{BD'CC'}\epsilon_{AD}. \tag{5.33}$$

The spinorial counterpart $\hat{L}_{AA'BB'}$ of the Schouten tensor admits, in turn, the decomposition

$$\hat{L}_{AA'BB'} = \Phi_{AA'BB'} - \frac{1}{24}R\epsilon_{AB}\epsilon_{A'B'} + \Phi_{AB}\epsilon_{A'B'} + \bar{\Phi}_{A'B'}\epsilon_{AB}$$

where $\Phi_{AA'BB'}$ represents the trace-free part of $\frac{1}{2}\hat{R}_{(ab)}$, while Φ_{AB} describes the antisymmetric tensor $\frac{1}{4}\hat{R}_{[ab]}$.

5.5 Conformal geodesics

This section discusses a class of invariants of the conformal structure of a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. To motivate the discussion let $x(\mathrm{s})$, $\mathrm{s} \in I \subset \mathbb{R}$, denote a curve on $\tilde{\mathcal{M}}$ with tangent given by $\boldsymbol{x}' \equiv \mathrm{d}x/\mathrm{ds}$. The curve $x(\mathrm{s})$ is a geodesic if it satisfies the equation $\tilde{\nabla}_{\boldsymbol{x}'}\boldsymbol{x}' = 0$. The transformation rule of the covariant derivative $\tilde{\boldsymbol{\nabla}}$ under the conformal rescaling (5.1) implies, in turn, the equation

$$\nabla_{\boldsymbol{x}'}\boldsymbol{x}' = 2\langle \boldsymbol{\Upsilon}, \boldsymbol{x}'\rangle \boldsymbol{x}' - \boldsymbol{g}(\boldsymbol{x}', \boldsymbol{x}')\boldsymbol{\Upsilon}^{\sharp}. \tag{5.34}$$

Let $\tau = \tau(\mathrm{s})$ denote a new parameter. Writing $\dot{\boldsymbol{x}} \equiv \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ and $\tau' \equiv \mathrm{d}\tau/\mathrm{ds}$, the chain rule yields $\boldsymbol{x}' = \tau'\dot{\boldsymbol{x}}$, so that Equation (5.34) implies

$$\tau'^2 \nabla_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = (2\langle \boldsymbol{\Upsilon}, \dot{\boldsymbol{x}}\rangle \tau'^2 - \tau'')\dot{\boldsymbol{x}} - \tau'^2 \boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{\Upsilon}^{\sharp}.$$

This last expression suggests choosing the parameter τ so that it satisfies the condition

$$\tau'' = 2\langle \boldsymbol{\Upsilon}, \dot{\boldsymbol{x}}\rangle \tau'^2.$$

As $\boldsymbol{\Upsilon}$ is known along the curve, this equation can be read as a second-order ordinary differential equation for τ. Thus, it can always be solved locally so that

$$\tau'^2 \nabla_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -\tau'^2 \boldsymbol{g}(\boldsymbol{x}', \boldsymbol{x}')\boldsymbol{\Upsilon}^{\sharp}.$$

It follows that only when the curve $x(\mathrm{s})$ is null (i.e. $\boldsymbol{g}(\boldsymbol{x}', \boldsymbol{x}') = 0$) is it possible to reparametrise so that $x(\mathrm{s})$ is a geodesic. *Hence, timelike or spacelike geodesics are not, in general, conformal invariants.*

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5.5.1 Basic definitions

A ***conformal geodesic*** on a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is a pair $(x(\tau), \boldsymbol{\beta}(\tau))$ consisting of a curve $x(\tau)$ on $\tilde{\mathcal{M}}$, $\tau \in I \subset \mathbb{R}$, with tangent $\dot{\boldsymbol{x}}(\tau)$ and a covector $\boldsymbol{\beta}(\tau)$ along $x(\tau)$ satisfying the equations

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{\beta}^\sharp, \tag{5.35a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\beta} = \langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\boldsymbol{\beta} - \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta})\dot{\boldsymbol{x}}^\flat + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot), \tag{5.35b}$$

where $\tilde{\boldsymbol{L}}$ denotes the *Schouten tensor* of the Levi-Civita connection $\tilde{\boldsymbol{\nabla}}$. Associated to a conformal geodesic, it is convenient to consider a frame $\{\boldsymbol{e}_{\boldsymbol{a}}\}$ which is *Weyl propagated* along $x(\tau)$ so that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{e}_{\boldsymbol{a}} = -\langle\boldsymbol{\beta}, \boldsymbol{e}_{\boldsymbol{a}}\rangle\dot{\boldsymbol{x}} - \langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\boldsymbol{e}_{\boldsymbol{a}} + \tilde{\boldsymbol{g}}(\boldsymbol{e}_{\boldsymbol{a}}, \dot{\boldsymbol{x}})\boldsymbol{\beta}^\sharp. \tag{5.36}$$

Initial data for the conformal geodesic Equations (5.35a) and (5.35b) consist of an initial position, an initial direction for the curve and an initial value for the covector:

$$x_\star \in \tilde{\mathcal{M}}, \quad \dot{\boldsymbol{x}}_\star \in T|_{x_\star}(\tilde{\mathcal{M}}), \quad \boldsymbol{\beta}_\star \in T^*|_{x_\star}(\tilde{\mathcal{M}}). \tag{5.37}$$

Piccard's theorem – see, for example, Hartman (1987) – ensures the existence of a unique conformal geodesic $(x(\tau), \boldsymbol{\beta}(\tau))$ near $x_\star$ satisfying for given $\tau_\star \in \mathbb{R}$

$$x(\tau_\star) \equiv x_\star, \quad \dot{\boldsymbol{x}}(\tau_\star) \equiv \dot{\boldsymbol{x}}_\star, \quad \boldsymbol{\beta}(\tau_\star) \equiv \boldsymbol{\beta}_\star.$$

A direct computation using Equations (5.35a) and (5.35b) yields the relations

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\left(\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\right) = -2\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}), \tag{5.38a}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle = -\langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle^2 + \frac{1}{2}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta}) + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}), \tag{5.38b}$$

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\big(\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta})\big) = \langle\boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta}) + 2\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \boldsymbol{\beta}^\sharp). \tag{5.38c}$$

In particular, from Equation (5.38a) it follows that if $\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 0$ at some point along the conformal geodesic, one has that $\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 0$ everywhere else. This null conformal geodesic can, in turn, be reparametrised so that it coincides with a null geodesic of $\tilde{\boldsymbol{g}}$.

