Preface

Analysis and geometry on infinite-dimensional spaces is an active research field with many applications in mathematics and physics. Examples for applications arise naturally even when one is interested in problems that on first sight seem genuinely finite dimensional. You might have heard that it is impossible to accurately predict the weather over a long time. It turns out that this can be explained by studying the curvature of certain infinite-dimensional manifolds (Arnold, 1966). This example shows that everyday phenomena are intricately linked to geometric objects residing on infinite-dimensional manifolds. In recent years the list of novel applications for infinite-dimensional (differential) geometry has broadened considerably. Among the more surprising novelties are applications in stochastic and rough analysis (rough path theory à la T. Lyons leads to spaces of paths in infinite-dimensional groups; see Friz and Hairer, 2020) and renormalisation of stochastic partial differential equations via Hairer's regularity structures (see Bogfjellmo and Schmeding, 2018).

The aim of this book is to give an introduction to infinite-dimensional (differential) geometry. Differential geometry in infinite dimensions comes in many flavours, such as Riemannian and symplectic geometry. One can study Lie groups and their actions as well as Kähler manifolds, or Finsler geometry. As should already be apparent from this very incomplete list, it is simply not possible to cover a sizeable portion of the diverse topics subsumed under the label (infinite-dimensional) differential geometry. Hence the present book will focus on two main areas: Riemannian geometry and Lie groups. These topics are arguably the most prominent and well studied of the above list. Moreover, certain basic but important examples in both topics can be approached based on basic results on manifolds of (differentiable) mappings. However, it is important to stress that the focus of this book is introductory in nature. We will usually refrain from discussing results in their most general form if this allows

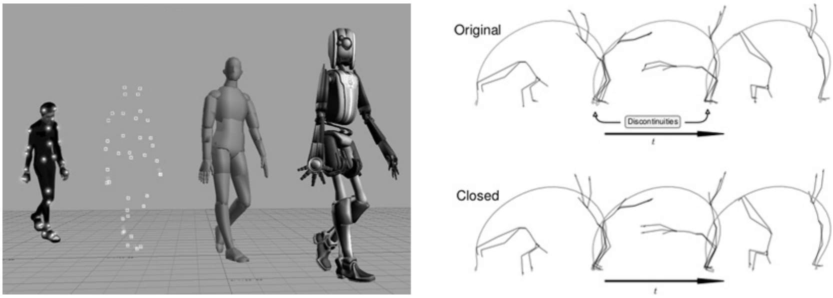

Figure 1 (a) Motion capturing using active markers to animate a figure. Based on the Wikimedia commons picture Activemarker2, public domain; see https://commons.wikimedia.org/wiki/File:Activemarker2.PNG, accessed: 07.12.21. (b) Removing discontinuities from animations via geometric methods. Reproduced from Celledoni et al. (2016) with permission of AIMS.

us to avoid lengthy technical discussions. Before we outline the programme of the book further, let us highlight two applications of infinite-dimensional geometric structures.

Shape analysis. Shapes are unparametrised curves in a vector space/on a manifold. Mathematically these are modelled by considering spaces of differentiable functions and quotienting out an appropriate action of a diffeomorphism group (modelling the reparametrisation). Spaces of differentiable functions can only be modelled using infinitely many parameters, whence they are prime examples of infinite-dimensional spaces. Notice that when talking about spaces of differentiable functions, the functions themselves are thought of as points in the infinite-dimensional space. This is a subtle point as a path between two points (aka functions) in this space will be a curve of curves. The aim of shape analysis is now to compare and transform shapes. For the comparison task one is interested in computing shortest distances between shapes. In the language of Riemannian geometry, one thus wants to compute geodesics (curves that are locally of shortest length) in these spaces. From the geometric formulation, a plethora of applications ranging from medical imaging to computer graphics can be dealt with. As a sample application we would like to highlight the processing of motion capturing data.

