Can. J. Math., Vol. XXII, No. 5, 1970, pp. 1082–1096

k-DEGENERATE GRAPHS

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1. Introduction. Graphs possessing a certain property are often characterized in terms of a type of configuration or subgraph which they cannot possess. For example, a graph is totally disconnected (or, has chromatic number one) if and only if it contains no lines; a graph is a forest (or, has point-arboricity one) if and only if it contains no cycles. Chartrand, Geller, and Hedetniemi [**2**] defined a graph to have property P_n if it contains no subgraph homeomorphic from the complete graph K_{n+1} or the complete bipartite graph

$$K\left(\left[\frac{n+2}{2}\right], \left\{\frac{n+2}{2}\right\}\right).$$

For the first four natural numbers n, the graphs with property P_n are exactly the totally disconnected graphs, forests, outerplanar and planar graphs, respectively. This unification suggested the extension of many results known to hold for one of the above four classes of graphs to one or more of the remaining classes. Chartrand, Geller, and Hedetniemi were very successful in this approach, but the methods of proof employed for values of n less than five generally do not extend for larger values.

In this paper we adopt a different viewpoint. Instead of studying graphs with property P_n, we consider the class Π_k of all k-degenerate graphs, for k a non-negative integer. The classes Π_0 and Π_1 are exactly the classes of totally disconnected graphs and of forests, respectively; the classes Π_2 and Π_5 properly contain all outerplanar and planar graphs, respectively. The advantage of this viewpoint is that many of the known results for chromatic number and point-arboricity (corresponding to the cases $k = 0$ and $k = 1$, respectively) have natural extensions, for *all* larger values of k. In several cases, the methods of proof were suggested by techniques employed by Chartrand, Geller, and Hedetniemi.

In § 2 we give basic definitions and establish some elementary properties for k-degenerate graphs. In § 3 the point-partition numbers ρ_k are defined. These concepts generalize the chromatic number ($k = 0$) and point-arboricity ($k = 1$) of a graph. The point-partition numbers of the complete n-partite graphs are developed in § 4. Section 5 treats graphs which are n-critical, n-durable, n-minimal, or n-permanent with respect to the parameter ρ_k.

Received February 23, 1970. The research of the first author was partially supported by grants from the National Science Foundation (GP-9435) and Western Michigan University (Faculty Research Fellowships).

Section 6 provides bounds for the point-partition numbers of an arbitrary graph. In § 7 it is shown that if a graph is a counter-example to the four-colour conjecture, then it must be 5-degenerate but not 4-degenerate.

2. Definitions and elementary properties. Those definitions not given in this section may be found in [**8**]. The graphs under consideration here are ordinary graphs; i.e. finite undirected graphs with neither loops nor multiple lines. The point set of a graph G is denoted by $V(G)$, while the line set is denoted by $E(G)$. The *degree* $d(v)$, of a point v of G is the number of lines incident with v. The smallest degree among the points of G is called the *minimum degree* of G and is denoted by $\delta(G)$. Similarly, the *maximum degree* of G is denoted by $\Delta(G)$.

A *subgraph* H of a graph G consists of a subset of the point set of G and a subset of the line set of G which together form a graph. The notation $H \leqq G$ will be employed to indicate that H is a subgraph of G. An important type of subgraph is the following: the *subgraph induced by a set* U of points of G, denoted by $\langle U \rangle$, has U as its point set and contains all lines of G incident with two points of U. Two subgraphs are said to be *disjoint* if they have no points in common.

The *complete* n-*partite graph* $K(p_1, \ldots, p_n)$ has its point set V partitioned into subsets V_i, with $|V_i| = p_i$, $i = 1, \ldots, n$; two points u and v are adjacent if and only if $u \in V_j$, and $v \in V_h$, where $j \neq h$. If $p_i = 1$, $i = 1, \ldots, n$, the graph is the *complete graph* on n points, and is denoted by K_n.

A graph G is said to be k-*degenerate*, for k a non-negative integer, if for each induced subgraph H of G, $\delta(H) \leqq k$. We use the symbol Π_k to denote the class of all k-degenerate graphs. The graph G illustrated in Figure 1 is a 2-degenerate graph which is not 1-degenerate.

G:

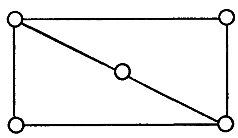

FIGURE 1

A *totally disconnected* graph is one which has no lines. It is evident that a graph is totally disconnected if and only if it is 0-degenerate. A *forest* is a graph without cycles, and these graphs are exactly the 1-degenerate graphs. An *outerplanar* graph is a graph which can be embedded in the plane so that each of its points lies in the boundary of the exterior region. Since every outerplanar graph has a point of degree at most two, and each subgraph of an outerplanar graph is outerplanar, every outerplanar graph is 2-degenerate. The graph G of Figure 1, however, is a 2-degenerate graph which is not outer-planar, so that the class of all outerplanar graphs is a proper subset of Π_2. A *planar* graph is one which can be embedded in the plane. Every planar graph has a point of degree at most five, and each subgraph of a planar graph is planar. It follows that a planar graph must be 5-degenerate. The complete graph K_5 is a 5-degenerate graph which is not planar, so that the class of all planar graphs is a proper subset of the class Π_5.