Expressions in abstract index notation

For later use, it is observed that the conformal geodesic equations can be written in abstract index notation using the tensor $S_{ab}{}^{cd}$ as

$$\dot{x}^c\tilde{\nabla}_c\dot{x}^a = -S_{ef}{}^{ac}\dot{x}^e\dot{x}^f\beta_c,$$

$$\dot{x}^c\tilde{\nabla}_c\beta_a = \frac{1}{2}S_{ca}{}^{ef}\beta_e\beta_f\dot{x}^c + \tilde{L}_{ca}\dot{x}^c,$$

$$\dot{x}^c\tilde{\nabla}_c e_{\boldsymbol{a}}{}^a = -S_{cd}{}^{af}e_{\boldsymbol{a}}{}^d\dot{x}^c\beta_f.$$

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5.5.2 Conformal geodesics and changes of connection

The motivation behind the notion of conformal geodesics is not directly apparent from the defining Equations (5.35a) and (5.35b). Their relevance becomes apparent only once one considers their transformation rules under conformal rescalings and transitions to Weyl connections.

As in the previous section, let $(x(\tau), \boldsymbol{\beta}(\tau))$ denote a solution to the conformal geodesic Equations (5.35a) and (5.35b) on a spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$. Given $\check{\boldsymbol{f}} \in T^*(\tilde{\mathcal{M}})$ one can define a Weyl connection $\check{\boldsymbol{\nabla}}$ via the relation

$$\check{\boldsymbol{\nabla}} \equiv \tilde{\boldsymbol{\nabla}} + \boldsymbol{S}(\check{\boldsymbol{f}}). \tag{5.39}$$

A computation using Equations (5.35a) and (5.35b) shows that $(x(\tau), \check{\boldsymbol{\beta}}(\tau))$ with

$$\check{\boldsymbol{\beta}}(\tau) \equiv \boldsymbol{\beta}(\tau) - \check{\boldsymbol{f}}(\tau) \tag{5.40}$$

is a solution to the $\check{\boldsymbol{\nabla}}$*-conformal geodesic equations*:

$$\check{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -2\langle\check{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} + \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\check{\boldsymbol{\beta}}^{\sharp},$$
$$\check{\nabla}_{\dot{\boldsymbol{x}}}\check{\boldsymbol{\beta}} = \langle\check{\boldsymbol{\beta}}, \dot{\boldsymbol{x}}\rangle\check{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\check{\boldsymbol{\beta}}, \check{\boldsymbol{\beta}})\dot{\boldsymbol{x}}^{\flat} + \check{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot),$$

where $\check{\boldsymbol{L}}$ denotes the Schouten tensor of the Weyl connection $\check{\boldsymbol{\nabla}}$. The latter is given by

$$\check{L}_{ab} = \tilde{L}_{ab} - \tilde{\nabla}_a \check{f}_b - \frac{1}{2} S_{ab}{}^{cd} \check{f}_c \check{f}_d.$$

Thus, one concludes that conformal geodesics are invariants of $[\tilde{\boldsymbol{g}}]$. Notice, in particular, that one could have chosen $\check{\boldsymbol{f}} = -\Xi^{-1}\mathbf{d}\Xi$ for some positive $\Xi \in \mathfrak{X}(\tilde{\mathcal{M}})$ so that the change of connections given by Equation (5.39) corresponds, in fact, to a conformal rescaling of $\tilde{\boldsymbol{g}}$.

Now, choosing $\check{\boldsymbol{f}}(\tau) = \boldsymbol{\beta}(\tau)$ one has that $\check{\boldsymbol{\beta}}(\tau) = 0$, so that the $\check{\boldsymbol{\nabla}}$-conformal geodesic equations reduce to:

$$\check{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0, \qquad \check{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot) = 0. \tag{5.41}$$

Moreover, the frame propagation Equation (5.36) yields

$$\check{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{e_a} = 0.$$

Hence, given a congruence of conformal geodesics on $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$, there exists a Weyl connection $\check{\boldsymbol{\nabla}}$ on $[\tilde{\boldsymbol{g}}]$ with respect to which the curves $x(\tau)$ are (affine) geodesics and the frame $\{\boldsymbol{e_a}\}$ is parallely propagated. This observation justifies the name *conformal geodesics* given to a solution to Equations (5.35a) and (5.35b). *Thus, conformal geodesics not only are an invariant of the conformal structure, but also single out a particular Weyl connection on the conformal class* $[\boldsymbol{g}]$.

5.5.3 Reparametrisations

Given two solutions to the conformal geodesic Equations (5.35a) and (5.35b), $(x(\tau), \boldsymbol{\beta}(\tau))$ and $(\bar{x}(\bar{\tau}), \bar{\boldsymbol{\beta}}(\bar{\tau}))$, it is natural to ask under which conditions $x(\tau)$ and $\bar{x}(\bar{\tau})$ coincide locally (as sets of points) so that $\tau = \tau(\bar{\tau})$ and $x(\tau(\bar{\tau})) = \bar{x}(\bar{\tau})$. Let $\dot{\boldsymbol{x}} \equiv \mathrm{d}x/\mathrm{d}\tau$ and $\bar{\boldsymbol{x}}' \equiv \mathrm{d}\bar{x}/\mathrm{d}\bar{\tau}$ denote the corresponding tangent vectors and assume that $\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \neq 0$ and $\tilde{\boldsymbol{g}}(\bar{\boldsymbol{x}}', \bar{\boldsymbol{x}}') \neq 0$. By definition, the tangent vector $\boldsymbol{x}'$ satisfies