The data from the motion capturing process leads to a skeletal wireframe. Since motion capturing is, in general, expensive, one would like to create algorithms that interpolate between different movements or remove discontinuities occuring in the looping of motions (see Figure 1 for a graphical example of the problem and its (numerical) solution).

Current groups and diffeomorphism groups. Groups arising from problems related to differential geometry, fluid dynamics and the symmetry of evolution equations are often naturally infinite-dimensional manifolds with smooth group operations. Prime examples are the diffeomorphism groups $\mathrm{Diff}(K)$ for K a smooth and compact manifold. We encountered them already as reparametrisation groups in the shape analysis example. However, they are also of independent interest, for example in fluid dynamics: if K is a three-dimensional torus, the motion of a particle in a fluid corresponds, under periodic boundary conditions, to a curve in $\mathrm{Diff}(K)$; see Ebin and Marsden (1970). This observation led Arnold to his discovery of a general method by which (certain) partial differential equations (PDEs) can be lifted to ordinary differential equations on diffeomorphism groups. Typically these PDEs arise in the context of hydrodynamics. Many of the applications of this technique, nowadays known as *Euler–Arnold theory for PDEs*, come from geometric hydrodynamics (see Khesin and Wendt, 2009 or Modin, 2019 for an introduction). The already mentioned relation of weather forecasts to infinite-dimensional manifolds arises in a similar fashion.

To deal with these examples, one frequently needs to leave the theory of differential geometry on Banach manifolds (Lang, 1999), since diffeomorphism groups cannot be modelled as Lie groups on Banach spaces. To understand this, consider the canonical action of the diffeomorphisms $\mathrm{Diff}(K)$ on a connected compact manifold K, that is,

$$\alpha\colon\ \mathrm{Diff}(K) \times K \to K, \quad (\varphi, k) \mapsto \varphi(k).$$

This action is effective (i.e. if $\varphi(k) = \psi(k)$ for all k, then $\varphi = \psi$), and with some work one can show that the action α is transitive (i.e. for every pair of points there is a diffeomorphism mapping the first to the second; see Michor and Vizman, 1994). We would expect that any canonical Lie group structure turns α into a smooth map. However, Omori established the following:

Theorem (Omori, 1978) *If a (connected) Banach–Lie group G acts effectively, transitively and smoothly on a compact manifold, then G must be a finite-dimensional Lie group.*

Thus, since there is no way for the diffeomorphism group being described only by finitely many parameters, the diffeomorphism group $\mathrm{Diff}(K)$ cannot be a Lie group. To treat these examples, one thus has to look beyond the realm of Banach spaces.

Another example are the so-called current groups $C^\infty(K, G)$ of smooth mappings from a compact manifold to a Lie group G. The group structure is here given by pointwise multiplication of functions. In physics, current groups describe symmetries in Yang–Mills theories. For example, in the theory of electrodynamics, the gauge transformations for Maxwell's equations form a (Lie) subgroup of a current group. Note that the manifold K in physically relevant theories typically models spacetime and is non-compact. However, we will restrict ourselves in this book to current groups (and function spaces) on compact manifolds K. This allows us to avoid an overly technical discussion while special properties of the main examples remain accessible. As an example, we mention the 'lifting' of geometric features from finite-dimensional target Lie groups to the infinite-dimensional current groups. This procedure works particularly well in the special case where $K = \mathbb{S}^1$ is the unit circle. In this case the current group is better known under the name *loop group* $LG := C^\infty(\mathbb{S}^1, G)$; see Pressley and Segal (1986). Loop groups and current groups are great examples for a general property of spaces of smooth functions which we shall encounter often: under suitable assumptions, the infinite-dimensional geometry arises from 'lifting the finite-dimensional geometry' of the target spaces.