It is easy to see that the complete graph K_{p+2} is $(p+1)$-degenerate, but not p-degenerate. Hence Π_k is a proper subset of Π_{k+1}, for each non-negative k. We note also that, for any graph G, Π_2 contains a graph H homeomorphic from G. (For instance, form H by replacing every line uv of G with a point w and the two lines uw and wv.) In particular, Π_2 contains a graph H homeomorphic from each of the "forbidden subgraphs" of Chartrand, Geller, and Hedetniemi.

We now make some elementary observations about k-degenerate graphs.

PROPOSITION 1. *A graph G is in Π_k if and only if the point set $V(G)$ can be ordered, say $v_1, \ldots, v_p$, such that $d(v_1) \leqq k$ and, in the induced subgraph $\langle\{v_n, \ldots, v_p\}\rangle$ of G, $d(v_n) \leqq k$, for each $n = 1, \ldots, p$.*

In other words, G can be reduced to the degenerate (i.e. trivial) graph K_1 by a sequence of removal of points of degree less than or equal to k.

PROPOSITION 2. (i) *If G is in Π_k, then G is in Π_n, for each $n \geqq k$.*

(ii) *For each graph G there is a minimal non-negative integer k such that G is in Π_k. Furthermore, $k \leqq \Delta(G)$.*

(iii) *A graph G is in Π_k if and only if each component of G is in Π_k.*

(iv) *If the graph G is in Π_k, then each subgraph of G is also in Π_k.*

One might expect that a graph G belongs to Π_k if and only if each block of G is in Π_k. Each block of the graph G of Figure 2, however, is 2-degenerate, even though the graph G itself is not 2-degenerate.

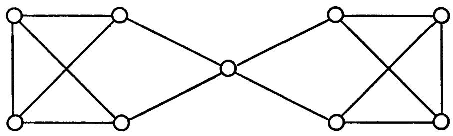

FIGURE 2

PROPOSITION 3. *Let G belong to Π_k and let G have p points, $p \geqq k$. Then G has at most $kp - \binom{k+1}{2}$ lines.*

The proof follows in a routine manner by induction on the order of G.

Let $\bar{G}$ denote the complement of G. If G is in Π_k, but $G + e$ is not in Π_k for every line e in $E(\bar{G})$, then G is said to be a *maximal k-degenerate graph*. Note that K_p is maximal k-degenerate, for $p \leqq k + 1$.

PROPOSITION 4. *Let G be a maximal k-degenerate graph with p points, $p \geqq k + 1$. Then $\delta(G) = k$.*

Proof. If G has $p = k + 1$ points, then $G = K_{k+1}$ and $\delta(G) = k$. If G has $p = k + 2$ points, then $G = K_{k+2} - e$, where e is a line of K_{k+2}, and $\delta(G) = k$.

Thus we assume that G has p points, $p > k + 2$, and that there is a point v of G with $d(v) < k$. Let u be any point of $G - v$ not adjacent to v in G. Let H be any induced subgraph of $G + uv$. If $v \notin V(H)$, then H is an induced subgraph of G and so $\delta(H) \leqq k$. If $v \in V(H)$, since $d(v) < k$ in G, $\delta(H) < k$. In either case, $\delta(H) \leqq k$ and thus $G + uv$ is in Π_k. This contradicts the fact that G is a maximal k-degenerate graph. Therefore $\delta(G) = k$.

PROPOSITION 5. *Let G be a maximal k-degenerate graph with p points, $p \geqq k + 1$. Let v be a point of degree k. Then $G - v$ is a maximal k-degenerate graph.*

Proof. Assume that v is a point of G with $d(v) = k$ and that $G - v$ is not a maximal k-degenerate graph. Hence there is a line e of $\overline{G - v}$, the complement of $G - v$, such that $G - v + e$ is also a k-degenerate graph. But since $d(v) = k$ and e is not incident with v, $G + e$ is also a k-degenerate graph. This contradicts the fact that G is a maximal k-degenerate graph.

COROLLARY 1. *Let G be a maximal k-degenerate graph with p points, $p \geqq k$. Then G has $kp - \binom{k+1}{2}$ lines.*

The inductive proof is routine, using Propositions 4 and 5.