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = -2\langle \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{x}}'\rangle \bar{\boldsymbol{x}}' + \tilde{\boldsymbol{g}}(\bar{\boldsymbol{x}}', \bar{\boldsymbol{x}}')\bar{\boldsymbol{\beta}}^{\sharp}, \tag{5.42a}$$

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{\beta}} = \langle \bar{\boldsymbol{\beta}}, \bar{\boldsymbol{x}}'\rangle \bar{\boldsymbol{\beta}} - \frac{1}{2}\tilde{\boldsymbol{g}}^{\sharp}(\bar{\boldsymbol{\beta}}, \bar{\boldsymbol{\beta}})\bar{\boldsymbol{x}}'^{\flat} + \tilde{\boldsymbol{L}}(\bar{\boldsymbol{x}}', \cdot). \tag{5.42b}$$

Now, letting $\tau' \equiv \mathrm{d}\tau/\mathrm{d}\bar{\tau}$ one has that

$$\bar{\boldsymbol{x}}' = \tau'\dot{\boldsymbol{x}}, \qquad \tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = \tau''\dot{\boldsymbol{x}} + \tau'^2\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}.$$

Substituting the latter into Equation (5.42a) and using (5.35a) to eliminate $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}$ one obtains

$$\tau''\dot{\boldsymbol{x}} + 2\tau'^2\langle \bar{\boldsymbol{\beta}} - \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle \dot{\boldsymbol{x}} + \tau'^2\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})(\boldsymbol{\beta}^{\sharp} - \bar{\boldsymbol{\beta}}^{\sharp}) = 0. \tag{5.43}$$

It follows from this last equation that the difference $\bar{\boldsymbol{\beta}}^{\sharp} - \boldsymbol{\beta}^{\sharp}$ has components only along $\dot{\boldsymbol{x}}$. Hence, one can write

$$\bar{\boldsymbol{\beta}} - \boldsymbol{\beta} = \alpha\dot{\boldsymbol{x}}^{\flat}, \tag{5.44}$$

for some scalar α. Substituting into Equation (5.43) one obtains the differential equation

$$\tau'' + \alpha\tau'^2\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 0. \tag{5.45}$$

Combining Equations (5.35a), (5.35b), (5.42b) and (5.44) one obtains

$$\dot{\alpha} = 2\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}}\rangle\alpha + \frac{1}{2}\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\alpha^2. \tag{5.46}$$

Equations (5.44), (5.45) *and* (5.46) *encode the requirement that the curves* $x(\tau)$ *and* $\bar{x}(\bar{\tau})$ *coincide as sets.* Using Equation (5.38a) together with Equation (5.46) one finds that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\left(\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\right) = \frac{1}{2}\left(\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\right)^2.$$

This last equation can be solved to give

$$\alpha\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = \frac{2\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star, \dot{\boldsymbol{x}}_\star)}{1 - \alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star, \dot{\boldsymbol{x}}_\star)(\tau - \tau_\star)},$$

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where $\alpha_\star \equiv \alpha(\tau_\star)$, $\dot{\boldsymbol{x}}_\star \equiv \dot{\boldsymbol{x}}(\tau_\star)$ and $\tau_\star$ denotes some fiducial value of the parameter τ. Using Equations (5.44) and (5.45) one finally finds that:

$$\bar{\boldsymbol{x}}' = \frac{4\varkappa}{1 + 2\varkappa\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star, \dot{\boldsymbol{x}}_\star)(\tau - \tau_\star)}\dot{\boldsymbol{x}}, \tag{5.47a}$$

$$\bar{\boldsymbol{\beta}} = \boldsymbol{\beta} + \frac{2\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star, \dot{\boldsymbol{x}}_\star)}{(1 - \alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}_\star, \dot{\boldsymbol{x}}_\star)(\tau - \tau_\star))\,\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})}\dot{\boldsymbol{x}}^\flat, \tag{5.47b}$$

$$\tau = \tau_\star + \frac{4\varkappa(\bar{\tau} - \bar{\tau}_\star)}{1 + 2\varkappa\alpha_\star\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})(\bar{\tau} - \bar{\tau}_\star)}, \tag{5.47c}$$

with $\varkappa$ a non-zero real constant. One can summarise the previous discussion in the following lemma:

Lemma 5.1 (***admissible reparametrisations of conformal geodesics***) *The admissible reparametrisations taking (non-null) conformal geodesics into (non-null) conformal geodesics are given by fractional transformations of the form*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \tag{5.48}$$

with $a, b, c, d \in \mathbb{R}$.

If $\alpha_\star = 0$, then Equation (5.47c) shows that the reparametrisation reduces to an affine parameter transformation. Notice also, that with a suitable choice of constants, it is always possible to choose a parametrisation such that $\tau \to \infty$ for a given value of $\bar{\tau}$. This property of conformal geodesics is in stark contrast to the behaviour of *standard* geodesics.

A final remark concerning the reparametrisation of conformal curves follows from evaluating Equations (5.47a) and (5.47b) at $\tau_\star$. One finds that $\bar{\boldsymbol{x}}'_\star = 4\varkappa\dot{\boldsymbol{x}}_\star$ and $\bar{\boldsymbol{\beta}}_\star = \bar{\boldsymbol{\beta}}_\star + \alpha_\star\dot{\boldsymbol{x}}^\flat_\star$. Consequently, the transformations of initial data given by

$$\dot{\boldsymbol{x}}_\star \mapsto 4\varkappa\dot{\boldsymbol{x}}_\star, \qquad \boldsymbol{\beta}_\star \mapsto \boldsymbol{\beta}_\star + \alpha_\star\dot{\boldsymbol{x}}^\flat_\star, \tag{5.49}$$

preserve the set of points covered by the conformal geodesics. *From the discussion in the previous paragraphs it follows that the transformation of initial data* (5.49) *implies a reparametrisation of the resulting curves.*