A common topic of the applications and mathematical topics mentioned above is an intimate connection between finite- and infinite-dimensional geometry. Indeed while infinite-dimensional differential geometry might seem like an arcane topic from the perspective of the finite-dimensional geometer, there are many sometimes surprising connections between the realms of finite- and infinite-dimensional geometry. We already mentioned Euler–Arnold theory and Arnold's insights relating curvature on infinite-dimensional manifolds to weather forecasts. Another example is Duistermaat and Kolk's proof of the third Lie theorem ('*every finite-dimensional Lie algebra is the Lie algebra of a Lie group*'; Duistermaat and Kolk, 2000, Section 1.14) or Klingenberg's investigation of closed geodesics (see Klingenberg, 1995). In both cases, infinite-dimensional techniques on path spaces were leveraged to solve the finite-dimensional problems. While many of the examples just mentioned can be studied while staying in the realm of finite-dimensional geometry, the link to infinite-dimensional geometry should not be disregarded as a purely academic exercise. A case in point might be the theory of rough paths discussed in §8. Here it generally suffices to consider truncated and thus finite-dimensional geometric settings. However, the infinite-dimensional limiting objects hint at deeper geometric insights hidden in the finite-dimensional perspective. It is my view that the infinite-dimensional perspective not only provides a convenient framework for these examples but exhibits important underlying structures and principles. These are worth exploring both for their own sake and for the connected applications.

In the context of the present book, we will explore the connection between the finite- and infinite-dimensional realm in the context of manifolds of differentiable mappings. These manifolds allow the lifting of geometric structures such as Lie groups and Riemannian metrics to interesting infinite-dimensional structures. For finite orders of differentiability, manifolds of differentiable mappings can be modelled on Banach spaces (see Palais, 1968). Thus they are within the grasp of differential geometry on Banach spaces (Lang, 1999). However, since certain constructions (such as the exponential law) become much more technical for finite orders of differentiability, we shall in the present book study exclusively manifolds and spaces of smooth (i.e. infinitely often differentiable) mappings. This places us immediately outside of the realm of Banach manifolds but enables important constructions such as the Lie group structure for diffeomorphism groups.

Finally, I would like to draw the reader's attention to the fact that besides many links and connections between finite- and infinite-dimensional geometry, there are quite severe differences between both settings of differential geometry. Many of the strong structural statements from finite-dimensional geometry simply cease to be valid when passing to manifolds modelled on infinite-dimensional spaces (many statements already break down in the more familiar Banach setting). These pitfalls are well known to experts but often surprise beginners in the field. In the spirit of providing an introduction to infinite-dimensional geometry, I have taken care to emphasise the differences and illustrate them with examples where possible. For example, the following problems are frequently encountered when passing to the infinite-dimensional setting:

- Infinite-dimensional spaces *cannot be locally compact* and they do not support a unique vector topology. Thus arguments building on compactness are not available (see Appendix A).
- Smooth bump functions do not need to exist (see Appendix A.4). Hence the usual local-to-global arguments become unavailable.
- Beyond Banach spaces, there is *no general solution theory for ordinary differential equations* and no general inverse function theorem (see Appendix A.5).
- Dual spaces become more difficult to handle. As a consequence, it is impossible to define differential forms as sections in dual bundles in general (see Remark 1.45).
- Equivalent definitions from finite-dimensional differential geometry are often not equivalent in the infinite-dimensional setting. This happens, for example, for submersions and immersions; compare §1.7.

The structure of the book is as follows. First of all, we shall provide the necessary foundational material for differential geometry in the general infinite-dimensional setting in the first two chapters. Here we emphasise

(a) calculus and manifolds on infinite-dimensional spaces beyond the Banach setting, and
(b) manifolds of differentiable functions.

Beyond Banach spaces there are several choices as to how one can generalise the concept of differentiability. We adopted the so-called Bastiani calculus based on iterated directional derivatives. This choice is different from the popular 'convenient calculus'. We shall compare the calculi in later chapters. Furthermore, we discuss several foundational topics, such as locally convex spaces, in a series of appendices. The material covered there should allow the reader to grasp most of the basics needed to follow the main part of the book. Moreover, the material in the appendices gives some insight into the common problems arising in the passage from finite- to infinite-dimensional settings already mentioned. While the setting in which we will be working is quite general, we will often not exhibit the most general definitions, results and settings which can possibly be treated in the framework. For example, only manifolds of mappings on a compact manifold are considered here. The non-compact case is interesting and deserving of attention, however the associated theory is much more involved and technical. Our philosophy here is that if the simplified case already admits applications and exhibits the character and problems of the infinite-dimensional setting, we will restrict our attention to the simple case. However, we shall comment on the more general case and provide pointers to the literature. Armed with the knowledge provided by the present book, the reader should be able to quickly learn the more general case should she so desire.