If G is a maximal 1-degenerate graph with p points, by Corollary 1, G must have $p - 1$ lines; that is, G is a tree and is therefore connected. This observation can be extended to maximal k-degenerate graphs. A graph G is said to be *n-connected* if the removal of any m points from G, $0 \leqq m < n$, results in neither a disconnected graph nor the trivial graph consisting of a single point. The 1-connected graphs are simply the connected graphs.

THEOREM 1. *Let G be a maximal k-degenerate graph with p points, $p \geqq k + 1$. Then G is k-connected.*

Proof. If $p = k + 1$, then $G = K_{k+1}$ and G is k-connected. If $p = k + 2$, then $G = K_{k+2} - e$, where e is a line of K_{k+2}, and again G is k-connected. We assume that any maximal k-degenerate graph with p points, $k + 1 \leqq p \leqq n$, is k-connected. Let G be a maximal k-degenerate graph with $n + 1$ points. Then Proposition 4 states that there is a point v of G with $d(v) = k$. Proposition 5 now states that $G - v$ is also a maximal k-degenerate graph, having n points. By the inductive assumption, then, $G - v$ is k-connected. Assume that G is not k-connected. Then there is a $(k - 1)$-cutset S of G. If $v \in S$, then $S - \{v\}$ is a $(k - 2)$-cutset of $G - v$, which contradicts the fact that $G - v$ is k-connected. Let C be the component of $G - S$ containing v. If there is a vertex $u \neq v$ in C, then S is a $(k - 1)$-cutset of $G - v$. This is also a contradiction. Thus $C = \{v\}$. But in this case, v is adjacent only to points of S and so $d(v) \leqq k - 1$. Again we have a contradiction, since $d(v) = k$. Therefore G is k-connected. By induction the result follows for all $p \geqq k + 1$.

https://doi.org/10.4153/CJM-1970-125-1 Published online by Cambridge University Press

We now prove another result dealing with the degrees of maximal k-degenerate graphs.

PROPOSITION 6. *Let G be a maximal k-degenerate graph with p points, $p \geqq k + 1 \geqq 2$. Then G has at least $k + 1$ points whose degrees do not exceed $2k - 1$.*

Proof. Since G is a maximal k-degenerate graph with p points, G has $kp - \binom{k+1}{2}$ lines. Thus the sum of the degrees of the points of G is $2kp - k(k + 1)$. If all the points of G had degree $2k$, then the sum of the degrees of the points of G would be $2kp$. Hence there must be enough points whose degrees are less than $2k$ so that the number of $2kp$ is reduced by $k(k + 1)$. However, every point of G has degree at least k, and thus the degree of no one point of G can reduce the number $2kp$ by more than k. Therefore, there must be at least $k + 1$ points whose degrees do not exceed $2k - 1$.

COROLLARY 2. *If G is in Π_k and if G has p points, $p \geqq k + 1 \geqq 2$, then G has at least $k + 1$ points whose degrees do not exceed $2k - 1$.*

We now show that the bound of Proposition 6 is the best possible, by constructing a graph with exactly $k + 1$ points of degree not exceeding $2k - 1$. Let the points of K_{k+1} be denoted by $v_1, \ldots, v_{k+1}$, and let G_{k+1} be the totally disconnected graph with points $u_1, \ldots, u_{k+1}$. Let H_{k+1} be the union of the graphs K_{k+1} and G_{k+1} with all additional lines $u_i v_j$, for $i \neq j$. Then the graph H_{k+1} has $k + 1$ points of degree k and $k + 1$ points of degree $2k$. It is easy to see that H_{k+1} is k-degenerate. See Figure 3 for the graph H_4.

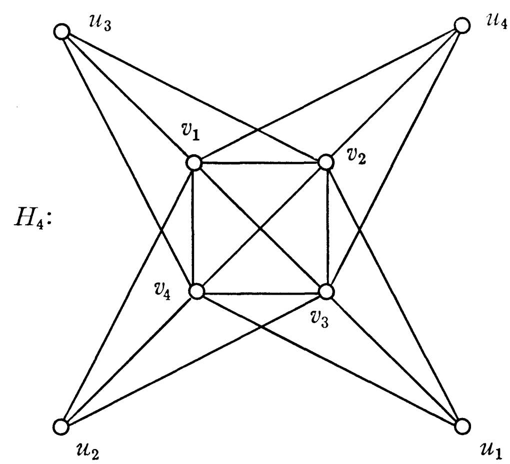

FIGURE 3

3. The point partition numbers. As mentioned in Proposition 2, for any

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graph G there is a non-negative integer k such that G is k-degenerate. For a given graph G and a given non-negative integer k, the graph G may not be k-degenerate, but it is clear that some subgraphs of G are k-degenerate. This observation leads to the following problem. For a given graph G and a given non-negative integer k, find the smallest number of disjoint induced subgraphs into which G can be divided so that each subgraph is k-degenerate.