5.5.4 Geodesics as conformal geodesics

It is of natural interest to investigate the relation between conformal geodesics and metric geodesics. For a null conformal geodesic this relation can be readily established. If $(\bar{x}(\bar{\tau}), \bar{\boldsymbol{\beta}}(\bar{\tau}))$ denotes a null conformal geodesic, it follows readily from Equation (5.42a) that

$$\tilde{\nabla}_{\bar{\boldsymbol{x}}'}\bar{\boldsymbol{x}}' = -2\langle\boldsymbol{\beta}, \bar{\boldsymbol{x}}'\rangle\bar{\boldsymbol{x}}'.$$

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Thus, using an argument similar to the one discussed at the beginning of Section 5.5, one finds that *null conformal geodesics are, up to a reparametrisation, null geodesics.*

The situation for non-null conformal geodesics is more complicated and requires restrictions of the Schouten tensor of the spacetime. One has the following result (see Friedrich and Schmidt (1987)):

Lemma 5.2 (***standard geodesics as conformal geodesics***) *Any non-null $\tilde{\boldsymbol{g}}$-geodesic in an Einstein spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is, up to a reparametrisation, a non-null conformal geodesic.*

Proof Let $x(\tau)$ denote a solution to the metric geodesic equation $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0$. Consider a reparametrisation of the curve of the form $\tau = \tau(\bar{\tau})$. The analysis in Section 5.5.3 suggests completing $x(\bar{\tau})$ to a conformal geodesic using an ansatz of the form $\bar{\boldsymbol{\beta}} = \alpha(\bar{\tau})\dot{\boldsymbol{x}}^\flat$. Writing, as in the previous section, $\bar{\boldsymbol{x}}' = \tau'\dot{\boldsymbol{x}}$, Equation (5.42a) readily leads to the condition

$$\tau'' + \alpha\tau'^2\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 0,$$

where it is noticed that $\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})$ is constant along the curve as it is a $\tilde{\boldsymbol{g}}$-geodesic. To obtain an equation for α one substitutes the ansatz for $\bar{\boldsymbol{\beta}}$ into (5.42b) and notices that $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\bar{\boldsymbol{\beta}} = \alpha'\dot{\boldsymbol{x}}^\flat$ so that

$$\alpha'\dot{\boldsymbol{x}}^\flat = \frac{1}{2}\alpha^2\tau'\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\dot{\boldsymbol{x}}^\flat + \tau'\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot).$$

The solvability of this equation depends on the available information about $\tilde{\boldsymbol{L}}$. In the case of an Einstein space one has that $\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \cdot) = \frac{1}{6}\lambda\dot{\boldsymbol{x}}^\flat$ so that one obtains

$$\alpha' = \frac{1}{2}\alpha^2\tau'\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + \frac{1}{6}\lambda\tau',$$

which can always be solved – at least locally. □

A partial converse of Lemma 5.2 is given by:

Lemma 5.3 (***conformal geodesics as metric geodesics***) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ be a Einstein spacetime and let $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$ be a further metric on $\tilde{\mathcal{M}}$. A conformal geodesic $(\bar{x}(\bar{\tau}), \bar{\boldsymbol{\beta}}(\bar{\tau}))$ with respect to the metric $\boldsymbol{g}$ is, up to a reparametrisation, a $\tilde{\boldsymbol{g}}$-geodesic if there exists a function $\alpha(\bar{\tau})$ such that*

$$\bar{\boldsymbol{\beta}} = -\boldsymbol{\Upsilon} + \alpha\bar{\boldsymbol{x}}'^\flat.$$

Proof The geodesic equation $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 0$ implies, under the conformal rescaling $\boldsymbol{g} = \Xi^2\tilde{\boldsymbol{g}}$, the equation

$$\nabla_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = 2\langle\boldsymbol{\Upsilon}, \dot{\boldsymbol{x}}\rangle\dot{\boldsymbol{x}} - \boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})\boldsymbol{\Upsilon}^\sharp. \tag{5.50}$$

It follows from the analysis of Section 5.5.3 that, to reparametrise the conformal geodesic equations for the metric $\boldsymbol{g}$ to yield Equation (5.50), one needs to have a parameter α such that $\bar{\boldsymbol{\beta}} = -\boldsymbol{\Upsilon} + \alpha \bar{\boldsymbol{x}}'^{\flat}$. □

5.5.5 Conformal factors associated to congruences of conformal geodesics

In what follows, for simplicity it will be assumed that the spacetime $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ can be covered by a *non-intersecting* congruence of conformal geodesics. The congruence of conformal geodesics can be used to single out a metric $\boldsymbol{g} \in [\tilde{\boldsymbol{g}}]$ by means of a conformal factor Θ such that

$$\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1, \qquad \boldsymbol{g} = \Theta^2 \tilde{\boldsymbol{g}}. \tag{5.51}$$

That is, the tangent vector field of the congruence of conformal geodesics is $\boldsymbol{g}$-normalised – *accordingly, the parameter τ of the geodesics corresponds to the $\boldsymbol{g}$-proper time*. It follows by applying $\tilde{\nabla}_{\dot{\boldsymbol{x}}}$ to the first equation in (5.51) and using the conformal geodesic Equation (5.35a) that

$$\dot{\Theta} = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle \Theta, \tag{5.52}$$

where $\dot{\Theta} \equiv \tilde{\nabla}_{\dot{\boldsymbol{x}}} \Theta$. Thus, by prescribing $\Theta_\star \equiv \Theta(\tau_\star)$ at some fiduciary value $\tau_\star \in \mathbb{R}$ along the conformal geodesic one finds that the value of Θ is fully determined by Equation (5.52). If the initial value $\Theta_\star$ is chosen to vary smoothly along the curves on the congruence, one readily obtains a conformal factor for the whole of the spacetime. It is important to remark that this conformal factor depends on the particular congruence of conformal geodesics; a different choice of congruence would lead to a different Θ and, hence, to a different conformal metric $\boldsymbol{g}$. *Thus, if the congruence of conformal geodesics is specified by a prescription of initial data of the form given in* (5.37) *on an initial hypersurface $\mathcal{S}$, then $\boldsymbol{g}$ is determined in an implicit way by the initial data for the congruence and by $\Theta_\star$*. In the remainder of this section it will be shown that for metrics $\tilde{\boldsymbol{g}}$ satisfying the vacuum Einstein equations this correspondence can be made explicit.