Having dealt with the general setting, we shall then study the main objects of interest in this book:

(a) (infinite-dimensional) Lie groups and Lie algebras, Chapter 3 and
(b) (weak) Riemannian geometry, Chapter 4.

Based on these building blocks, we shall explore several applications of infinite-dimensional geometry in the next chapters. These range from shape analysis to connections with higher geometry (in the guise of Lie groupoids) to Euler–Arnold theory for PDEs and the geometry of rough path spaces. These later chapters can be read mostly independently from each other. I have selected the topics for the advanced chapters with a view towards developments within the field over the last years. The broad selection of topics and the introductory nature of the present book prevent an in-depth discussion of these advanced topics. Therefore these chapters gracefully omit many of the more technical and subtle points. However, the reader will be able to gain an impression of the role infinite-dimensional differential geometry plays in these applications. Moreover, there are ample references to the literature which should enable the interested reader to follow up on a topic after perusing the respective chapter.

Before we begin, let us set some conventions that will be in effect for all that is to come.

Conventions

- Write $\mathbb{N} := \{1,2,3,\ldots\}$ for the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- All topological spaces are assumed to be Hausdorff, that is, for every pair of two (distinct) points in the space, there are disjoint open neighbourhoods of these points.
- If nothing else is said, products of topological spaces will always carry the product topology.
- If $U \subseteq X$ is an open subset of a topological space, we also write shorter $U \Subset X$.

We shall exclusively work over the real numbers $\mathbb{R}$, all vector spaces are to be understood over the real numbers. However, many results carry over to complex vector spaces. For example, it is no problem to define the notion of complex differentiability (see e.g. Glöckner, 2002).

Recommended Further Reading

As mentioned above, infinite-dimensional differential geometry is a vast topic, which we cannot hope to cover in this book. Thus we have concentrated on an introduction to infinite-dimensional Lie groups, (weak) Riemannian metrics and their interplay. For further reading on the topics of this book, we would like to recommend the following works, which have also influenced the presentation in the present book:

- **(infinite-dimensional) Lie theory** (Neeb, 2005) – lecture notes; (Neeb, 2006) – extensive survey; and, once it becomes available (Glöckner and Neeb, forthcoming).
- **Manifolds of mappings** (Wockel, 2014); on non-compact domains (Michor, 1980).
- **(weak) Riemannian geometry** (Bruveris, 2018, 2019).

Furthermore, there is a host of topics that could not be covered in this book. Albeit there is again no hope that we could do justice to the vast body of research on these topics, we would like to mention a few works that are either introductory or exhibit new phenomena which are genuinely infinitely dimensional in nature:

- **Symplectic geometry** (Abraham et al., 1988 (Banach setting); Kriegl and Michor (1997, Section 48)).
- **Sub-Riemannian geometry** (Grong et al., 2015; Agrachev and Caponigro, 2009).
- **Kähler geometry** (Sergeev, 2020).
- **Poisson geometry** (Beltiţă et al., 2018).
- **Finsler geometry** (Larotonda (2019) on diffeomorphism groups; Eftekharinasab and Petrusenko (2020) in the context of bounded Fréchet geometry).

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I have profited from an early draft of the forthcoming book by H. Glöckner and K.-H. Neeb on infinite-dimensional Lie groups (Glöckner and Neeb, forthcoming). Their lucid presentation of calculus directly influenced §1. Material from the thesis by A. Mykleblust ('Limits of Banach spaces', UiB 2020, advised by A. Schmeding) is included in Appendix A.

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