The *point partition number* $\rho_k(G)$, $k \geqq 0$, is the minimum number of sets into which the point set $V(G)$ can be partitioned so that each set induces a k-degenerate subgraph of G.

The point partition number $\rho_0(G)$ is the extensively studied *chromatic number* of G, while the point partition number $\rho_1(G)$ is the more recently investigated *point-arboricity* of G. For values of $k \geqq 2$, no special name has yet been given to the parameter $\rho_k(G)$ and these concepts have not yet been considered as a topic of research. Each of the parameters $\rho_k(G)$ may be thought of as a colouring number, since it gives the minimum number of colours in any colouring of the points of G so that each colour induces a k-degenerate subgraph of G.

It is helpful, in determining the numbers $\rho_k(G)$, to note that we may restrict ourselves to connected graphs (see Proposition 2 (iii)).

PROPOSITION 7. *The value of* $\rho_k(G)$ *is the maximum of the values of* $\rho_k(C_i)$ *for the components* C_i *of* G.

Since $\rho_k(G) \geqq 1$ for each graph G and each non-negative integer k, and $\rho_n(G) \leqq \rho_m(G)$ for $n \geqq m$, we make the following elementary observation.

PROPOSITION 8. *If G is a k-degenerate graph and $m \geqq k$, then* $\rho_m(G) = 1$.

4. The point partition numbers of the complete n-partite graphs. We now consider the point partition numbers in more detail. As one would expect, for most graphs G and for small values of k, the numbers $\rho_k(G)$ are difficult to determine. However, for one important class of graphs, the complete n-partite graphs, the numbers $\rho_k(G)$ are easily calculated. The following proof generalizes the approach employed by Chartrand, Kronk, and Wall [**4**].

THEOREM 2. *The point partition number* ρ_k *of the complete n-partite graph* $K(p_1, \ldots, p_n)$, $1 \leqq p_1 \leqq \ldots \leqq p_n$, *is given by*:

$$\rho_k(K(p_1, \ldots, p_n)) = n - \max\left\{j : \sum_{i=0}^{j} p_i \leqq (n-j)k\right\},$$

where we define $p_0 = 0$.

Proof. Since $\rho_0(K(p_1, \ldots, p_n)) = n$, Proposition 2(i) implies that $\rho_k(K(p_1, \ldots, p_n)) \leqq n$. We employ induction on n. For $n = 1$, $\rho_k(K(p_1)) = 1$, since $K(p_1)$ is totally disconnected. Assume that the formula holds for n, $n \geqq 1$, and consider the graph $G = K(p_1, \ldots, p_n, p_{n+1})$ with point set $V(G)$

and subsets $V_1, \ldots, V_{n+1}$ as described in the definition of an $(n+1)$-partite graph. For the subgraph $H = K(p_1, \ldots, p_n)$, suppose that

$$\sum_{i=0}^{t} p_i \leqq (n-t)k,$$

but,

$$\sum_{i=0}^{t+1} p_i > (n-(t+1))k.$$

Then, by hypothesis, $\rho_k(H) = n - t$. Since H is a subgraph of G, $\rho_k(H) \leqq \rho_k(G)$. Also, since the additional set V_{n+1} of p_{n+1} points used in forming G from H induces a totally disconnected (and thus k-degenerate) graph, $\rho_k(G) \leqq \rho_k(H) + 1$. We now consider two cases.

Case (i). Suppose that

$$\sum_{i=0}^{t+1} p_i > ((n+1)-(t+1))k = (n-t)k.$$

This implies that

$$(n+1) - \max\left\{j : \sum_{i=0}^{j} p_i \leqq ((n+1)-j)k\right\} = (n+1) - t.$$

Thus, in this case, we must show that $\rho_k(G) = \rho_k(H) + 1$. Assume that this is not the case, i.e. $\rho_k(G) = \rho_k(H)$. The complete $(n+1)$-partite graph $G' = K(p_1, \ldots, p_t, p_{t+1}, p_{t+1}, \ldots, p_{t+1})$ is a subgraph of the graph G and so $\rho_k(G') \leqq \rho_k(G) = \rho_k(H) = n - t$. Let $V_1', \ldots, V_{n+1}'$ be the subsets of $V(G')$ as in the definition of a complete $(n+1)$-partite graph. The graph G' contains

$$\sum_{i=0}^{t+1} p_i + (n-t)p_{t+1} = p^*$$

points, which implies that in any partition of the point set $V(G')$ into $n - t$ (or fewer) subsets, at least one such subset must contain at least $p^*/(n-t)$ points. By hypothesis,

$$\sum_{i=0}^{t+1} p_i > (n-t)k,$$

and so at least one of the subsets of the partition, say U_1, contains at least $p_{t+1} + (k+1)$ points. But any of the subsets $V_1', \ldots, V_{n+1}'$ of $V(G')$ can contain at most p_{t+1} points. Thus, for any $v \in V(\langle U_1 \rangle)$ (say that $v \in V_i$) there are at least $k+1$ points in $\langle U_1 \rangle - V_i$. Hence $d(v) \geqq k+1$ in $\langle U_1 \rangle$; i.e. $\delta(\langle U_1 \rangle) > k$, and $\langle U_1 \rangle$ is not k-degenerate, a contradiction. Thus $\rho_k(G) = \rho_k(H) + 1$.