A direct consequence of Equations (5.38a) and (5.52) is that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}} \left(\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \right) = 0.$$

Hence, one sees that a conformal geodesic that is, respectively, timelike, null or spacelike at a given point in $\tilde{\mathcal{M}}$ preserves its causal character throughout the whole curve. Further computations using the conformal geodesic Equations (5.35a) and (5.35b) and the relations (5.38a)–(5.38c) and (5.52) show that

$$\ddot{\Theta} = \frac{1}{2} \Theta \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta}) + \Theta \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}), \tag{5.53a}$$

$$\dddot{\Theta} = \left(\tilde{\nabla}_{\dot{\boldsymbol{x}}} (\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})) + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \boldsymbol{\beta}^\sharp) \tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) + \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) \right) \Theta. \tag{5.53b}$$

Moreover, if $\{\boldsymbol{e_a}\}$ denotes a $\boldsymbol{g}$-orthonormal frame, that is, $\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b}) = \eta_{\boldsymbol{ab}}$, propagated according to Equation (5.36) with $\boldsymbol{e_0} = \dot{\boldsymbol{x}}$, one readily finds that

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\Theta\langle\boldsymbol{\beta}, \boldsymbol{e_a}\rangle) = \Theta\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}}, \boldsymbol{e_a}) + \frac{1}{2}\Theta\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta})\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \boldsymbol{e_a}). \tag{5.54}$$

Notice that for the frame $\{\boldsymbol{e_a}\}$ one has, in addition, that $\tilde{\nabla}_{\dot{\boldsymbol{x}}}(\boldsymbol{g}(\boldsymbol{e_a}, \boldsymbol{e_b})) = 0$. The expressions discussed in the previous paragraph lead to the following result first proven in Friedrich (1995):

Proposition 5.1 (***the canonical conformal factor associated to a conformal geodesic***) *Let $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ denote an Einstein spacetime. Suppose that $(x(\tau), \boldsymbol{\beta}(\tau))$ is a solution to the conformal geodesic equations (5.35a) and* (5.35b) *and that $\{\boldsymbol{e_a}\}$ is a $\boldsymbol{g}$-orthonormal frame propagated along the curve according to Equation (5.36). If $\boldsymbol{g} = \Theta^2\tilde{\boldsymbol{g}}$ is such that $\boldsymbol{g}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = 1$, then the conformal factor Θ satisfies*

$$\Theta(\tau) = \Theta_\star + \dot{\Theta}_\star(\tau - \tau_\star) + \frac{1}{2}\ddot{\Theta}_\star(\tau - \tau_\star)^2, \tag{5.55}$$

where the coefficients $\Theta_\star \equiv \Theta(\tau_\star)$, $\dot{\Theta}_\star \equiv \dot{\Theta}(\tau_\star)$ and $\ddot{\Theta}_\star \equiv \ddot{\Theta}(\tau_\star)$ are constant along the conformal geodesic and are subject to the constraints

$$\dot{\Theta}_\star = \langle\boldsymbol{\beta}_\star, \dot{\boldsymbol{x}}_\star\rangle\Theta_\star, \qquad \Theta_\star\ddot{\Theta}_\star = \frac{1}{2}\tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\beta}_\star, \boldsymbol{\beta}_\star) + \frac{1}{6}\lambda. \tag{5.56}$$

Furthermore, along each conformal geodesic

$$\Theta\beta_{\boldsymbol{0}} = \dot{\Theta}, \qquad \Theta\beta_{\boldsymbol{i}} = \Theta_\star\beta_{\boldsymbol{i}\star}, \tag{5.57}$$

where $\beta_{\boldsymbol{a}} \equiv \langle\boldsymbol{\beta}, \boldsymbol{e_a}\rangle$.

Proof For an Einstein spacetime the Schouten tensor is given by $\tilde{\boldsymbol{L}} = \frac{1}{6}\lambda\tilde{\boldsymbol{g}}$. Substituting this expression into Equation (5.53b), one finds that $\dddot{\Theta} = 0$ so that Equation (5.55) follows. The constraints (5.56) follow from Equations (5.52) and (5.53a). Finally, the relations in (5.57) follow from (5.52) and (5.54). □

5.5.6 The $\tilde{g}$-adapted equations

As a consequence of the normalisation condition (5.51), the parameter τ is the $\boldsymbol{g}$-proper time of the curve $x(\tau)$. In some computations it is more convenient to consider a parametrisation in terms of a $\tilde{\boldsymbol{g}}$-proper time $\tilde{\tau}$. To this end, consider the parameter transformation $\tilde{\tau} = \tilde{\tau}(\tau)$ given by

$$\frac{\mathrm{d}\tau}{\mathrm{d}\tilde{\tau}} = \Theta, \qquad \text{so that} \qquad \tilde{\tau} = \tilde{\tau}_\star + \int_{\tau_\star}^{\tau} \frac{\mathrm{ds}}{\Theta(\mathrm{s})}, \tag{5.58}$$

with inverse $\tau = \tau(\tilde{\tau})$. In what follows, write $\tilde{x}(\tilde{\tau}) \equiv x(\tau(\tilde{\tau}))$. It can then be verified that

$$\tilde{\boldsymbol{x}}' \equiv \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{\tau}} = \frac{\mathrm{d}\tau}{\mathrm{d}\tilde{\tau}}\frac{\mathrm{d}x}{\mathrm{d}\tau} = \Theta\dot{\boldsymbol{x}}, \tag{5.59}$$