Case (ii). Suppose that

$$\sum_{i=0}^{t+1} p_i \leqq (n-t)k.$$

Since

$$\sum_{i=0}^{t+1} p_i > (n - (t+1))k,$$

it follows that

$$\sum_{i=0}^{t+2} p_i > (n - (t+1))k = ((n+1) - (t+2))k,$$

so that

$$(n+1) - \max\left\{j : \sum_{i=0}^{j} p_i \leqq ((n+1) - j)k\right\} = (n+1) - (t+1) = n - t.$$

Thus, we must prove, in this case, that $\rho_k(G) = \rho_k(H)$. Since

$$\sum_{i=0}^{t+1} p_i \leqq (n-t)k,$$

or equivalently,

$$\begin{aligned}|V_1 \cup V_2 \cup \ldots \cup V_{t+1}| &\leqq ((n+1) - (t+2) + 1)k \\ &= (n-t)k,\end{aligned}$$

we can exhaust the set $V_1 \cup V_2 \cup \ldots \cup V_{t+1}$ by adding at most k of its points to each of the sets $V_{t+2}, \ldots, V_{n+1}$. Since, for each set V_j, $t + 2 \leqq j \leqq n + 1$, at most k points from the set $V_1 \cup V_2 \cup \ldots \cup V_{t+1}$ have been added, the resulting set induces a k-degenerate subgraph of G. Hence $\rho_k(G) \leqq n - t$. But then $n - t = \rho_k(H) \leqq \rho_k(G) \leqq n - t$, and $\rho_k(G) = n - t$.

We now list the point partition numbers of the complete graphs and the complete bipartite graphs, as two corollaries to Theorem 2. The notation $\{r\}$ indicates the least integer greater than or equal to r.

COROLLARY 3. *For the complete graph K_p with p points,*

$$\rho_k(K_p) = \left\{\frac{p}{k+1}\right\}.$$

COROLLARY 4. *For the complete bipartite graph $K(p_1, p_2)$, with $p_1 \leqq p_2$,*

$$\rho_k(K(p_1, p_2)) = \begin{cases} 1, & \text{if } p_1 \leqq k, \\ 2, & \text{if } p_1 > k. \end{cases}$$

Since every graph with p points can be considered as a subgraph of K_p, we obtain the following upper bound for $\rho_k(G)$.

COROLLARY 5. *For every graph G with p points and for every non-negative integer k,*

$$\rho_k(G) \leqq \left\{\frac{p}{k+1}\right\}.$$

5. Critical and durable graphs. In general, the bound given in Corollary 5 for $\rho_k(G)$ is not particularly good. We will sharpen this bound in § 6, using two of the results of this section.

A graph G is said to be *n-critical with respect to ρ_k* if $\rho_k(G) = n$, but $\rho_k(G - v) = n - 1$ for each point v of G. A graph G is said to be *n-minimal with respect to ρ_k* if $\rho_k(G) = n$, but $\rho_k(G - e) = n - 1$ for each line e of G. Graphs which are critical with respect to ρ_0 (chromatic number) have been studied extensively, particularly by Dirac (see [**5**; **6**; **7**]). In [**9**], we considered graphs which are n-critical and n-minimal with respect to ρ_0 and ρ_1 (point-arboricity).

Proposition 9. *If G is n-minimal with respect to ρ_k and if G has no isolated points, then G is n-critical with respect to ρ_k.*

Proof. Let G be n-minimal with respect to ρ_k and assume that G has no isolated points. Then for any point v of G, v is incident with at least one line e of G. Then

$$\rho_k(G - v) \leqq \rho_k(G - e) = n - 1,$$

so that $\rho_k(G - v) = n - 1$, and G is n-critical with respect to ρ_k.

It is well known that any graph having chromatic number n contains an n-critical induced subgraph. Chartrand and Kronk [**3**] have established the analogue for point-arboricity. In the following theorem, we generalize these results for all values of k.

Theorem 3. *Let G be a graph such that $\rho_k(G) = n$, where $n \geqq 2$. Then G contains an induced n-critical subgraph.*

Proof. If G is n-critical with respect to ρ_k, there is nothing to prove. Otherwise, there is a point v_1 in G such that $\rho_k(G - v_1) = n$. Now, if $G - v_1$ is n-critical, our proof is complete. Otherwise, there is a point v_2 in $G - v_1$ such that $\rho_k(G - v_1 - v_2) = n$. Continuing this process, we eventually arrive at an induced n-critical subgraph.

It also seems natural to study graphs G for which $\rho_k(G - v) = \rho_k(G)$ for each point v of G, and graphs H which satisfy $\rho_k(H - e) = \rho_k(H)$ for each line e of H. Such graphs are called *durable with respect to ρ_k* and *permanent with respect to ρ_k*, respectively. If $\rho_k(G) = n$ and G is durable with respect to ρ_k, then G is said to be *n-durable with respect to ρ_k*. Similarly, if $\rho_k(H) = n$ and H is permanent with respect to ρ_k, then H is called *n-permanent with respect to ρ_k*. In [**9**], we investigated graphs which are n-durable and n-permanent with respect to ρ_0 (chromatic number) and ρ_1 (point-arboricity).

Proposition 10. *If G is n-durable with respect to ρ_k, then G is n-permanent with respect to ρ_k.*

Proof. Let G be n-durable with respect to ρ_k and let e be any line of G. Let v be an endpoint of e. Then

$$\rho_k(G - e) \geqq \rho_k(G - v) = n.$$

Therefore G is n-permanent with respect to ρ_k.

In contrast to Theorem 3, we have the following result.

PROPOSITION 11. *Let G be a graph such that $\rho_k(G) = n$. Then G is an induced subgraph of an n-durable graph.*

Proof. Let H be the graph having two components, each isomorphic to G. Then H is n-durable, and G is an induced subgraph of H.

From the proofs of Theorem 3 and Proposition 11, it is clear that, if $\rho_k(G) = n$, then G also contains an induced n-minimal subgraph (if $n \geqq 2$) and is an induced subgraph of an n-permanent graph.

It is well known that if G is an n-critical graph with respect to chromatic number, then the minimum degree of G satisfies the inequality $\delta(G) \geqq n - 1$. Chartrand and Kronk [**3**] proved that if a graph G is n-critical with respect to point-arboricity, then $\delta(G) \geqq 2(n - 1)$. We now extend these results to graphs G which are n-critical with respect to ρ_k, for arbitrary non-negative integers k.

THEOREM 4. *If the graph G is n-critical with respect to ρ_k, then $\delta(G) \geqq (k + 1)(n - 1)$.*

Proof. There are no non-trivial 1-critical graphs with respect to ρ_k, and so we assume that $n \geqq 2$. Suppose that G is n-critical with respect to ρ_k and that G contains a point v with $d(v) < (k + 1)(n - 1)$. Since G is n-critical with respect to ρ_k, $\rho_k(G - v) = n - 1$, and there is a partition $V_1, \ldots, V_{n-1}$ of $V(G - v)$ such that each induced subgraph $\langle V_i \rangle$, $i = 1, \ldots, n - 1$, is a k-degenerate subgraph of $G - v$. Since $d(v) < (k + 1)(n - 1)$, at least one of these subsets, say V_j, contains at most k points adjacent to v. Thus, the set $V_j \cup \{v\}$ necessarily induces a k-degenerate subgraph of G. Hence, $V_1, \ldots, V_j \cup \{v\}, \ldots, V_{n-1}$ is a partition of $V(G)$ such that each subset induces a k-degenerate subgraph of G. This implies that $\rho_k(G) < n$, which contradicts the fact that G is n-critical with respect to ρ_k. Therefore, $\delta(G) \geqq (k + 1)(n - 1)$.

The observation that any graph G having at most $k + 1$ points is k-degenerate leads to the fact that if $\rho_k(G) = n$, then G has at least $(k + 1)(n - 1) + 1$ points. Hence, any graph G which is n-critical (n-minimal) with respect to ρ_k has at least $(k + 1)(n - 1) + 1$ points. In fact, the following proposition shows that the unique graph G with $\rho_k(G) = n$ and $(k + 1)(n - 1) + 1$ points is $G = K_{(k+1)(n-1)+1}$.

PROPOSITION 12. *If G is a graph with $(k + 1)(n - 1) + 1$ points, then $\rho_k(G) = n$ if and only if $G = K_{(k+1)(n-1)+1}$.*

Proof. That $\rho_k(K_{(k+1)(n-1)+1}) = n$ follows immediately from Corollary 3. Conversely, assume that $\rho_k(G) = n$ and that G has $(k + 1)(n - 1) + 1$ points. If $n = 1$, then trivially $G = K_1$. Otherwise, let u and v be any pair of points of G. If u and v are not adjacent in G, then we may partition $V(G)$ into $n - 2$ sets each containing $k + 1$ points and one set containing u and v as well as k other points. Since any graph with $k + 1$ points is k-degenerate, and any subgraph of $K_{k+2} - e$ is also k-degenerate, it follows that the point set $V(G)$ can be partitioned into $n - 1$ sets, such that each set induces a k-degenerate subgraph of G. Thus $\rho_k(G) < n$, which is a contradiction. Therefore each pair of points of G is adjacent, and $G = K_{(k+1)(n-1)+1}$.