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so that $\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}') = 1$. Hence, $\tilde{\tau}$ is, indeed, the $\tilde{\boldsymbol{g}}$-proper time of the curve $\tilde{x}$. Now, consider, consistent with Equation (5.47b), the split

$$\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} + \varpi \dot{\boldsymbol{x}}^\flat, \qquad \varpi \equiv \frac{\langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle}{\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}})}, \tag{5.60}$$

where the covector $\tilde{\boldsymbol{\beta}}$ satisfies

$$\langle \tilde{\boldsymbol{\beta}}, \dot{\boldsymbol{x}} \rangle = 0, \qquad \boldsymbol{g}^\sharp(\boldsymbol{\beta}, \boldsymbol{\beta}) = \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle^2 + \boldsymbol{g}^\sharp(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}}). \tag{5.61}$$

It can be readily verified that

$$\tilde{\boldsymbol{g}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}) = \Theta^{-2}, \qquad \langle \boldsymbol{\beta}, \dot{\boldsymbol{x}} \rangle = \Theta^{-1}\dot{\Theta}, \qquad \varpi = \Theta\dot{\Theta}. \tag{5.62}$$

Using the split (5.60) in Equations (5.35a) and (5.35b) and taking into account the relations in (5.59), (5.61) and (5.62), one obtains the following ***$\tilde{g}$-adapted equations for the conformal geodesics***:

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'} \tilde{\boldsymbol{x}}' = \tilde{\boldsymbol{\beta}}^\sharp, \tag{5.63a}$$

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'} \tilde{\boldsymbol{\beta}} = \beta^2 \tilde{\boldsymbol{x}}'^\flat + \tilde{\boldsymbol{L}}(\tilde{\boldsymbol{x}}', \cdot) - \tilde{\boldsymbol{L}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}')\tilde{\boldsymbol{x}}'^\flat, \tag{5.63b}$$

with $\beta^2 \equiv -\tilde{\boldsymbol{g}}^\sharp(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}})$ – observe that as a consequence of (5.61) the covector $\tilde{\boldsymbol{\beta}}$ is spacelike, and, thus, the definition of β^2 makes sense. The Weyl propagation Equation (5.36) can also be cast in a $\tilde{\boldsymbol{g}}$-adapted form. A calculation shows that

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}(\Theta \boldsymbol{e}_{\boldsymbol{a}}) = -\langle \tilde{\boldsymbol{\beta}}, \Theta \boldsymbol{e}_{\boldsymbol{a}} \rangle \tilde{\boldsymbol{x}}'.$$

Equation (5.63a) provides a clear-cut interpretation of the covector $\tilde{\boldsymbol{\beta}}$ – it corresponds to the *physical acceleration* of the conformal curve. Recalling that $\tilde{\boldsymbol{g}} = \Theta^2 \boldsymbol{g}$ and using (5.61) together with Equation (5.57) of Proposition 5.1 one finds that

$$\beta^2 = -\tilde{\boldsymbol{g}}^\sharp(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}}) = -\Theta^2 \boldsymbol{g}^\sharp(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}}) = \Theta^2 \delta^{\boldsymbol{ij}} \beta_{\boldsymbol{i}} \beta_{\boldsymbol{j}} = \Theta_\star^2 \delta^{\boldsymbol{ij}} \beta_{\boldsymbol{i}\star} \beta_{\boldsymbol{j}\star}. \tag{5.64}$$

That is, β^2 is a constant along the conformal geodesic. Using Equation (5.63a) to eliminate $\tilde{\boldsymbol{\beta}}$ in Equation (5.63b), one obtains a third-order differential equation for the curve $\tilde{x}(\tilde{\tau})$:

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'} \tilde{\nabla}_{\tilde{\boldsymbol{x}}'} \tilde{\boldsymbol{x}}' = \beta^2 \tilde{\boldsymbol{x}}' + \tilde{\boldsymbol{L}}^\sharp(\tilde{\boldsymbol{x}}', \cdot) - \tilde{\boldsymbol{L}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{x}}')\tilde{\boldsymbol{x}}'. \tag{5.65}$$

A computation making use of the expressions derived in this section shows that

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\big(\tilde{\boldsymbol{g}}(\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{\beta}})\big) = 2\tilde{\boldsymbol{L}}(\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{\beta}}).$$

Consequently, unless $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is an Einstein spacetime the acceleration of the curve cannot be constant. This is related to an open question concerning the behaviour of conformal geodesics discussed in Tod (2012): if a conformal geodesic γ enters every neighbourhood of a point p, does γ necessarily pass through p with a finite limiting velocity and acceleration? This potential pathological behaviour is known as ***spiralling***; see Figure 5.1. This does not happen for

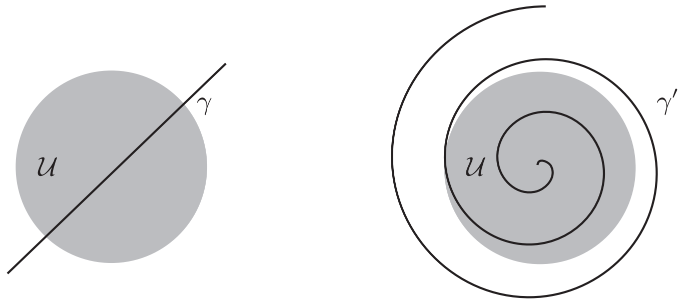

Figure 5.1 Spiralling of conformal geodesics: (left) a standard geodesic γ entering a geodesically convex ball $\mathcal{U}$ must leave it in finite proper time; (right) by contrast, a conformal geodesic γ' may not leave $\mathcal{U}$ and spiral towards a point.

standard geodesics, for if a geodesic enters a *geodesically convex ball*, then it must leave it too; see Section 11.6.2 for a discussion of the notion of geodesically convex ball. Using Piccard's existence theorem for ordinary differential equations – see, for example, Hartman (1987) – on Equation (5.65), it follows that spiralling can occur only if either $\tilde{\boldsymbol{\beta}}$ or $\tilde{\boldsymbol{x}}'$ diverge.

5.5.7 The conformal geodesic deviation equations

An important issue arising in applications involving congruences of conformal geodesics is that of deciding whether the congruence develops caustics, that is, points where it becomes singular. To address this one needs to consider the *conformal geodesic deviation equations* for the congruence. The deviation of these equations is analogous to the one leading to the geodesic deviation equation for standard geodesics; see Section 2.4.5.