Since the unique smallest (with respect to the number of points) graph with $\rho_k(G) = n$ is $G = K_{(k+1)(n-1)+1}$, the smallest n-critical (n-minimal) graph with respect to ρ_k is also $K_{(k+1)(n-1)+1}$. It follows that any n-durable (n-permanent) graph with respect to ρ_k has at least $(k + 1)(n - 1) + 2$ points. This leads to the following proposition.

PROPOSITION 13. *If G is a graph with $(k + 1)(n - 1) + 2$ points, where $n > 1$ and $k > 0$, then G is n-durable (n-permanent) with respect to ρ_k if and only if $G = K_{(k+1)(n-1)+2}$.*

The proof, being similar to that of Proposition 12, is omitted.

Proposition 13 does not cover the case of durability and permanence with respect to chromatic number ($\rho_k(K_{(k+1)(n-1)+2}) = n$, by Corollary 3, *unless* $k = 0$). In [**9**] we have shown that the unique smallest n-durable graph with respect to ρ_0 is the graph $2K_n$, consisting of two disjoint copies of K_n. The smallest n-permanent graph with respect to ρ_0 is the graph $K_n \cdot K_n$; that is, the one point union of two copies of K_n.

6. Bounds for the parameters ρ_k. As mentioned earlier, the upper bound given in Corollary 5 is not particularly good. We now present a generally sharper upper bound for the parameter ρ_k, together with a lower bound. For a graph G and a non-negative integer k, let M_k denote the maximum number of points in G which induce a k-degenerate subgraph of G. The number M_0 is also called the *point-independence number* of G.

THEOREM 5. *Let G be a graph with p points and let k be a non-negative integer. Then*

$$\frac{p}{M_k} \leqq \rho_k(G) \leqq \left\{\frac{p - M_k}{k + 1}\right\} + 1.$$

Proof. We first establish the lower bound. Let $S_1, \ldots, S_n$ be a minimum partition of $V(G)$ such that S_i, $i = 1, \ldots, n$, is a k-degenerate subgraph of G.

Thus $\rho_k(G) = n$ and $|S_i| \leqq M_k$, $i = 1, \ldots, n$. Therefore,

$$\sum_{i=1}^{n} |S_i| \leqq nM_k = \rho_k(G)M_k.$$

But

$$\sum_{i=1}^{n} |S_i| = p,$$

so that $\rho_k(G) \geqq p/M_k$.

In order to establish the upper bound, let S be a set of points of G such that S is a k-degenerate subgraph of G and $|S| = M_k$. Let $G - S$ denote the subgraph of G obtained by removing the points in S. Then $\rho_k(G - S) \geqq \rho_k(G) - 1$. Since $G - S$ has $p - M_k$ points, we apply Corollary 5 to obtain

$$\rho_k(G - S) \leqq \left\{\frac{p - M_k}{k + 1}\right\}.$$

Therefore,

$$\rho_k(G) \leqq \left\{\frac{p - M_k}{k + 1}\right\} + 1.$$

We now give additional upper bounds for $\rho_k(G)$. It is well known that $\rho_0(G) \leqq 1 + \Delta(G)$. Chartrand, Kronk, and Wall [**4**] showed that

$$\rho_1(G) \leqq 1 + [\tfrac{1}{2}\Delta(G)].$$

As a corollary to the next theorem, it will follow that

$$\rho_k(G) \leqq 1 + \left[\frac{\Delta(G)}{k + 1}\right]$$

for each non-negative integer k.

We use the notation $H < G$ to indicate that H is an induced subgraph of G. In [**11**], Szekeres and Wilf proved that

$$\rho_0(G) \leqq 1 + \max_{H<G} \delta(H),$$

where the maximum is taken over all induced subgraphs H of G. Chartrand and Kronk [**3**] extended the result to point-arboricity:

$$\rho_1(G) \leqq 1 + \left[\tfrac{1}{2} \max_{H<G} \delta(H)\right].$$

These results are now generalized for all of the parameters ρ_k.

THEOREM 6. *For any graph G,*

$$\rho_k(G) \leqq 1 + \left[\frac{\max_{H<G} \delta(H)}{k + 1}\right],$$

where the maximum is taken over all induced subgraphs H of G.