In what follows let

$$(x_\eta(\tau), \boldsymbol{\beta}_\eta(\tau)) \equiv (x(\tau,\eta), \boldsymbol{\beta}(\tau,\eta))$$

denote a family of conformal geodesics depending smoothly on a parameter $\eta \in \mathbb{R}$. Following the notation used in previous sections for fixed η, let $\dot{\boldsymbol{x}}$ denote the tangent vector to the curves of the congruence. The ***deviation vector*** and ***deviation covector*** are defined, respectively, by

$$\boldsymbol{z} \equiv \partial_\eta x, \qquad \boldsymbol{\zeta} \equiv \tilde{\nabla}_{\boldsymbol{z}} \boldsymbol{\beta}. \tag{5.66}$$

A short computation shows that

$$[\dot{\boldsymbol{x}}, \boldsymbol{z}] = \tilde{\nabla}_{\dot{\boldsymbol{x}}} \boldsymbol{z} - \tilde{\nabla}_{\boldsymbol{z}} \dot{\boldsymbol{x}} = 0, \tag{5.67}$$

so that $\boldsymbol{z}$ is a well-defined *deviation vector*; compare Section 2.4.5. Moreover, making use of the definition of the Riemann tensor given by Equation (2.9), one has that

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$$\boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\dot{\boldsymbol{x}} = \tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\boldsymbol{z}}\dot{\boldsymbol{x}} - \tilde{\nabla}_{\boldsymbol{z}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}}. \tag{5.68}$$

Hence,

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\boldsymbol{z}}\dot{\boldsymbol{x}} = \tilde{\nabla}_{\boldsymbol{z}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} + \boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\dot{\boldsymbol{x}},$$

as a consequence of Equations (5.67) and (5.68). Now, using the conformal geodesic equation (5.35a) in the form $\tilde{\nabla}_{\dot{\boldsymbol{x}}}\dot{\boldsymbol{x}} = -\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})$, where $\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})$ corresponds to $S_{ab}{}^{cd}\dot{x}^a\dot{x}^b\beta_c$ in abstract index notation, one finds that

$$\begin{aligned}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z} &= -\tilde{\nabla}_{\boldsymbol{z}}(\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})) + \boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\dot{\boldsymbol{x}}\\ &= -\boldsymbol{S}(\tilde{\nabla}_{\boldsymbol{z}}\boldsymbol{\beta};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) - 2\boldsymbol{S}(\boldsymbol{\beta};\tilde{\nabla}_{\boldsymbol{z}}\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) + \boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\dot{\boldsymbol{x}}.\end{aligned} \tag{5.69}$$

A similar computation shows that

$$\begin{aligned}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\zeta} &= \tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\boldsymbol{z}}\boldsymbol{\beta} = \tilde{\nabla}_{\boldsymbol{z}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\beta} - \boldsymbol{\beta}\cdot\boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\\ &= \frac{1}{2}\tilde{\nabla}_{\boldsymbol{z}}(\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot)) + \tilde{\nabla}_{\boldsymbol{z}}(\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}},\cdot)) - \boldsymbol{\beta}\cdot\boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\\ &= -\boldsymbol{\beta}\cdot\boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z}) + \tilde{\nabla}_{\boldsymbol{z}}\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}},\cdot) + \tilde{\boldsymbol{L}}(\tilde{\nabla}_{\boldsymbol{z}}\dot{\boldsymbol{x}},\cdot) + \frac{1}{2}\tilde{\nabla}_{\boldsymbol{z}}\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot)\\ &\quad + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{S}(\tilde{\nabla}_{\boldsymbol{z}}\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot) + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z},\cdot)\end{aligned} \tag{5.70}$$

where, in the third line, Equation (5.35b) in the form

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\beta} = \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot) + \tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}},\cdot)$$

has been used. Finally, taking into account the definitions in (5.66) in Equations (5.69) and (5.70), one obtains the ***conformal geodesic deviation equations***:

$$\tilde{\nabla}_{\dot{\boldsymbol{x}}}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z} = \boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z})\dot{\boldsymbol{x}} - \boldsymbol{S}(\boldsymbol{\zeta};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}}) - 2\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z}), \tag{5.71a}$$

$$\begin{aligned}\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{\zeta} &= -\boldsymbol{\beta}\cdot\boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\dot{\boldsymbol{x}},\boldsymbol{z}) + \tilde{\nabla}_{\boldsymbol{z}}\tilde{\boldsymbol{L}}(\dot{\boldsymbol{x}},\cdot) + \tilde{\boldsymbol{L}}(\tilde{\nabla}_{\boldsymbol{z}}\dot{\boldsymbol{x}},\cdot) + \frac{1}{2}\boldsymbol{\zeta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\cdot)\\ &\quad + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\zeta};\dot{\boldsymbol{x}},\cdot) + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{S}(\boldsymbol{\beta};\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z},\cdot),\end{aligned} \tag{5.71b}$$

where

$$\boldsymbol{S}(\boldsymbol{\beta};\boldsymbol{u},\boldsymbol{v}) \equiv \langle\boldsymbol{\beta},\boldsymbol{u}\rangle\boldsymbol{v} + \langle\boldsymbol{\beta},\boldsymbol{v}\rangle\boldsymbol{u} - \tilde{\boldsymbol{g}}(\boldsymbol{u},\boldsymbol{v})\boldsymbol{\beta}^\sharp,$$

$$\boldsymbol{\alpha}\cdot\boldsymbol{S}(\boldsymbol{\beta};\boldsymbol{u},\cdot) \equiv \langle\boldsymbol{\alpha},\boldsymbol{u}\rangle\boldsymbol{\beta} + \langle\boldsymbol{\beta},\boldsymbol{u}\rangle\boldsymbol{\alpha} - \tilde{\boldsymbol{g}}^\sharp(\boldsymbol{\alpha},\boldsymbol{\beta})\boldsymbol{u}^\flat,$$

for $\boldsymbol{u},\boldsymbol{v}\in T(\tilde{\mathcal{M}})$ and $\boldsymbol{\alpha}\in T^*(\tilde{\mathcal{M}})$. In standard abstract index notation $\boldsymbol{S}(\boldsymbol{\beta};\boldsymbol{u},\boldsymbol{v})$ corresponds to the expression $S_{ab}{}^{cd}u^a v^b\beta_c$, while $\boldsymbol{\alpha}\cdot\boldsymbol{S}(\boldsymbol{\beta};\boldsymbol{u},\cdot)$, to $S_{ab}{}^{cd}u^a\beta_c\alpha_d$.