Proof. If G is a k-degenerate graph, then the result is obvious; thus we assume that $\rho_k(G) = n \geqq 2$. There exists an induced subgraph H of G which is n-critical with respect to ρ_k. It follows that

$$\delta(H) \leqq \max_{G'<H} \delta(G'),$$

since H itself is an induced subgraph of H. Moreover, any induced subgraph of H is also an induced subgraph of G, and so

$$\max_{G'<H} \delta(G') \leqq \max_{G'<G} \delta(G').$$

Since H is n-critical with respect to ρ_k, $\delta(H) \geqq (k+1)(n-1)$, and so

$$\max_{G'<G} \delta(G') \geqq (k+1)(n-1) = (k+1)n - (k+1) = (k+1)\rho_k(G) - (k+1).$$

The desired result follows from this inequality.

Since $\delta(G') \leqq \Delta(G)$ for any induced subgraph G' of G, we have the following corollaries.

COROLLARY 6. *For any graph G,*

$$\rho_k(G) \leqq 1 + \left[\frac{\Delta(G)}{k+1}\right].$$

COROLLARY 7. *If G is a planar graph, then:* $\rho_1(G) \leqq 3$, $\rho_k(G) \leqq 2$ *for* $k = 2, 3, 4$, *and* $\rho_k(G) = 1$ *for* $k \geqq 5$.

There are many other bounds for the chromatic number and point-arboricity of a graph that can be generalized to include all point partition numbers ρ_k. For example, if λ is the largest eigenvalue of the adjacency matrix A of graph G, then

$$\rho_k(G) \leqq 1 + \left[\frac{\lambda}{k+1}\right].$$

The case where $k = 0$ (chromatic number) was proved by Wilf [**12**], and Mitchem [**10**] has extended this result to the case $k = 1$ (point-arboricity).

7. Remarks concerning the four-colour conjecture. One of the truly famous unsolved problems in mathematics is the four-colour conjecture. In the terminology of this paper the conjecture may be stated as follows. For any planar graph G, $\rho_0(G) \leqq 4$. In this section, we show that if G is a counter-example to this conjecture, then $G \in \Pi_5 - \Pi_4$. The proof of the following theorem is essentially part of that employed by Behzad and Chartrand [**1**] in giving a version of Kempe's incorrect "proof" of the four-colour "theorem".

THEOREM 7. *If G is a planar 4-degenerate graph, then* $\rho_0(G) \leqq 4$.

Proof. We use induction on p, the order of G. For $p = 1$, the result is trivial. Thus, let G be a planar 4-degenerate graph with p points. Since G is 4-degenerate, $\delta(G) \leqq 4$; let $v \in V(G)$, with $d(v) \leqq 4$. By the induction hypothesis, since $G - v$ is also planar and 4-degenerate, $\rho_0(G - v) \leqq 4$; i.e. the graph $G - v$ can be four-coloured. Let a proper four-colouring of $G - v$ be given. If less than four colours are required in G for the points adjacent to v, v may be coloured with a fourth colour, so that $\rho_0(G) \leqq 4$. Suppose, however, that v is adjacent to v_1, v_2, v_3, v_4 in G, with v_i given colour i in the four-colouring of $G - v$, $i = 1, 2, 3, 4$. Without loss of generality, suppose these points to be arranged cyclically around v as indicated in Figure 4. Consider the subgraph

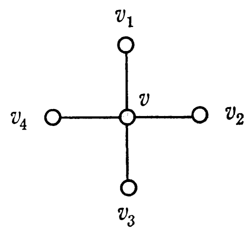

FIGURE 4

H of $G - v$ induced by those points coloured either 1 or 3. If v_1 and v_3 are in different components of H, then there is no path all of whose points are coloured 1 or 3 joining v_1 and v_3 in $G - v$. In this case, the colours of the points in the component containing v_1 may be interchanged, and then v may be given the colour 1, showing that $\rho_0(G) \leqq 4$. On the other hand, if v_1 and v_3 are in the same component of H, then there is a path joining v_1 and v_3 in $G - v$, all of whose points are coloured 1 or 3. Then this path together with v and the lines v_3v, vv_1 forms a cycle in G enclosing v_2. It therefore follows that there is no path joining v_2 to v_4 in $G - v$, all of whose points are coloured 2 or 4. Let F be the subgraph of $G - v$ induced by those points coloured either 2 or 4; then v_2 and v_4 are in different components of F. Now interchange the colours of the points in the component of F containing v_2. Then v may be given the colour 2, showing that $\rho_0(G) \leqq 4$. As this exhausts the possibilities, the proof is complete.

Since any planar graph is 5-degenerate, and any planar 4-degenerate graph is four-colourable, it follows immediately that any planar graph which cannot be four-coloured must be 5-degenerate, but not 4-degenerate.

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