A ***caustic*** in a conformal geodesic is a point along the curve for which $\boldsymbol{z}=0$. Caustics of conformal geodesics are more complicated than caustics of metric geodesics since, for a given tangent vector, there exists a three-parameter family of conformal geodesics with the same tangent vector. Moreover, the analysis of Equation (5.71a) requires the simultaneous consideration of the evolution equation of the deviation covector $\boldsymbol{\zeta}$, Equation (5.71b). This feature can be useful in applications: Equation (5.71a) has two extra terms, $-\boldsymbol{S}(\boldsymbol{\zeta};\dot{\boldsymbol{x}},\dot{\boldsymbol{x}})$ and $-2\boldsymbol{S}(\boldsymbol{\beta};\dot{\boldsymbol{x}},\tilde{\nabla}_{\dot{\boldsymbol{x}}}\boldsymbol{z})$, not appearing in the standard geodesic deviation equation; under suitable circumstances these terms may be used to counteract the natural tendency of the curvature to develop caustics.

The $\tilde{\boldsymbol{g}}$-adapted conformal geodesic deviation equations

Following the strategy discussed in Section 5.5.6, one can rewrite the conformal geodesic deviation equations in a way adapted to the metric $\tilde{\boldsymbol{g}}$. To this end define the ***$\tilde{g}$-adapted deviation vector and covector***

$$\tilde{\boldsymbol{z}} \equiv \partial_\lambda \tilde{x}, \qquad \tilde{\boldsymbol{\zeta}} \equiv \tilde{\nabla}_{\tilde{\boldsymbol{z}}}\tilde{\boldsymbol{\beta}}.$$

Now, observing that $[\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{z}}] = 0$, a computation taking into account the $\tilde{\boldsymbol{g}}$-adapted conformal geodesic Equations (5.63a) and (5.63b) and the commutator of covariant derivatives leads to the following ***$\tilde{g}$-adapted conformal geodesic deviation equations***:

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{z}} = \boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{z}})\tilde{\boldsymbol{x}}' + \tilde{\boldsymbol{\zeta}}^{\sharp}, \tag{5.72a}$$

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{\zeta}} = -\tilde{\boldsymbol{\beta}}\cdot \boldsymbol{Riem}[\tilde{\boldsymbol{g}}](\tilde{\boldsymbol{x}}', \tilde{\boldsymbol{z}}) + (\tilde{\nabla}_{\tilde{\boldsymbol{z}}}\beta^2)\tilde{\boldsymbol{x}}'^{\flat} + \beta^2\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{z}}^{\flat}. \tag{5.72b}$$

A computation exploiting the fact that the connection $\tilde{\boldsymbol{\nabla}}$ is assumed to be torsion free gives

$$\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\nabla}_{\tilde{\boldsymbol{z}}}\beta^2 = \tilde{\nabla}_{\tilde{\boldsymbol{z}}}\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\beta^2 = 0,$$

where the last equality follows from the fact that β^2 is constant along a given conformal geodesic; see Equation (5.64). Hence, the components of the terms with $\tilde{\boldsymbol{x}}'^{\flat}$ and $\tilde{\nabla}_{\tilde{\boldsymbol{x}}'}\tilde{\boldsymbol{z}}^{\flat}$ in Equation (5.72b) are constant and can be evaluated at some fiducial time.

5.6 Further reading

Basic references for applications of conformal geometry in general relativity are Penrose and Rindler (1984, 1986) and Stewart (1991). A discussion of the properties of the Weyl and Cotton tensor can be found in García et al. (2004).

The first systematic treatments of conformal geodesics in the context of general relativity can be found in Schmidt (1986) and Friedrich and Schmidt (1987). A discussion of Weyl connections making use of the more general language of fibre bundles is given in Friedrich (1995); a brief presentation of the subject in the spirit of this chapter can be found in Friedrich (2002). A discussion of the properties of conformal geodesics in the context of general relativity can be found in Friedrich (2003a); a more technical discussion can be found in the earlier

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reference Friedrich (1995). Properties of conformal geodesics have been explored from a different perspective in Tod (2012).

The results of Proposition 5.1 strongly depend on the hypothesis that $(\tilde{\mathcal{M}}, \tilde{\boldsymbol{g}})$ is an Einstein space – in other words, $\tilde{\boldsymbol{g}}$ satisfies the vacuum Einstein equations. To get around this restriction, a more general class of curves has been introduced in Lübbe and Valiente Kroon (2012). These curves are a suitable generalisation of the conformal geodesics which allow the recovery of the conclusions of Proposition 5.1 for general spacetimes and, thus, provide a systematic way of identifying the conformal boundary of non-vacuum spacetimes. A discussion of the associated deviation equations with explicit expressions for the case of warped-product spacetimes is given in Lübbe and Valiente Kroon (2013a).

A detailed mathematical theory of conformal connections can be found in Ogiue (1967) and Kobayashi (1995). A more recent monograph on the subject is Fefferman and Graham (2012). Conformal geometry is naturally related to twistor theory; a discussion of this and related topics such as *tractors* can be found in Eastwood (1996).

The reader interested in surveys on research in conformal geometry is referred to Kulkarni and Pinkall (1988), Chang et al. (2007) and Branson et al. (2004) as suitable entry points to the literature in the subject.